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Visualization as an Intuitive Process in Mathematical Practice

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Abstract

In the field of the philosophy of mathematics, in recent years, there has been a resurgence of two processes: intuition and visualization. History has shown us that great mathematicians in their inventions have used these processes to arrive at their most brilliant proofs, theories and concepts. In this article, we want to defend that both intuition and visualization can be understood as processes that contribute to the development of mathematical knowledge as evidenced in the history of mathematics. Like intuition, visualization does not have a definition, and its role has become more prominent both in pure mathematics and in educational research. For us, both visualization and intuition are processes that start from the real world of those who "intuit" or "visualize," require experience and knowledge of concepts and theories (the more expertise in the subject, the more profound the results will be) and must, in the end, be validated by the specialized academic community. In this article, we defend the importance of visualization in mathematical practice and its role in the advances of great scientists (Euclid, Euler, Galileo, Descartes, Newton, Maxwell, Riemann, Einstein, Feynman, among others) as an alternative and valuable way to symbolic thinking, which has "reigned" in the academic and scientific community.

Keywords: *visualization; intuition; dynamic process; mathematical practice*

I. Introduction

In recent decades, there has been a pressing need to extend the theory of mathematical knowledge that addresses epistemological issues, including "conceptual fecundity, evidence, visualization, diagrammatic reasoning, understanding, explanation, and other aspects of the theory of mathematical knowledge that are orthogonal to the problem of access to 'abstract objects.'"[1] The renewal of the philosophy of mathematics must include a fundamental aspect such as mathematical practice. Some philosophical problems become relevant only when a certain area of mathematics is taken into consideration: "for example, geometry, node theory, and algebraic topology are sure to arouse interest (and philosophical bewilderment) on the subject of diagrammatic reasoning and visualization."[2] And precisely the issue of visualization, the subject of this article, seems to be useful to address important problems in the philosophy of mathematics.

Visualization processes have become a central topic of interest thanks to the development of computer images in differential geometry, chaos theory, topology, geometry, and complex analysis. In recent research we find arguments that indicate that mathematical visualization has played an epistemic role, since visual resources are fundamental in the cognitive grasp of structures.[3] In recent discussions on the philosophy of mathematics, the topic of visualization and schematic reasoning has become relevant.

Now, the first objective of this article is to present arguments that support one of the theses that we want to defend, namely that visualization, as well as intuition, can be understood as fundamental processes in the development of the epistemology of mathematics.[4] We are particularly interested in affirming that visualization is a process that allows the mathematician (or student) to build and expand their knowledge system in mathematics. Thus, we will take the following definition as a starting point:

> Intuition is a process, where the real world and the individual's prior knowledge play an important role; and in the course of this process, the need for logic to formalize the findings obtained by intuition cannot be ignored.[5]

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[1] Paolo Mancosu, "Algunas Observaciones Sobre La Filosofía de La Práctica Matemática," *Disputatio Philosophical Research Bulletin* 5, no. 6 (2016): 131-156.

[2] Ibid., 132.

[3] Zachary Hawes et al., "Relations between Numerical, Spatial, and Executive Function Skills and Mathematics Achievement: A Latent-Variable Approach," *Cognitive Psychology* 109 (2019): 68-90.

[4] Robert James Brown, "Naturalism, Pictures and Platonic Intuitions," in *Visualization, Explanation and Reasoning Styles in Mathematics*, eds. Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen, 57-73 (Dordrecht: Springer, 2005).

[5] Lina María Peña-Páez, "Consideraciones Sobre La Intuición Matemática," *Agora-Papeles de Filosofía* 39, no. 2 (2020): 127-141.

And the following:

> Visualization is the capacity, process and product of the creation, interpretation, use and reflection on figures, images, diagrams, in our mind, on paper or with technological tools with the purpose of representing and communicating information, thinking and developing ideas and advance understanding.[6]

We also consider that both visualization and intuition are dynamic processes that require individual experience (particularly in the mathematical context) that cannot be ignored in any of the stages of this process. That is, whoever is not familiar with the concepts, the statements, the diagrams, in general, with mathematical or visual thinking, will have certain difficulties in inventing theories, analyzing a graph or building new mathematical knowledge. Thus, intuition, like visualization, requires experience, practice, and solid mathematical knowledge.

Both intuition and visualization have enabled mathematicians during their practice to deduce and "discover" advanced properties and concepts. As well as intuition, visualization in mathematics has had a resurgence in recent decades, due to the development of different areas such as computer science, mathematics education, science itself, psychology and philosophy.[7]

Great mathematicians have used mathematical intuition to "arrive" at their great ideas, as they themselves have evidenced in their works.[8] However, his ideas regarding definition and intuition are different. Something similar happens with the idea of visualization:

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[6] Abraham Arcavi, "The Role of Visual Representations in the Learning of Mathematics," *Educational Studies in Mathematics* 52, no. 3 (2003): 215-241.

[7] George Polya, *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving* (New York: John Wiley & Sons, 1980); Richard Tieszen, *Mathematical Intuition: Phenomenology and Mathematical Knowledge* (Dordrecht: Kluwer Academic Publishers, 1989); Philip Kitcher, *The Nature of Mathematical Knowledge* (New York: Oxford University Press, 1984); Efraim Fischbein, *Intuition in Science and Mathematics: An Educational Approach* (Dordrecht: D. Reidel, 2002); and Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen., eds., *Visualization, Explanation and Reasoning Styles in Mathematics* (Netherlands: Springer, 2023).

[8] Kurt Gödel, *Obras Completas,* trans. Jesús Mosterín (Madrid: Alianza, 2006); Henri Poincaré, "La Intuición y La Lógica En Las Matemáticas," in *El Valor de La Ciencia*, trans. Carlos S. Chinea (Madrid: Espasa-Calpe, 1964), 1-9; Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* (New York: Donver, 1954).

> In the end, the term visualization certainly does not have a "usual meaning." It was used in the literature as a noun to describe a graphic representation, as a verb to describe the process of creating a graphic representation, and commonly as a synonym for visual image.[9]

The history of mathematics has shown the importance of visualization and its predominant role in many of the advances of great scientists (Euclid, Euler, Galileo, Descartes, Newton, Maxwell, Riemann, Einstein, Feynman, among others). However, verbal thought and symbolic language have prevailed as the "best options" to present the results to the scientific and academic community.

Visualization is a complex process that implies the organization of all the available information and the reconfiguration of the previous information, thus, new points of view will be generated to address problems that, in the end, could be solved or not. Fischbein states that when visualization is incorporated into cognitive activity, it becomes an essential factor contributing to intuitive understanding. Likewise, visual representations allow the organization of information in synoptic representations, which entail an important factor of globalization.[10]

In the second part of this article, we will focus on visualization and its role in the practice of well-known mathematicians, showing how visual thinking has allowed significant advances in mathematics (which does not imply ignoring that symbolic language and its demonstrations have also allowed great advances). Since the 90s, program design has involved formal reasoning systems that use diagrams to establish their validity. Indeed, there are reasons to avoid becoming formal: in a formalized version of a proof, the original intuitive train of thought may be obscured by a multitude of painstaking steps.[11]

Trying to give visualization the rigorous and dogmatic character of formalization "seems to deprive visualization of its effectiveness and simplicity, which are, on the contrary, its most interesting aspects from a cognitive point of view."[12] The "visual tests" also have a step by step like the language tests. Likewise, in the process of "discovery" we are

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[9] Linda M. Phillips, Stephen P. Norris, and John S. Macnab, *Visualization in Mathematics, Reading and Science Education* (New York: Springer, 2010), 18.

[10] Fischbein.

[11] Paolo Mancosu, *The Philosophy of Mathematical Practice* (New York: Oxford University Press, 2008).

[12] Valeria Giardino and Gian Carlo Rota, "Intuition and Visualization in Mathematical Problem Solving," *Topoi* 29, no.1 (2010): 29-39.

finding a justification for what we want to prove. Cases in history show examples of this and "demonstrate how intuitive thinking or visualizations are adequate elements in the process of finding a solution to a problem or feeling justified in our beliefs."[13] Therefore, we can reject the premise that leads to an opposition between visual, intuitive and linguistic processes in mathematical reasoning.[14]

Mathematics is a complex phenomenon and goes beyond the proof or the dogma of logic. If the case of teaching and research is taken, there is no need (for example, in Poincaré or Gödel) of having to choose between modern logic (formalization, rigor) and multimodal merit (practical reasoning). It is important to recognize that display objects are elements that can lead to mathematical proofs. Any mathematics teacher can confirm that explaining a full proof is not useful for the immediate understanding of the student, "In fact, often images or informal arguments will play an 'ideal' explanatory role, whereas a full proof will be no explanation at all in that context."[15]

When the dogma of logic is not left, priority is given to the activity of "proving" belittling the idea of "looking for reasons." The development of mathematics seems to show that the need for a theorem is found after digging deep and focusing attention on the possibilities that that theorem offers. In this same sense, many mathematicians are not only interested in proving their conjectures but in finding the reasons why the conjecture is true. Proving a proposition does not provide reasons why it "works."

II. Visualization and intuition

Relating vision with intuition is an idea that we have found since Plato with his "intellectual eyes,"[16] going through Kant, who used "visual imagination as a means to obtain intuitive awareness of abstract objects,"[17] even mathematicians like Gödel[18] for whom intuition is

[13] Ibid., 33.

[14] Henri Poincaré, "La Intuición y La Lógica En Las Matemáticas," in *El Valor de La Ciencia*, trans. Carlos S. Chinea (Madrid: Espasa-Calpe, 1964), 1-9.

[15] Giardino and Rota, 32.

[16] Karl Popper and John Eccles, *El Yo y Su Cerebro*, trans. Carlos Solís Santos (Barcelona: Labor, 1993), 51.

[17] Elijah Chudnoff, "Intuition in Mathematics," in *Rational Intuition: Philosophical Roots, Scientific Investigations*, eds. Lisa M. Osbeck and Barbara S. Held, 174-191 (Cambridge: Cambridge University Press, 2014).

[18] Gödel, *Obras Completas*.

"a guide or global vision, which does not grant immediate or fallible knowledge."[19] The philosophy of science does not escape this relationship either: "we feel the fundamental need to 'see' with our mind, as we see with our eyes."[20] Therefore, "an alternative way of describing mathematical intuition would be to define it as the ability to perpetuate the function of vision, but by means other than the eyes."[21]

Reviewing the literature regarding the notion of mathematical intuition, we find that it does not have a definition in which both mathematicians, philosophers and even educators fully agree. For example, some have assumed it as "the third eye" that only prodigious mathematicians like Ramanujan have. Others have used it to represent "informal, or loose, or visual, or holistic, or incomplete, or perhaps even convincing despite lack of evidence."[22]

One of the characteristics of intuition is its apparent immediacy, which refers to the fact that after reviewing a theory several times when we see some of its results it seems obvious to us, but it is because of all the mathematical experience behind this theory. In this context, Fischbein[23] suggests that the main factor contributing to this immediacy effect is visualization. And although it seems trivial, it is still true "that one naturally tends to think in terms of visual images and that what one cannot visually imagine is difficult to achieve mentally."[24] So much so that mathematicians like Poincaré called geometers those who for him had a more intuitive thought and "Hilbert, when escryibing the ways in which a mathematician thinks, reminds us of the fundamental role of images."[25]

Hence, some authors strongly associate intuition with vision: "mathematical practice reveals that intuitions play an indispensable role and that visualizations are important tools for generating strong intuitions." This occurs not only in geometry, but also in algebraic theories.[26] In special cases, it is possible to infer correct mathematical theories or propositions from images, in the same way that after an

[19] Lina María Peña-Páez, "Filosofía de La Matemática: La Intuición En El Pensamiento de Kurt Gödel," *Filosofia Unisinos* 22, no. 2 (2021): 1-13.

[20] Fischbein, 7.

[21] Giardino and Rota, 30.

[22] Tieszen, 11.

[23] Fischbein, 7.

[24] Ibid., 103.

[25] Ibid.

[26] Leon Horsten and Irina Starikova, "Mathematical Knowledge: Intuition, Visualization, and Understanding," *Topoi* 29, no. 1 (2010): 2.

intuitive process we could reach true conclusions. Visual representations have a role in knowledge, they allow us to recognize and identify properties, make inferences and, why not, make mistakes. Hence, we can conceive visual objects as devices that contribute to the process of mathematical intuition.

For Fischbein, it is not possible to think of geometric points or lines without visualizing them, and we are "trapped" in intuitive representations, since it seems impossible to think of time without spatializing it. The point is that these representations are not possible to manipulate conceptually.[27] For Bergson, spatialized time is different from the time of consciousness, which he calls duration[28] and only intuition is capable of grasping this duration: "we consider that the spatialized representation of time is also a matter of intuitive elaboration."[29] The individual is constantly translating the operations into spatial representations that are then converted into images (into visual representations, for the subject of this chapter). So:

> Visualizations can be realistic or schematic and can represent the directly visualizable or the non-visualizable. Furthermore, the effectiveness of visual representations is related to the contexts in which they are used; there is no direct path from visualization to understanding.[30]

A visual representation could be one of those ways that intuition shows its conclusions and generalities. Assuming that neither intuition nor visualization are forms of immediate knowledge of mathematical facts, we will understand mathematical activity as "the result of the interconnections between acquired knowledge and unstable beliefs: the mathematical knowledge system is dynamic and always open to reconfiguration." In fact, the results of mathematical practice show that "intuition and visualization are interrelated parts of a vast network of knowledge."[31]

Thanks to the intuitive process, the interconnections are preserved, allowing us to reach generalities, conclusions and the stability of certain beliefs. It can be stated that:

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[27] Henri Bergson, *Ensayo Sobre Los Datos Inmediatos de La Conciencia*, trans. Juan Miguel Palacios (Salamanca: Ediciones Síguema, 1999).

[28] Henri Bergson, *Introducción a La Metafísica y La Intuición Filosófica*, trans. M. Hector Alberti (Buenos Aires: Ediciones Leviatan, 1956).

[29] Fischbein, 8.

[30] Phillips, Norris, and Macnab, 9.

[31] Giardino and Rota, 39.

> Intuitive processes and visualization appear as something profoundly natural, both in the birth of geometric thought and in the discovery of new relationships between mathematical objects and also, naturally, in the transmission and communication typical of mathematical activity.[32]

Visualization research suggests that in mathematical practice images are necessary for the development of intuition.[33] Although somewhat relegated to them, recent studies also show that graphs provide something additional and important to mathematical knowledge and proofs.[34]

Historically, visualization and intuition have been given greater importance in geometry. However, Cayley's graphs are a good example of how to expand their importance to algebra and real analysis to understand the notion of a group: "the evolution of geometric group theory strongly suggests that mathematical intuition, in certain cases, such as a matter of empirical fact, it has depended on pictorial representations for its growth and development."[35] Here the idea of intuition is being used "as something that is capable of development through systematic theoretical reasoning and an increasingly deep and variable understanding of concepts."[36]

Therefore, we are not assuming that, when observing a graphical representation, the concept will be evident to us or we will immediately understand a theory. Like intuition, a good graph requires some preconceptions, a knowledge of what you want to exemplify or demonstrate with said visual representation. If we do not have an idea, for example, of what a group means, or an educated intuition in this field of study, it will not be easy to understand Cayley's graphs. If a student who has never looked at something like the definition of a group is pre-

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[32] Inés Gómez-Chacón, *Visualización Matemática: Intuición y Razonamiento* (Madrid: Universidad Complutense, 2012), 203.

[33] Horsten and Starikova, "Mathematical Knowledge," 1-2; Giuseppe Longo and Arnaud Viarouge, "Mathematical Intuition and The Cognitive Roots of Mathematical Concepts," *Topoi* 29, no. 1 (2010): 15-27; and Luciano Boi, "The Role of Intuition and Formal Thinking in Kant, Riemann, Husserl, Poincare, Weyl, and in Current Mathematics and Physics," *Kairos – Journal of Philosophy & Science* 22, no. 1 (2019): 1-53.

[34] Johanna Pejlare, *On Axioms and Images in The History of Mathematics* (Uppsala: Uppsala University, 2007).

[35] Irina Starikova, "Why Do Mathematicians Need Different Ways of Presenting Mathematical Objects? The Case of Cayley Graphs," *Topoi* 29, no. 1 (2010): 41.

[36] Ibid., 41-42.

sented with a Cayley graph, can they figure out what that graph means? Can you understand its construction? The answer is no. As in intuition, knowledge is produced after a process, it is not immediate knowledge,

> Let us consider a representation of groups as symmetries of geometric objects. It is easy to understand how the group acts when it is determined by some (geometric) object. The medium for this insight could be a picture or paper model of an equilateral triangle, on which twists and rotations can be performed. This gives a physical or geometric intuition of the group operation: the composition of the movements; the axiom of the existence of an inverse element would be intuited as performing a backward transformation: if we rotate a triangle 120 degrees clockwise we can rotate it backwards and obtain the initial state.[37]

That is, the graph is the result of an intuition process. Many of the "visual discoveries" have implicit mathematical considerations. And precisely the discovery is reached because all the concepts, theorems, propositions and other elements available to the mathematician are activated when making said discovery: "what triggers the activation of these dispositions is the conscious, in fact, attentive visual experience; but the presence and functioning of these dispositions is hidden from the subject." [38] And while the visual identification process seems easy or immediate, a sense of obviousness occurs. This sensation of the obvious is also present in intuition and is the result "from the exercise of [the] conceptual skills that we have acquired [...]. Or perhaps it derives from the indoctrination that we received in our mathematical youth."[39] Intuitions are introduced by epistemology and there is no reason to believe that a brilliant mathematician (including Gödel) has a "special" ability to have intuitions. Now, neither visualization nor intuition are obvious or immediate processes; only those who have been familiar with "hidden" mathematical concepts could understand what you are trying to prove. Visualization also allows us to understand a problem globally:

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[37] Ibid., 46.

[38] Marcus Giaquinto, "From Symmetry Perception to Basic Geometry," in *Visualization, Explanation and Reasoning Styles in Mathematics*, eds. Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen, 31-55 (Netherlands: Springer, 2005).

[39] Kitcher, 61.

> One of the main functions of pictorial representations in reasoning processes is to produce a global, simultaneous and panoramic account of what is actually a process, a succession of events. Globalization does not necessarily lead to intuitive acceptance, but it can help produce or enhance intuitive acceptance. It can be assumed that the effects of various globalization mechanisms are often combined.[40]

When the visual images materialize, a sensation of evidence or immediacy appears in the individual, or perhaps, as has often been believed, intuition. What is behind the visualization, however, is an organization of the available data into structures that are already meaningful to the mathematician or student. The graphs can serve as a guide – just like one of Gödel's interpretations of intuition – to develop a solution.[41] In the words of Fischbein: "visual representations are an essential anticipatory device."[42] In this sense, we can understand immediacy not as "something" that is perceived directly, but rather that involves the individual, from his emotionality or his mathematical reality and his experiences in other areas.

> Intuition, as we have often stressed, implies a kind of empathy, a type of cognition through direct internal identification with a phenomenon. A visual representation with its rich and concrete details mediates such a personal participation, usually much better than a concept or a formal description [...]. Visual representations and, in general, mental images play a considerable role in creative activity.[43]

So we have that "intuition, as well as visualization, are not a kind of direct access to mathematical facts, but are mediated by knowledge and experience."[44] That is, the experience and mathematical knowledge of the individual are required both for intuition and for the proper analysis of the visual process.

When a problem is posed and it wants to be represented with a figure, the mathematician must be clear about several concepts that will

[40] Fischbein, 120.

[41] Gödel, *Obras Completas*; Kurt Gödel, *Ensayos Inéditos*, trans. and ed. Fransisco Rodríguez Consuegra (Barcelona: Mondadori, 1994).

[42] Fischbein, 104.

[43] Ibid.

[44] Giardino and Rota, 33.

intervene in the solution. The figure is not a simple isolated fact but is an element of a vast system of knowledge. This does not mean that errors do not occur, let's remember that both intuition and visualization are processes that can be fallible. However, by being intertwined with the rest of the shared system of knowledge, practices and procedures, there is a certain guarantee of reliability.

However, it will always be necessary to verify that "(i) the hypotheses introduced are correct and coherent with the knowledge system that is assumed (checking of pre-visual errors), and that (ii) the visual medium does not introduce its own restrictions on the representation of the target area (checking for post-visual errors)."[45] Checking (i) and (ii) can be done in the course of practice. Furthermore, this is precisely what mathematicians have done in their daily work.

As this visualization process is fallible, errors can occur in the deductions, one could be by raising a wrong hypothesis about how to draw a figure or the wrong hypothesis about the properties of the figure. Haven't false properties been deduced using symbolic language? The history of mathematics has also shown us that it is not necessarily true that knowing the definitions implies visualizing the correct path, what is needed is knowing how to use said definitions and propositions to visualize properly.

The stigma of not allowing the advancement of science could be attributed to visualization (for example, the case of Ptolemy and Copernicus), however, the "backwardness" of scientific advances is not necessarily linked to the way in which the mathematician "comes to his theories" or how he presents them to the academic community (with symbolic or graphic language), in many cases, and in particular, in that of Copernicus it is also due to "non-mathematical" beliefs (religion, politics, philosophy, economics), and others to "errors" in mathematics that influence their own development. We cannot forget that it was precisely a diagram, in the book *De Revolutionibus Orbium Caelestium Libri VI* that "revolutionized" science and the world.

Regarding the experience for mathematical knowledge, presenting concrete objects to study abstract objects allows greater familiarization with the latter and more significant interpretations:

> The absorption of the techniques, as well as the more intuitive practices, such as visualization, are controlled by experience. There is nothing like ex nihilo mathematical intuition: it all depends on how familiar we are with the relationships

[45] Ibid., 38.

> in our mathematical knowledge network, as well as how experienced we are with mathematical manipulations.[46]

In this sense, Fischbein[47] reminds us that visual representations are not knowledge in themselves: “visual images are an important factor in immediacy, but immediacy is not a sufficient condition to produce the specific structure of a cognition. Intuitive.”[48] Even if the schematic of an electronic device is perceived, a deep understanding of its operation is not guaranteed, this will only be possible if special training has been received. That is, mathematical experience or the “real world” is required to understand how it works.

We will understand visual images as that device that facilitates the process of intuition. Hence, we can establish a connection between sensory experience and observation in mathematics which, in Gödel’s terms, is analogous to mathematical intuition: “they are connected in another sense: one sees a diagram (sensory perception) which induces an intuition (mathematical perception) of something very different. This is what happens when an image is not simply a heuristic aid, but a real proof.”[49]

Visualization and intuition should not be considered solely as automated reaction systems. They seem automatic because the mind is educated in concepts and theories, because there is a previous mathematical experience, and the internalization of mathematical statements leads to an apparently obvious and immediate reaction. In reality, they are belief systems with autonomous expectations where experience plays a fundamental role

> because, in certain circumstances, it configures stable expectations. Such expectations become so stable, so firmly attached to certain circumstances that their empirical origin can apparently disappear from the subject’s consciousness.[50]

Thus, experience can generate stable visual insights and organized and seemingly autonomous belief systems.

[46] Ibid., 39.

[47] Fischbein.

[48] Ibid., 103.

[49] Brown, 66.

[50] Fischbein, 88.

As we have mentioned, some given situations in mathematical practice are not generated in a natural and direct way from scientific or mathematical notions, sometimes a visual representation could be a "bridge" for understanding or be itself a generator of knowledge. This last point can be evidenced with the aforementioned example of the Cayley's graphs. These graphs have been considered as mathematical objects, not only as useful tools for visualizing groups.

> The richness and variety of insights from Cayley's graphs produce a link to other areas of mathematics, such as graph theory, but they also produce a fruitful link between algebra and geometry. The most important and intriguing impact of GCs on algebra is the new geometry: they are related to the notion of "group."[51]

In addition to the idea of group structure, thanks to these graphs, we can demonstrate in practice how geometric elements/diagrams are combined with algebraic concepts and the same idea of group.

> The main function of intuition in this case highlights the structure of mathematical objects. This was achieved by introducing a "new presentation" of abstract mathematical objects, groups that are not easily intuited, through objects from other areas of mathematics (in our case, graphics).[52]

This is a great example of how diagrams are powerful tools that can facilitate the intuitive process and that in turn can be a good start for new insights that will eventually lead "to advanced conceptual links with geometry and the introduction of a wide arsenal for geometric algebra."[53] Furthermore, it is also clear that the use and understanding of these graphs implies a baggage, a mathematical experience of the individual who is faced with this new knowledge.

We will end this article by describing some examples of how mathematicians have used visualization in their practice.

[51] Starikova, 47.

[52] Ibid., 51.

[53] Ibid.

III. Visualization and mathematical practice

The focus of the philosophy of mathematics is centered on theory. Many philosophers are interested, for example, in checking if a system is consistent, if theorems are true in the nature of objects under a certain theory, among others. And although in recent years there has been a growing interest in claiming the mathematical practice,[54] "when philosophers of mathematics are asked to consider the activity Mathematics, as opposed to bodies of established mathematics, tend to think of the investigative activity of professional mathematicians, typically proving theorems."[55] It seems that this is the only activity they can do. They ignore a whole field of other possibilities, such as creativity, applications, new knowledge, partially true justifications, and the explanation about the understanding of the objects of mathematics, among others.

In this range of possibilities, Mancosu,[56] in the introduction to his book The Philosophy of Mathematical Practice, states that:

> Visualization processes (for example, by means of mental images) are fundamental to our mathematical activity and recently this has once again become a central issue due to the influence of computer images on differential geometry and chaos theory. and the call for visual approaches to geometry, topology, and complex analysis.[57]

For the author, the heuristic use of visual representations is increasingly significant, and it cannot continue to be an ignored topic when mathematical thought is studied. Its importance is clear: "even the algorithms that we are taught in secondary school for the calculation of several digits are visuo-spatial in nature."[58] Let us remember that the visual thought that contributes to the "discovery" is essential for the development of the epistemology of mathematics and, despite this, it is a path that still has a lot to explore.

[54] Jessica Carter, "Philosophy of Mathematical Practice: Motivations, Themes and Prospects," *Philosophia Mathematica* 27, no. 1 (2019): 1-32.

[55] Marcus Giaquinto, "Mathematical Activity," in *Visualization, Explanation and Reasoning Styles in Mathematics*, eds. Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen, 75-87 (Netherlands: Springer, 2005), 75.

[56] Mancosu, "The Philosophy of Mathematical Practice."

[57] Ibid., 14.

[58] Ibid., 39.

Thanks to the incursion of computerized systems for the visualization of complex graphs, it was possible to have a guide to arrive at mathematical demonstrations of very complex statements. Recent decades have seen a "revolution" against purely symbolic mathematics, calling attention to visual methods: "its call for a return to intuition and visualization runs deeper and is rooted in an appreciation of the importance of visual intuition in areas such as geometry, topology and complex analysis."[59]

However, visual demonstrations are also becoming more relevant, not only in the field of mathematics itself but also in education.[60] A demonstration based on images or diagrams, that is, without words, can help to "understand why a mathematical statement is true; They vividly show us why a property is true, and sometimes even suggest how to prove it in a formal way."[61] These types of demonstrations have been forgotten thanks to the impetus and almost obsession of modern mathematics for rigor, "for a few decades, first rescued by didactics and now vindicated from computer computing and experimental mathematics, occupy their deserved space."[62]

Visualization in the history of science has a long history from Euclid's geometry, through the idea of perspective, cartesian geometry, eulerian graph theory. and computer graphics today. Scientists such as Galileo, Descartes, Newton, Maxwell, Riemann, Einstein, Feynman, among others have used visualization expanding its scope thanks to attempts to represent certain natural phenomena that, in many cases, are almost impossible to observe directly:

> So why do scientists bother with visualization? The empirical nature of science means that scientists are often busy making sense of the data they have collected and communicating with other scientists about it. Visualization can

[59] Paolo Mancosu, "Visualization in Logic And Mathematics," in *Visualization, Explanation and Reasoning Styles in Mathematics*, eds. Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen, 13-30 (Netherlands: Springer, 2005), 20.

[60] Demetrios Sampson, J. Michael Spector, and Dirk Ifenthaler, eds., *Learning Technologies Learning, and Large-Scale Teaching, for Transforming Assessment* (Netherlands: Springer, 2019); Zehavit Kohen et al., "Self-Efficacy and Problem-Solving Skills in Mathematics: The Effect of Instruction-Based Dynamic Versus Static Visualization," *Interactive Learning Environments* 4, no. 30 (2022): 759-778; and Sevinç Mert Uyangör, "Investigation of the Mathematical Thinking Processes of Students in Mathematics Education Supported with Graph Theory," *Universal Journal of Educational Research* 7, no. 1 (2019): 1-9.

[61] Bartolo Luque, "Demostraciones Visuales," *Investigación y Ciencia* 445 (2013): 89.

[62] Ibid., 89.

> facilitate these processes by presenting the data in a more accessible way than, say, a table of numbers or a verbal account.[63]

Perhaps it is not a nuisance but a long tradition in which verbal and linguistic thinking has been accepted as the "best" way of presenting the results of science. Now, in contrast to this tradition, the history of science is full of examples where great thinkers have used images to illustrate their findings, even though the demonstrations of their theories included only symbolic language.

For example, Galileo embodied in his drawings the principles of perspective and his interpretation of certain physical phenomena. Descartes and his illustrations on magnetic force and human optics, Newton and his rigorous way of presenting the specific physical states of phenomena. We can continue with Maxwell's drawings of the distribution of magnetic forces in space (a good way to understand that tradition of graphically representing data that is not observed through the senses). We also have Riemann and his complex analysis graphs. Then there is Feynman and the set of diagrams of his representing the interaction between particles (diagrams representing probability functions geometrically), which was an important departure from previous visualization ideas, to the extent that they tried to represent invisible phenomena. Next, we will present some significant facts of history for our study.

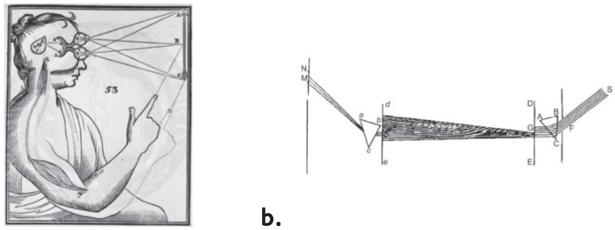
Let's start with Descartes and Newton.[64] They made use of visualization to represent the structure and relationships between the scientific phenomena examined, considering them of great interest for their advances in the field of mathematics and physics. When scientists use images they are not only interested in showing what the world looks like, but how it works:

> Descartes and Newton are two scientists who used numerous illustrations in their scientific work. While most of his scientific theories have long since been superseded, many of his discoveries and achievements are still referenced in contemporary science education. This is certainly the case

[63] John Braga, Linda M. Phillips, and Stephen P. Norris, "Visualizations and Visualization in Science Education," in *Reading for Evidence and Interpreting Visualizations in Mathematics and Science Education*, ed. Stephen P. Norris, 123-145 (Rotterdam: Sense Publishers, 2012), 126.

[64] Jesús Alcolea, "On Mathematical Language: Characteristics, Semiosis and Indispensability," in *Language and Scientific Research*, ed. Wenceslao J. Gonzalez, 223-245 (Cham: Palgrave Macmillan, 2021), 234-237.

> in optics where his works, and occasionally his display objects, are still found.[65]



a. b.

Fig. 1. a. Descartes' illustration of the optical process. **b.** Newton's illustration of the optical process. Both figures appear in Braga, Phillips, and Norris.[66]

From figure 1, we note that in addition to its realism, it literally focuses on the phenomenon of vision. Descartes's illustrations had another purpose as well; convincing not only scientists, but laymen about how the world worked, his intention was to help change the concept. In this illustration, emphasis must be placed on the interpretative demands that are sought, that is, it must identify which elements are important and which are irrelevant for human vision. This implies the need for a textual complement, that is, knowledge of optics is required to better understand the image. For example, it should be understood that the lines that penetrate the eyes are vitally important because they show the path of light rays from the arrow to your eyes. Finally, we recognize that the analogy in the illustration is more direct (although more distractions appear) and, in turn, requires less cognitive demand.

In contrast to the figure of the French philosopher, we now find figure 2. Here we note that Newton's style is more diagrammatic. For the English physicist, the important thing is to concentrate on the phenomenon that the vision describes. Here, the illustrations were aimed at scientists, whose intention was to help them figure out how to reproduce their experiments, as a way of solving problems. Furthermore, he himself established his own conventions: "since Newton's conventions are the direct ancestors of ours, his image may seem less strange despite being much less realistic than Descartes's image."[67]

Newton's figure is completely schematic, the correspondence between its elements and reality is not very precise. It is evident that the interpretive demands of this figure are more challenging than those of

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[65] Braga, Phillips, and Norris, "Visualizations," 136.

[66] Ibid., 137-138.

[67] Ibid., 137.

the Descartes figure. Newton managed to eliminate the distractions, although the cognitive interpretation is much more demanding.

In general, the advancement of science has led scientists to focus their efforts on aspects of reality that are far from the visual experience. Mathematicians have had to search for a "language" to describe unobservable objects, which has given mathematics its function of being the language of science, but, in addition, it has been given another task: to visualize the mathematical expressions that appear in the use of scientific graphics.

Many scientific graphs show a high degree of abstraction, moving further and further away from reality, which entails the difficulty of connecting the elements of the graph with the physical phenomena they represent. We recently found that "theorists and researchers now use 'visualization' as a label for strikingly different processes within the learning of mathematics."[68]

For great mathematicians like Dedekind, Hilbert, and Russell, visual intuitions were unreliable.[69] Furthermore, they affirmed that in a good book there should be no figures. However, at the end of the 20th century there was a shift in favor of visualization, which can be evidenced by the titles of some books such as "visual geometry and topology" or "complex visual analysis." A special pedagogical attention to visualization begins and computer- generated images begin to bear fruit in research. What we do not yet have an agreement on is the role of visual thinking in the epistemology of mathematics.

Now, sometimes, even if the proof of a mathematical statement is correct, it does not necessarily imply that we are convinced that we understand said argument, it seems that something more is needed. "One of the many reasons accepted in practice for preferring one formulation over another is that one way of framing and approaching an issue may be more fruitful than another."[70] For some, that one argument is more fruitful than another, is something that is assumed as a "natural" matter that seems to lead to "easier" understanding.

Under this framework, we have the example of Riemann and the use of visual devices to represent complex functions. At the beginning

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[68] Elaine Simmt et al., "Curriculum Development to Promote Visualization and Mathematical Reasoning: Radicals," in *Reading for Evidence and Interpreting Visualizations in Mathematics and Science Education*, ed. Stephen P. Norris, 147-163 (Rotterdam: Sense Publishers, 2012), 148.

[69] Roy Cook and Geoffrey Hellman, eds., *Hilary Putnam on Logic and Mathematics* (Cham: Springer, 2018); Mancosu, Frovin, and Pedersen.

[70] Jamie Tappenden, "Proof Style and Understanding in Mathematics I: Visualization, Unification and Axiom Choice," in *Visualization, Explanation and Reasoning Styles in Mathematics*, eds. Paolo Mancosu, Klaus Frovin Jørgensen, and Stig Andur Pedersen, 147-214 (Dordrecht: Springer, 2005), 152.

of the 20th century, a division arose regarding the test methods in complex analysis. The two "currents" were led by Weierstrass and Riemann, the first with a purely algorithmic approach focused on finding explicit representations of functions, and the second, focused on the conceptual: "it aimed to describe functions in terms of general properties and demonstrate results of existence of indirect functions that need not be linked to explicit representations."[71]

The Riemannian approach involves the use of surfaces that allow easy visualization of complex functions,[72] his reference to visualization contributed to the fecundity of the connections as the examples became more elementary and manageable. Thus, a difficult case like complex analysis, on a smaller scale, is exemplified in the application of classical projective geometry in graphical statics. Here we find a possibility to visualize the arguments and the analysis of the theoretical framework.

Riemann surfaces were not only very useful for being consistent, but have consistently continued to facilitate understanding and discovery. His visual devices gave novel and unexpected results, which is why the academic community accepted them as an adequate context to study functions of interest in complex analysis, turning them into "an indispensable essential component of the theory; not a supplement, more or less artificially distilled from the functions, but their native soil, the only soil in which the functions grow and thrive."[73]

For the case that we are addressing, the fact that Riemann's methodology is more natural and that its results are fruitful has nothing to do with the subjectivity of the individual, moreover, new interesting knowledge has been built on its results. When the "more natural" formulations are studied, it is done in the context of discovery and for some mathematicians this is part of psychology and not of methodology or mathematical practice. But these judgments need to be broader: "the advantages and shortcomings of the Riemannian approach to complex analysis compared to the Weierstrass approach is just one of many concrete examples that illustrate and anchor the point."[74]

Visualization is part of mathematical practice and can be a good way to formulate a problem or a theory. For example, the intuitive geometric aspect has influenced topology and although we cannot say

[71] Ibid., 149.

[72] Boi, "The Role of Intuition and Formal Thinking," 1-53.

[73] Tappenden, 152.

[74] Ibid., 154.

that it was its main impulse, it is the result of visualizing other problems such as complex analysis with Riemann, mechanics with Poincaré and group theory with Denh. It is possible to describe the importance of visualization in mathematical reasoning, leaving aside the nature of visualization itself and recognizing its usefulness and effectiveness in mathematical practice.

With visualization often happens what happens with mathematical intuition, some philosophers have ignored it and find it uninteresting because they assume that they are accidental, pragmatic, subjective or psychological "phenomena."[75] Some only give visualization the place of support to remember some "complicated" propositions. What is interesting in the case of Riemann is not only to recognize that the "connection with the vision is an interesting and useful advantage, [but] that the issues raised by the Riemann-Weierstrass opposition are of interest independently."[76]

An important point that has been observed in the advance of the most outstanding physical and mathematical theories of the last centuries is due to the idea of unification. We have, for example, Newton and the unification of the celestial theory with the terrestrial, Maxwell and the unification of optical, magnetic and electrical phenomena and the current physical theories that try to unify quantum mechanics and gravitation. Under this idea of unification, we find Riemann's approach to the theory of complex functions, in which "a variety of points of view is admitted, in part because he effected the unification of the theory of complex functions with the theory of curves and complex surfaces."[77] Likewise, one of the hallmarks that identified the german mathematician's proposal was the appropriate choice of definitions and basic unifying principles. Riemann's example is an invitation to improve the idea "of how this kind of indirect connection with vision can inform our choice of theoretical frameworks."[78]

Although in some cases, the visualization of a representation that occurs in the mathematician's mind does not lead directly to a rigorous proof, it does lead to an outline, or in Poincaré's words: to a "sort of moral certainty." An example of this case is found with Klein, who, when studying abstract questions in the theory of functions, replaces his Riemann surface with a metallic surface whose electrical conductiv-

[75] Mario Bunge, *La Ciencia, Su Método y Su Filosofía* (Buenos Aires: Fundación Promotora Colombiana, 2002)

[76] Tappenden, 157.

[77] Ibid., 159.

[78] Ibid., 180.

ity varies according to certain laws. In addition, he connects two points with two batteries: "the signal, he says, must pass, and the distribution of this current on the surface will define a function whose singularities will be precisely those requested by the statement of the problema."[79] For Klein, this situation is not only a passive representation of reality, but that representation that he had visualized in his mind contributed to a solution that could be global, which was preliminary, which was still unfinished but which in turn would show the way. of the final solution.

> The visual representation was more than an image; it was the intuitive solution to a problem in which the sensory-mental structure played a fundamental role. In fact, it is not just visual images that help structure intuitions - although they are certainly the most common form of imaginary support. Sounds, in the case of musicians; muscular, motor and tactile representations, in the case of sculptors, etc. They play a fundamental role in artistic creative activity. In a discussion with Max Wertheimer, one of the founders of Gestalt psychology, Einstein once stated referring to the creation of the theory of relativity: "These thoughts did not come in any verbal formulation. I rarely think in words. A thought comes, and then I can try to put it into words" (Wertheimer, 1961, p. 228). Mental images are, in fact, part of a more complex psychological domain [...], namely, the domain of mental models.[80]

We can deduce that he is trying to defend that the idea of visualization, as well as that of intuition, are constructive processes, which have meanings in themselves. What is interesting is the role that these processes can play at the time of a philosophical explanation of mathematical knowledge or in mathematics education.[81]

The history of mathematics has shown us some episodes in which visual reasoning has led to errors that have later been corrected symbolically. These situations have led great mathematicians to emphasize symbolic proofs over visual ones.

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[79] Poincaré, "La Intuición y La Lógica," 2.

[80] Fischbein, 105-106.

[81] Yacin Hamami and Rebecca Lea Morris, "Philosophy of Mathematical Practice: A Primer for Mathematics Educators," *ZDM – Mathematics Education* 52, no. 6 (2020): 1113-1126.

> The existence of the delusions of the senses is not an obstacle to our knowledge of physics; it is an obstacle to the thesis that the sensory processes that actually guarantee our beliefs could continue to do so, no matter what experience we were to have. Similarly, the paradoxes of set theory do not challenge the possibility of mathematical knowledge, but rather threaten apriorism.[82]

Despite these "deceptions," recent research results show benefits of visualization in the learning and application of mathematical knowledge, namely, they generate structure to explore visual operations, they serve as reference points to derive theorems, they allow visual generalizations, provide a way to trace cases and alternatives and help expand spatial pattern memory.[83]

The visual nature of geometry, the use of graphs in group theory, the graphs of functions comprise all those mental skills related to understanding and visually reorganizing relationships. The drawings or graphs are close, in many cases to real objects, which allow highlighting some aspect of them, but they can also symbolically represent a process:

> Using geometric shapes to represent real objects or events, diagrams can show the relationship between objects or events or represent the process of an activity. In such cases, they may not present the entire object; instead, they can focus the reading's attention on a particular aspect, part, or relationship.[84]

Though for years the use of diagrams and images was left for a heuristic level of mathematics and not for the formal, the appearance and increase of visualization techniques in computing and its subsequent impact on mathematics, has made visualization as a more complex thought process gain relevance. Computer graphics or tables are a way to have a quick visual comprehension. But "the epistemic function of visualization in mathematics can go beyond merely heuristics and ac-

[82] Kitcher, 63.

[83] Norris, *Reading for Evidence*.

[84] Rhonda D. L. Booth and Michael O. J. Thomas, "Visualization in Mathematics Learning: Arithmetic Problem-Solving and Student Difficulties," *Journal of Mathematical Behavior* 18, no. 2 (1999): 169-190.

tually be a means of discovery."[85] That is, graphical representations help to visualize complex objects and thus capture their properties. But about the influence of technology in the philosophy of science there will be much more to investigate in future studies.

IV. Conclusions

Keeping in mind the history of mathematics and the definition of mathematical intuition as a dynamic process that starts from the real context of the individual (in terms of their mathematical and even personal experience), whose results must be validated by the mechanisms of mathematics and that finally it will be the scientific community that will determine their immersion in the formal system of mathematics; we have shown how visualization can also be understood as a process that requires the visual experience of the person who draws or interprets a graph. And whose results must be validated by the scientific community.

The examples presented show that visual thinking has been essential for great mathematicians during their practice. The visualization contributes to the understanding of formulas, algorithms, but it also contributes to the decision of whether a test method is correct or not. However, history is also full of examples where the pre-eminence has been in symbolic language, as the means par excellence to present both scientific and academic results.

On this subject of visualization, much remains to be said, computational advances open a new field that should be of interest to the philosophy of mathematics, likewise, it is our interest to continue investigating the close relationship between mathematical practice and its impact on practice of mathematics education. Can classes be developed from activities that "educate" intuition? How to develop in students skills beyond the techniques of computer management? What should be the approach to the philosophy of mathematics for future educators or mathematicians to improve their practice?

It is not a secret that currently modeling tools and visualization mediated by technological resources have made great contributions to research in both the scientific and educational fields and their influence is increasing. These tools "help to understand and illustrate problems, since phenomena in applied fields can be described by quite complex mathematical models."[86] Thanks to the incursion of technology in the

[85] Mancosu, "Visualization in Logic and Mathematics," 22.

[86] János Karsai et al., "Visualization and Art in the Mathematics Classroom," *ZDM – Interna-*

classroom, our students can explore, experiment and visualize complex concepts, and they can strengthen their knowledge and put it into practice. Of course, this will perhaps bring other types of challenges and difficulties to education, which opens up a new topic of interest: the influence of technology on educational processes, bearing in mind what the philosophy of science has to say about.

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