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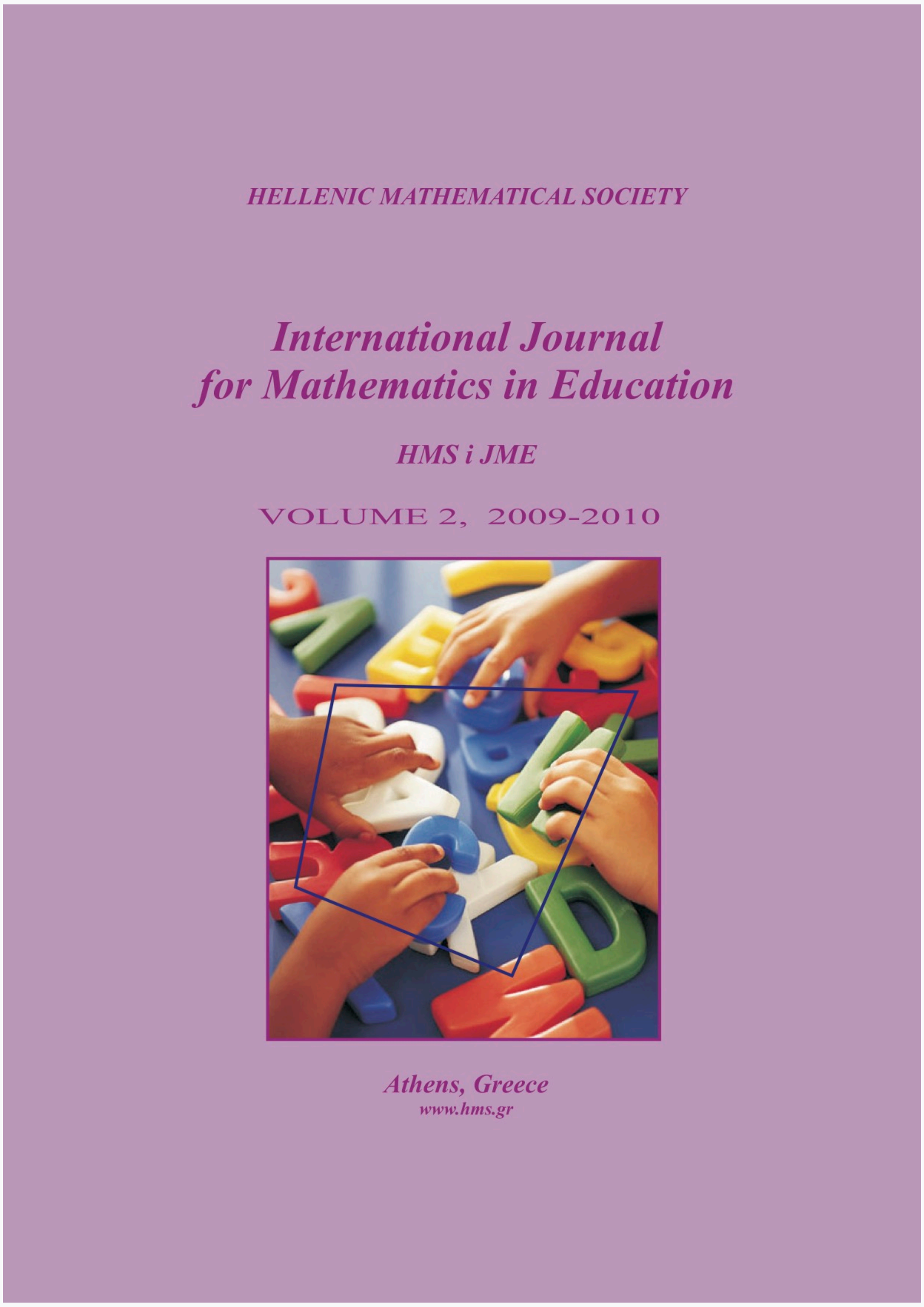

HELLENIC MATHEMATICAL SOCIETY

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VOLUME 2, 2009-2010

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Contents

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GREEK MATHEMATICAL SOCIETY
34, Panepistimiou (Eleftheriou Venizelou) Street, GR. 106 79 - Athens - HELLAS
Tel. +30 210 3616532 - +30 210 3617784 - Fax: +30 210 3641025
e-mail : info@hms.gr - www.hms.gr

International Journal for Mathematics in Education (*HMS i JME*)

The Hellenic Mathematical Society (HMS) decided to add this Journal, the seventh one, in the quite long list of its publications, covering all aspects of the mathematical experience. The primary mission of the HMS International Journal for Mathematics Education (***HMS i JME***) is to provide a forum for communicating novel ideas and research results in all areas of Mathematics Education with reference to all educational levels.

The proposals must be written almost exclusively in English but may be admitted, if necessary, also in French, in German or perhaps in Spanish.

The proposals could concern: research in didactics of mathematics, reports of new developments in mathematics curricula, integration of new technologies into mathematics education, network environments and focused learning populations, description of innovative experimental teaching approaches illustrating new ideas of general interest, trends in teachers' education, design of mathematical activities and educational materials, research results and new approaches for the learning of mathematics.

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Address: *Hellenic Mathematical Society*
34, Panepistimiou (Eleftheriou Venizelou) Street
GR. 106 79 - Athens Greece e-mail : info@hms.gr
Tel. (0030210) 3616532 - (0030210) 3617784 - Fax: (0030210) 3641025

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NOTES FOR AUTHORS

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The first page should contain the author's e-mail address and keywords. Papers should be written in English but also, if necessary, in French, in German or in Spanish.

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The submitted papers are reviewed by two members of the International Scientific Committee (blind review). Two positive reviews are necessary for a paper to be accepted.

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GR. 106 79 - Athens - Greece
Tel. 3616532 - 3617784 - Fax: (0030210) 3641025
e-mail : info@hms.gr

Introduction

The second volume of the HMS International Journal for Mathematics in Education includes four research papers.

The first invited paper by Peter Appelbaum *"Mathematics as sculpture of utopia: changing how we think of models and modeling in mathematics education"* concerns three kinds of approaches to models used in mathematics classrooms. As the writer mentions, it is "an invitation to collaborate on a 'museum of models' – a collection of sample uses of models by teachers and pupils in and out of school".

The second paper written by Emmanouil Nikoloudakis "*A Proposed Model to Teach Geometry to First-Year Senior High School Students"* combines and enriches the phases of the van Hiele theory with the methods of Cognitive Apprenticeship proposing a Model of p-m Combinations. This model was used to the teaching of geometry courses to 15-year old senior high school (Lyceum) students before they are taught how to write formal proofs.

The purpose of the third paper by Sonia Kafoussi, Petros Chaviaris & Rijkje Dekker *"Factors that influence the development of students' regulating activities as they collaborate in mathematics"* is to investigate the issue of how 10-11 year old students regulate their behavior during their mathematical activity as they reflect on their small-group interaction by observing and discussing their video-recorded collaboration.

Finally, Michael Voskoglou by his article ***"****Mathematizing the process of learning a subject matter in the classroom"* is trying to build a Markov model for the description of the process of learning a subject matter by a group of students in the classroom. The results are illustrated by a classroom experiment for learning mathematics performed at the School of Technological Applications of the Graduate Technological Educational Institute of Patras, in Greece.

Hellenic Mathematical Society
International Journal for Mathematics in Education

Mathematics as sculpture of utopia: changing how we think of models and modeling in mathematics education

Peter Appelbaum

Abstract

Physical materials, diagrams, charts, symbolic representations, and exemplary problem types model concepts and relationships in mathematics classrooms. There are at least three kinds of approaches to models, characterized by metaphors that represent our assumptions, values, fears, desires, structures of discourse, and so on: the architect, scientist or artist, as explicated by sculptor Josiah McElheny. Teachers and pupils act in each of these ways at various times, but even when working in the styles of an architect or scientist, they must also use models in the manner of McElheny's artists, as 'invitations'. Models in McElheny's 'artistic' sense provoke questions and conversation; confusion and fascination; contemplation; new philosophic inquiries, imaginations; fantasies; and repulsions. His primary example is Isamu Noguchi, whose proposals for modernist playgrounds mostly remained in the realm of fantasy and enchantment, rather than as constructs in 'the real world'. This article is itself an invitation to collaborate on a 'museum of models' – a collection of sample uses of models by teachers and pupils in and out of school.

Keywords: *mathematics, mathematics teaching, modeling, metaphor, conceptual art*

Representations & Modeling in a Living, Growing Discipline

Physical materials, diagrams, charts, symbolic representations, exemplary problems of a 'type', etc., dominate mathematics teaching and learning. Much of our pedagogy rushes to the representations. We want pupils to become very good at using these to *model concepts and relationships*. This, indeed, seems to be at the heart of 'mathematics as a living, growing discipline'. Pupils who can move from one model to another are usually taken as 'understanding' the mathematics. At times it seems that the very act of working *with* the representations as *models* might actually be a specific characteristic *of* mathematics. This process of treating the representation as the subject of analysis, repeated often, might be what we need to carefully explore and understand if we are to better comprehend the possibilities for mathematics as a living, growing discipline.

It is with these thoughts in mind that I ask us to consider the processes of modeling and their implications for our work, as individuals, and as a potential network of international mathematics educators with the power to influence and transform teaching and learning worldwide. I suggest that we too often assume we know what we mean by representation and by models, and that we need to consider that there are multiple ways to conceive of them and to apply them in our work as researchers, as developers of curriculum materials, as teachers of mathematics, and as mathematicians. There are at least three approaches to the act of modeling – independent of the medium of representation, as explicated by the sculptor Josiah McElheny (2007), and I want to propose that we use his analysis of models as an opening for our reflections on models and the act of modeling concepts.

"Czech Modernism Mirrored and Reflected Infinitely" by sculptor Josiah McElheny

McElheny works as a conceptual artist, and I invite us to consider how we can learn from the kinds of thinking that conceptual artists invoke as we explore the roles of concepts in mathematics. I have been pursuing the study of possibilities for mathematics education through such connections with mathematics-as-art, pupils-as-artists, etc., for some time (Appelbaum 2007a, 2007b, 2008, 2009). *This essay* is *not* so grand a fantasy: it is a smaller question or set of questions that pop up when we look at how some sculptors think about what a model is, what a concept is, and what the purposes of their work are. Others have made connections between aesthetics and mathematics as related to teaching and learning (Fujita,et al. 2004, Hickman & Huckstep 2003, Sinclair & Pimm 2009, Sinclair 2006, Sinclair, Pimm and Higginson 2006, Sullivan 1956). This essay has more in common with such searches for the commonalities among the arts and mathematics as modes of reasoning and communicating.

Three Approaches to Modeling and Using Models

Josiah McElheny says there are three kinds of models. Each type of model enlists metaphors that, in turn, represent our assumptions, values, fears, desires, structures of discourse, and so on. They are labeled in ways that stereotype the labels, but which characterize in certain ways the kinds of work that is often valued in a particular craft or profession – architect, scientist, or artist. Moreover, one might work in a stereotyped way that is associated with these crafts or professions in *any* field, and we can use them as metaphors to interpret the work involved.

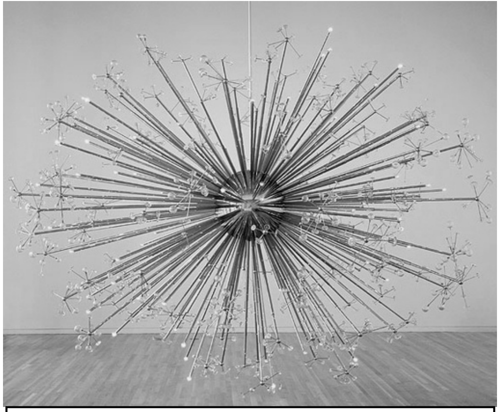
We could imagine, for example, that a teacher or pupil might work as an architect-mathematician, scientist-mathematician, or artist-mathematician

Architect: uses a model to convey information to garner ideological, financial, political, or institutional support

Scientist: uses models to pose new systems of understanding.

Artist: can exist outside these considerations - depicts something not intended to be built, creates an imaginary space.

at different times during a school day, in order to achieve various goals. One might employ blocks, pictures or equations as part of an argument to convince others of a conclusion or the reasonableness of a result, metaphorically acting like an architect who employs a blueprint or a scale model of a building to convince clients to use the architect's plans. Or, one might invent a way of representing a system or set of relationships, and then proceed to analyze the representation for what it further implies regarding these relationships, much as a scientist uses models to better comprehend the relationships among natural phenomena. An architect-mathematician might employ a construction by straightedge and compass to convince another pupil that the perpendicular bisectors of the sides of a triangle will intersect in one unique point; they might create an animation with Geometer's Sketchpad, or instead create a series of logical statements that include details about these parts of a triangle in algebraic symbols. If pupils are graphing parabolic data, then - if they use the graph to study parabolic functions (noting various characteristics of the types of relationships that are demonstrated among the variables that are depicted by their graph) - they could be metaphorically labeled as scientist-mathematicians; similarly, drawing a picture of a person and their shadow from a lamppost as they walk, marking items that could be measured at discrete distances from the lamppost along the ground, a scientist-mathematician could then use the picture to generate a chart or graph, from which they could further describe precisely particular aspects of the relationships involved, i.e., proportionality between heights of the light source and the person walking, length of shadow, distance from

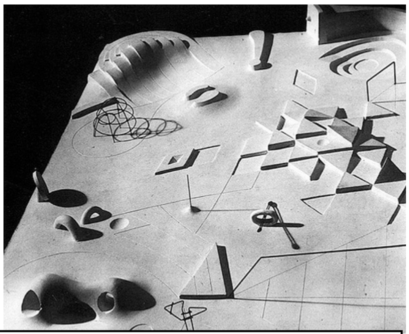

"Big Bang" by sculptor Josiah McElhney accurately represents current theories in astrophysics

lamppost, and so on. Much rarer in school mathematics do we find artist-mathematicians of the type that McElheny describes. For this type of work with models to occur, we have to think a bit more abstractly about what we do mathematically when we are ‘thinking mathematically’, and, in fact, I believe that this is more central to the work of mathematicians than the other two, at least, more at the heart of what the mathematical experience is all about. How we use models is not shackled by the assumption that we must convince others of what we believe, nor is it tied to the application of the models to solve a problem through modeling of a situation. Or, to express this differently: (a) mathematical work is not always done within the framework of what McElheny calls the architect or scientist; and (b) more importantly, there is a way in which even the work of an architect or scientist follows from the previous work of acting in the way that McElheny calls an ‘artist’, or eventually leads to this artist-like way of working. The artist-aspects of such work are ‘prior’ to, ‘anterior’ to, or independent of the work of the architect or scientist.

Before we describe such work, however, I will share some preliminary concerns regarding the difficulties in working with the artist-approach to models as outlined by McElheny. In some ways, the artist-mathematician seems to be less concerned with the real world and with applications of their ideas. This is not really the case, but the issue deserves some attention. The history of mathematics is filled with people who have extolled the virtues of ‘pure mathematics’’, such as G.H Hardy, famed for his unabashed *Mathematician’s Apology* (1940) --those whose efforts seem to exist independent of the practicalities and necessities of an architect, or whose work has no intended scientific application. That is not what I want to address here. Even a pure mathematician employs the habits and skills of an architect when convincing others of their conclusions or proofs, and those of a scientist in elaborating and evolving models of systems of relationships. Given that there are many publications and that there exists a great deal of research about modeling and models in mathematics education, I suspect that my argument in this paper is challenging to some mathematics educators. Most of us already have a working understanding of models and representations, and most of us employ a number of assumptions about these in our own work. I am asking us to reconceptualize some aspects of mathematics that are very fundamental to what we do. This may take a great deal of effort. In my recent discussions with a number of mathematics educators, I found the conversations collapsing back into a discourse that presumes a scientist-mathematician framework that does not allow for the comparisons that this paper makes. For example, if a teacher presupposes that most activity in a classroom should have the purpose of solving a problem or of practicing methods of solving types of problems, then that teacher is going to think of models only in the way of a scientist-mathematician, since the main purposes of models in such a classroom are to accurately present a mathematized analogy for the situation occurring in the problem to be solved. This means that such a teacher will have to reorient themselves in order to take advantage of the points in this paper. He or she might, for example, begin to introduce new types of activities in their classroom, during which pupils are not solving problems or practicing methods of solving problems; such activities would involve pupils comparing and contrasting models, with no intention of using the models to solve a problem. Other intentions for the models would have to be present in the conversations that unfold in this classroom.

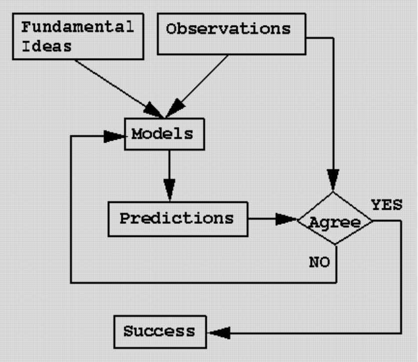
Sculpture of Utopia

McElheny's 'artist' uses models to create imaginary, new worlds, imaginary spaces of learning outside of time and space. The models become 'proposals' – invitations to come and play and explore the ideas. The other kinds of models drag us down into realms of accuracy, correctness, and so on:"Is it a 'good model'?" is too often taken to mean, "Is this model a true replica of the real world?" Such models are tossed aside as soon as they fail to love up to the demand that they precisely 'mirror nature'. McElheny refers to the designs for playgrounds by Isamu Noguchi. One famous model for a U.N. Playground in New York was for an intended playground that would actually be built, so that the original model was that of a literal architect. Unfortunately, the playground was never realized in real life. But the model became a well-known work of art included in museum exhibits. Used in *this* way, the model for the playground becomes an invitation to imagine what playgrounds could be, to think about our assumptions about what a playground should or could be, and to question the decisions that we generally make about what should take place in a playground space. The artist's model has very different value. Not tied to true mirroring, its own pleasures and value are in itself, and in the way it allows us to dwell in the very act of modeling itself. Conceptual art leads a person interacting with it to reflect on the process of making the art and the concepts that are invoked with the art. It is in this sense that I believe McElheny's 'artist' is like our mathematicians contributing to a living, growing discipline. Noguchi's playground is now a playground of the mind, generating all sorts of experiments and questions in a time and space not in the real world, but potentially influencing that world through the ways we might act in the future, even more than one playground might have affected the lives of whoever had played there. It is in this sense that the model becomes a 'sculpture of utopia', because a utopia is an ideal conception that does not exist in the world. A mathematician or artist who establishes this sort of standpoint on his or her use of models evokes what Brian Rotman (1993) once called a 'meta-subject' – someone who is compelled to consider the relationship to the act of idea, mathematics, or art creation itself; the work provokes reflection on the meaning of the work, and the very action of constructing the model, rather than accurately mirroring reality or serving as a pleasing object of our



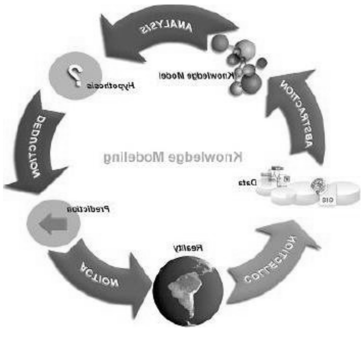
Isamu Noguchi's model for the U.N. playground in New York gaze. For mathematics education, this attention to the act of model creation is critical, because it enables the teacher and student to talk about the specific point of mathematical ideas, as well as to reflect on the processes of idea-development usually lost in the black box of pedagogical theory: it lays out in the open those aspects of learning and concept development most difficult to address in ordinary class practices. Both the teacher and the pupil now have the possibility to examine and discuss what is typically left to chance, the actual processes of creating, representing, and modifying ideas.

Utopia in fiction, philosophy and art challenges our assumptions about the way the world works, offering alternatives unhindered by our ideological commitments. Mathematics needs to work for us and our pupils in this way, offering unhindered provocations that enable us to construct new conceptions of relationships and systems. Models in the more traditional sense of applications of mathematics to the real world do not challenge presumptions; they provide algorithms for obtaining solutions to problems. Such models do not help people focus on the mathematics, but instead on the algorithms that provide recipes for answers. We are left dissatisfied that our pupils are merely memorizing lists of steps toward a formulaic solution, rather than genuinely understanding the mathematics. Models in McElheny's sense provoke questions and conversation, confusion and fascination, contemplation, new philosophic inquiries, fantasies, repulsions, and more. Noguchi's model provokes new questions: What *is* a playground? What *could be* a playground? Why do our playgrounds look as they do, and not differently? Analogously, a collection of base-ten blocks might provoke such questions as: Why do we work in base-ten? How does a base-ten way of organizing numbers of things influence the ways that we think? How does thinking about numbers in terms of three-dimensional volume lead us to different questions and conclusions when compared with the types of questions that emerge when working with 100s-charts or number lines? The analogy also helps us see that a comparison of algebraic representations, graphs, and tables for the same functional relationship could also lead to provocative considerations of the relationship between relationships and the representations for them. Whether we are thinking about base-ten numeral systems of functional relationships, this new use of models generates opportunities to interrogate the meanings within the mathematics while simultaneously inventing our own algorithms, both of which make it possible to easily approach standard procedural knowledge critically and meaningfully – to appreciate their power as well as their limitations.



“Old-school” approach to models, #1

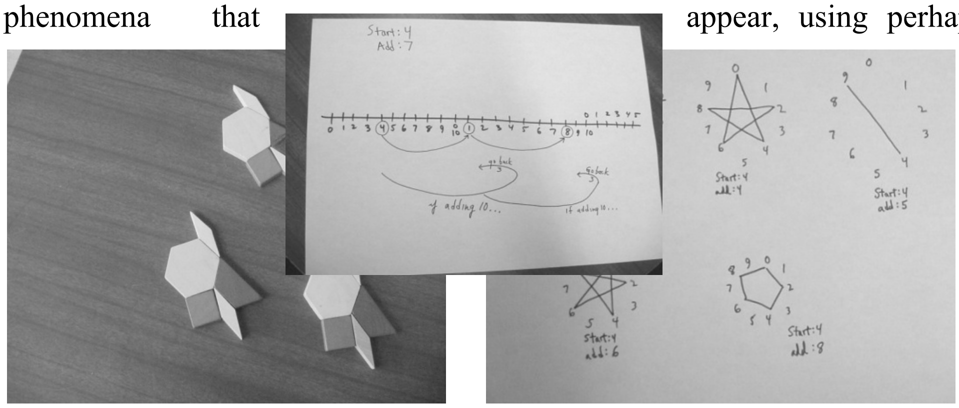
I am essentially arguing for a new positioning of the teacher and the pupil vis-à-vis the mathematics and the doing of mathematics. The ‘old-school’ style of working with models focuses on how accurately the model uses representations to create an analogy for ‘reality’. In the ‘new-school’ point-of-view, pupils do not mainly work with models as part of a traditional knowledge cycle of model-development, but instead critique models as if they are works of mathematical art. While the ‘new-school’ pupils sometimes work in ‘old-school’ ways, they would more often be found juxtaposing, appraising, analyzing & creating models; placing models in historical context; and using models to provoke emotions, such as joy, nostalgia, outrage, constancy, in an audience who is listening to a presentation. Each representation – every picture, diagram, chart, graph, equation, etc., would in such classrooms would always be taken as a potential model of many concepts & relationships. And this notion that a particular representation is always potentially many different models all at once becomes very important, because these different uses



“Old-school” approach to models, #2

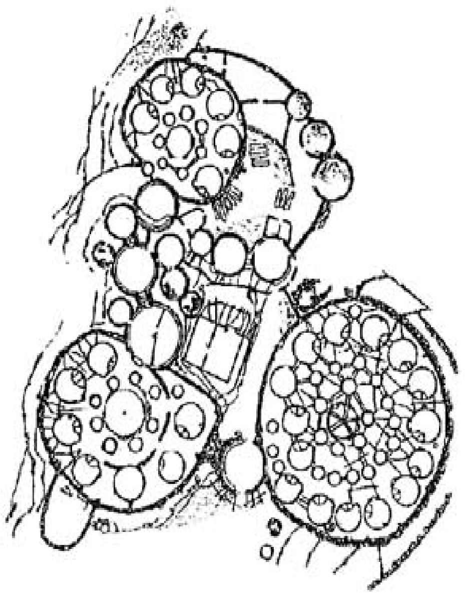
of the representation as a model can be discussed, explored, applied, critiqued, modified, etc. Seemingly unrelated concepts & relationships modeled by a common representation are drawn together into new worlds of similarity & difference within classroom conversations and pupils' investigations. This all occurs outside of the time & space of the literal modeling processes that are within the usual/traditional focus, in other word, within a *utopian* place both inside and outside of time and space.

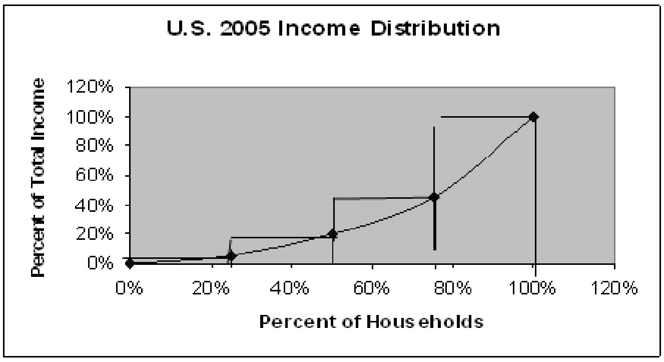
An old-school approach to the study of repeated addition on a calculator, for example, might lead pupils through a series of efforts to model the phenomena that appear, using perhaps a number line, a 100s-chart, a collection of tesslation tiles, or a circular representation of modular arithmetic. Students would use the models as part of being guided to deduce that the patterns in the units digits of such repeated addition, and the lengths



"New-school" approach to models, #1

of those patterns, are related in particular ways to the seed starting number and the constant adding number. The new-school aproach would use the context of the study of repeated addition on the calculator to facilitate important discussions regarding the differences among the types of models for what they reveal and obscure.





“New-school” approach to models, #2

Similarly, new-school attention to sketches of an African village's architecture, shadows cast by a sculptor's collections of human trash, or a graph of recent income distribution in a country would help pupils understand the kinds of relationships and questions that are able to be examined and posed with each type of representation as a model of real-world phenomena as much as or more than help them to obtain answers to particular mathematical problems.

Proposal

I propose a newly created international collaboration can and should take on the project of promoting a better understanding of the role of the 'artist-mathematician' in the teaching and learning of mathematics, and to facilitate greater, more effective uses of this approach with mathematics education internationally in our various countries. One way to do this is to create a *'Museum of Models'* on the internet. Such a 'museum' would include a section that exhibits examples of modeling in all three approaches discussed in this article in different classrooms internationally. Teacher-produced lesson plans would be accompanied by short video excerpts and by commentary from other teachers related to the example. The website would collect examples for as many ages and types of mathematical content as possible to exhibit in this gallery of 'exemplary classroom models'. Another space within the museum would be for shared dialogue among teachers, researchers, policy-makers and other interested members of the mathematics education community to discuss models and modeling in the world of mathematics and mathematics education more generally, and specifically about individuals' own personal attempts to understand those approaches to modeling that are most difficult to understand. A third area of the museum would feature collaborations among mathematics educators, pupils of mathematics and conceptual artists, in order to help us further the kinds of research in this area that can build a foundation of theoretical understanding to later develop such work crossing the boundaries of mathematics education and sculpture – both literally and metaphorically. Existing outside the time and space of our daily lives, in the ether of the internet, the museum of models would be a *utopia* where we together sculpt new worlds of mathematics education. Those interested in pursuing a museum of models should contact the author at Appelbaum@arcadia.edu.

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Peter Appelbaum is Professor of Education and Director-at-Large of Undergraduate Curriculum at Arcadia University in Philadelphia, USA. He is the President of the American Association for the Advancement of Curriculum Studies (AAACS), and a member of the International Commission for the Study and Improvement of Mathematics Education (CIEAEM). Recent books include *Embracing Mathematics: On Becoming a Teacher and Changing with Mathematics* (Routledge, 2008) and *Children's Books for Grown-Up Teachers: Reading and Writing Curriculum Theory* (Routledge, 2007).

Arcadia University, Philadelphia, USA
Appelbaum@arcadia.edu
http://gargoyle.arcadia.edu/appelbaum/

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A Proposed Model to Teach Geometry to First-Year Senior High School Students

by Emmanouil Nikoloudakis

Abstract

We combined the phases of the van Hiele theory with the methods of Cognitive Apprenticeship and enriched these combinations adding the three following ideas: (a) a special worksheet, named Structured Form Worksheet, which we used when teaching geometry (SFW), (b) a matrix named Reasoning Control Matrix for the Proving Process (RECOMPP), which helped students with reasoning production and (c) the concepts of simple and partial proof to write the formal proof. We called the above mentioned combination Model of p-m Combinations. Then, we used this model to teach geometry courses to 15-year old senior high school (Lyceum) students. In this article we claim that students should be able to write simple and partial proofs before they are taught how to write formal proofs.

Key-words and phrases: *Formal, Simple, Partial Proofs; Phases-Methods Combinations Model; Structured Form Worksheet; Euclidean Geometry; Reasoning Control Matrix for the Proving Process.*

1. Introduction

Research related to the understanding of geometric concepts by students has shown that students have difficulties in defining and recognizing geometric shapes and in the use of deductive thinking in geometry (Pyshkalo 1968; Burger, 1982; APU, 1982; Hart, 1981). Despite the importance of proofs, research has shown that students have great difficulty with the task of proof construction (Senk, 1985; Schoenfeld, 1985; Martin and Harel, 1989; Harel and Sowder, 1998). Due to students' difficulty to write proofs successfully, numerous projects focus on the teaching of geometrical proof (Hanna, 2000; Martin & Harel, 1989; Leron, 1985; Recio & Godino, 2001; Senk, 1985; Usiskin, 1982). The inefficiency to teach the notion of "proof" is almost global (Hadas et al., 2000).

In Greece, Euclidean geometry is taught under a theoretical framework, during the first two years of Lyceum. Junior high-school students usually count and calculate, based on specific situations, whilst they seldom make use of abstract procedures. Thus, first-year Lyceum students, who move from specific procedures to more abstract ones, are not familiar with the role of axioms, definitions, and theorems. Instead, they have to cope with the concept of proof in a purely theoretical context. The fact that the students of Lyceum cannot learn the proof processes correctly seems to influence their future ability as undergraduates to solve mathematic problems. So, university teachers realize that the processes which first-year undergraduate students follow, when solving a mathematic problem, are the typical ones they have learnt in preparatory schools or private lessons (Kalavassis, 1996).

2. Simple proposition

According to Dimakos and Nikoloudakis (2008) a proof is constituted and is analysed in simple justifications. We develop this aspect here briefly, because this analysis represents a necessary component for this article. Initially we give two examples to explain what we mean by the words

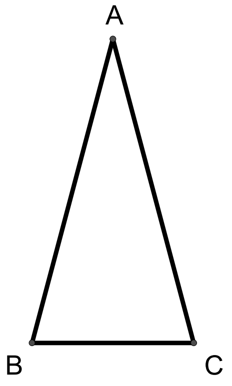

Figure 1

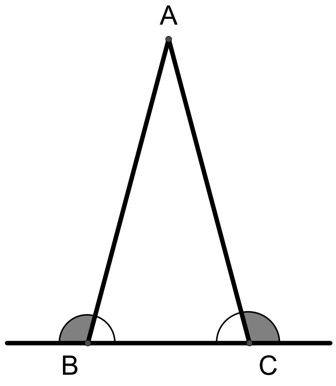

Figure 2

"statement", "justification" and "partial proof". Hence, let us consider the following two propositions:

Proposition (P-1): If AB=AC, prove that the triangle ABC is isosceles (See Figure 1).

Proposition (P-2): The exterior base angles of an isosceles triangle ABC are equal (See Figure 2).

Generally, we maintain that *every proposition contains a statement* and *every proof consists of two parts, a statement (which needs a justification) and a justification (of this statement)*.

Especially, we can maintain that *every proposition contains a statement* and every proof (of this proposition) consists of two parts, *a statement* (the statement of the proposition which needs a justification) and *a justification* (of this statement) (see Figure 3).

• For proposition P-1 the *statement* is: the triangle ABC is isosceles.

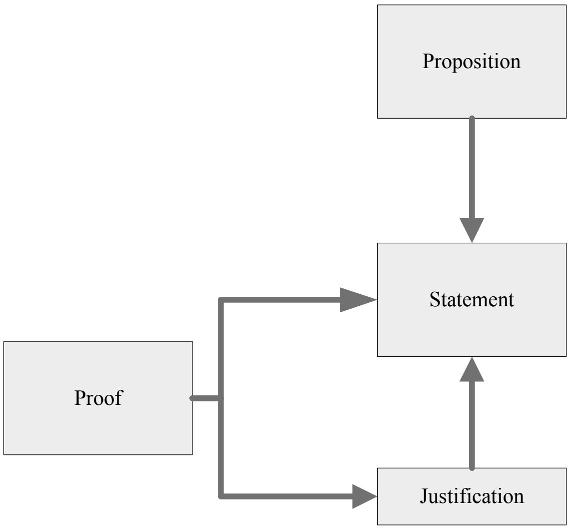

Figure 3

It is also noted that for the proof of the statement of proposition P-1 we have:

i. (i) *Statement:* the triangle ABC is isosceles

ii. (ii) *Justification:* because AB=AC

So when we say that the triangle ABC is isosceles, because AB=AC, then **we have fully reasoned** the statement: the triangle ABC is isosceles for proposition P-1. Thus we have proved proposition P-1.

• For proposition P-2 the *statement* is: the exterior base angles of an isosceles triangle ABC are equal.

It is also noted that for the proof of the statement of proposition P-2 we have:

i. *statement* “the exterior base angles of an isosceles triangle ABC are equal”, and
ii. *justification* “because they are supplementary to the equal angles B and C”.

Nevertheless, we haven’t fully reasoned the statement (i) with the justification (ii) because we have not reasoned that the angles B and C are equal. As a result justification (ii) (because they are supplementary to the equal angles B and C) is a statement and is consequently **a new proposition.** Since any proposition in geometry, except for definitions, postulates and axioms, needs a proof, **the new proposition needs a proof as well**.

The *statement* of the new proposition is: the supplementary angles B and C are equal and its proof components are:

iii. *statement* “the supplementary angles B and C are equal”, and
iv. *justification* “because the angles B and C are equal”.

Also, we have not fully reasoned the statement (iii) with the justification (iv). Now we must explain why the angles B and C are equal. Thus, the justification (iv) is another new proposition with statement: the angles B and C are equal and its proof components are:

v. *statement* “ the angles B and C are equal”.
vi. *justification* “because the triangle ABC is isosceles”.

Similarly we must explain why the triangle ABC is isosceles, so we have the proposition with the statement: the triangle ABC is isosceles and its proof components are:

vii. *statement* “the triangle ABC is isosceles”.
viii. *justification* “it is given”.

The following are observed: justification (ii) of statement (i) in proposition P-1 does not need further justification for statement (i) to be valid so the proof is fully reasoned by (i)-(ii).However, this is not the case in proposition P-2 for justifications (ii), (iv) and (vi) of this proposition. In detail, for statement (i) to be valid, justification (ii) has to be valid. For justification (ii) to be valid, justification (iv) has to be valid and so on. So with just (i) and (ii) the proof of P-2 is not complete.

When the justification of a statement, like in proposition P-1, does not need further justification for the statement to be valid, then the justification is referred to as *simple justification*. In particular:

Definition: A justification is called *simple justification* when no further justification is needed in order to prove its truth. Simple justification will also be called *simple proof.*

Definition: We shall say that a justification of a statement is a *non-simple justification* or *a partial proof* when its truth depends on yet another justification.

Proofs (i)-(ii), (iii)-(iv), (v)-(vi) of proposition P-2 are non - simple justifications but the last part of the proof of P-2, that is proof (vii)-(viii), is indeed a simple justification. To conclude, the proof of P-2 consists of some partial proofs (i)-(ii), (iii)-(iv),(v)-(vi) and a simple justification (vii)-(viii).

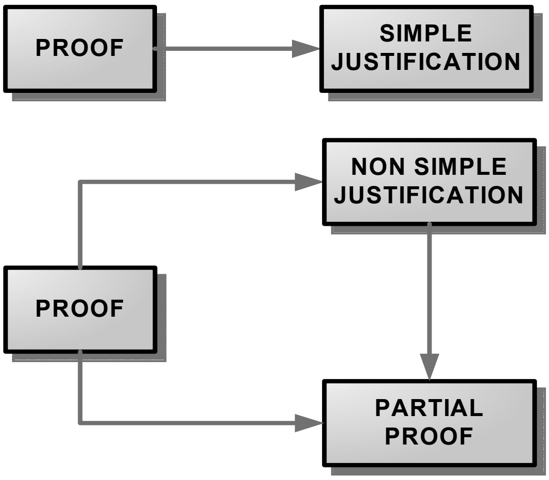

Figure 4

Taking the above in to account we can say that every proof is consists of two components. These parts are both a *statement* and a *justification*, or a *statement* and *a justification, which corresponds to a partial proof* (see Figure 4). Nevertheless, the partial proof is a proof itself. So, it can be further analysed *to a simple justification* or to *a partial proof and* *so on*. ***This way a proof consists of and is analysed in simple justifications*** (see Figure 5).

Also, we define as simple proposition a proposition whose proof is a simple proof. In this article it is claimed that students should be able to write simple and partial proofs before they are taught how to write formal proofs

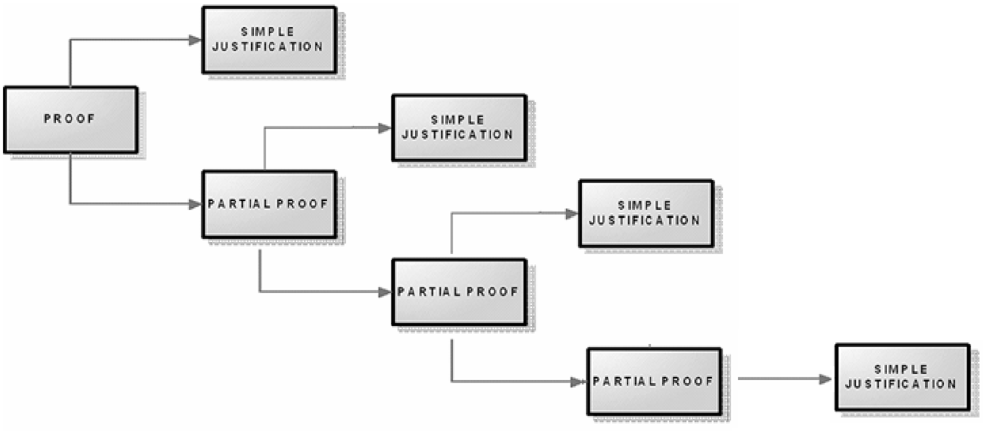

Figure 5

3. Two well known theories

Dutch educators Pierre van Hiele and Dina van Hiele-Geldof developed a theory of five levels of geometric thought (Anderson, Reder and Simon, 1996). According to the van Hiele theory there are five hierarchical levels that students pass through as they progress from merely recognizing a figure to being able to write a formal geometric proof. Alan Hoffer (1981), named the first level Recognition, the second level Analysis, the third level Informal Deduction, the fourth level Deduction and the fifth level Rigor. Along the levels van Hiele proposed five phases to help the students progress from one level to the next one. Van Hiele has called these phases Familiarization or Information, Guided Orientation, Verbalization or Explicitation Free Orientation, and Integration.

Cognitive apprenticeship is the application of the principles of apprenticeship to learning cognitive skills. Collins et al. (1989) comment :

> *We propose in alternative model of instruction that is accessible within the framework of the typical American classroom. It is a model of instruction that goes back to apprenticeship but incorporates elements of schooling. We call this model cognitive apprenticeship.*
>
> *(Collins, Brown, and Newman, 1989).*

They, also, claim that *Cognitive Apprenticeship makes thinking visible*. We believe that this aspect helps a mathematician teach the students to write successful proofs in geometry.

In our research we attempted to teach the course of Geometry using ideas from the above theories. In detail we combined the phases proposed by the Theory of van Hiele with the methods of Cognitive Apprenticeship and we enriched these combinations with our own ideas so that our instruction would be more coherent with Vygotsky's ideas (i.e. zone of proximal development, etc). We have used the term "Model of Phases-Methods Combinations" or "Model of p-m combinations" to describe the proposed model.

3.1. Model of Phases-Methods Combinations

Regarding the analysis of phases, van Hiele (1986, p. 177) mentions the following: "I have not mentioned a specific form of instruction. The ideas that have been used here have a place in every method of teaching". Also, Collins et al. (1991) believed that there are more than one ways to apply the methods of Cognitive Apprenticeship and that, ultimately, the teacher is the one who is responsible for determining the ways in which cognitive apprenticeship can be applied in the range of his/her teaching.

We attempt to teach the course of Geometry combining the phases proposed by the theory of van Hiele with the methods of Cognitive Apprenticeship taking into consideration the above statements of van Hiele (1986 p. 177), of Collins et al. (1991) and also that:

- Students find it difficult to understand the course of geometry and its processes (Van Hieles, 1986; Hoffer, 1981; Usiskin 1982;1987; Burger and Shaughnessy, 1986; Crowley 1987; Fuys, Geddes, and Tischler 1988; Gutierrez, Jaime, and Fortuny 1991; Mason 1997; Wirszup, 1976)
- Students find it very difficult to successfully write simple geometry proofs (Weber, 2003). This happens, when they repeat proofs taken from their coursebook as well (Burger & Shaughnessy 1986; Hoffer 1983; Wirszup, 1976)
- The van Hiele theory of levels of geometric thought specially refers to the course of Geometry
- Cognitive Apprenticeship, according to its creators Collins et al. (1989) and Collins et al. (1991) makes the thought visible
- According to Fuys et al. (1988) the progress from one level to another depends on the teaching method followed by the instructor, regardless of the age of the students or their biological maturity

The combination of the phases of instruction of the van Hiele theory with the methods of Cognitive Apprenticeship was based on the participants' characteristics, actions, and roles in the teaching process in both theories.

More specifically:

Phase 1 "Information" of van Hiele's theory was combined with the method of Modeling of Cognitive Apprenticeship.

Phase 2 "Bound Orientation" of van Hiele's theory was combined with the method of Coaching of Cognitive Apprenticeship.

Phase 3 "Explicitation" of van Hiele's theory was combined with the method of Articulation of Cognitive Apprenticeship.

Phase 4 "Free Orientation" of van Hiele's theory was combined with the method of Exploration of Cognitive Apprenticeship.

Phase 5 "Integration" of van Hiele's theory was combined with the method of Reflection of Cognitive Apprenticeship.

All the above phases of van Hiele's theory were combined with the method of Scaffolding of Cognitive Apprenticeship.

3.1.1. SFW and RECOMPP

To implement this combination we coined a special worksheet, named "Structured Form Worksheet" (SFW). (Dimakos, Nikoloudakis, 2008). The SFW and an important component of SFW (Dimakos, Nikoloudakis, 2007) called "Reasoning Control Matrix for the Proving Process" (RECOMPP) are briefly described below.

3.1.1.1. *Structured Form Worksheet (SFW)*

The SFW consists of the following three sections :

a) The Reminder Notes.
b) The Process.
c) The Assessment.

The Reminder Notes

In the first section, named "Reminder Notes", the teacher reminds the students of some theorems. These are some essential theorems, based on the students' prior knowledge, which help students understand the new cognitive object. In this section, what takes place is the combination of the first phase of van Hiele's model (Inquiry/Information) with the method of Modeling of the Cognitive Apprenticeship model.

The Process

In the second section, named "Process", students have to conjecture, to discover, to argue, to prove, and to express their opinion on how to solve certain problems, that the teacher has prepared for them beforehand. In this section the following combinations take place:

- the combination of the 2nd phase of van Hiele's model (Directed Orientation) with the method of Coaching of the Cognitive Apprenticeship model
- the combination of the 3rd phase of van Hiele's model (Explication) with

the method of Articulation of the Cognitive Apprenticeship model

- the combination of the 4th phase of van Hiele's model (Free Orientation) with the method of Exploration of the Cognitive Apprenticeship model. In this combination students make use of a matrix that we coined, dubbed "reasoning control matrix for the proving process" (RECOMPP).

The Assessment

In the third section, named "Assessment", students have to tell each other what they have done in the prior section, they have to describe the way they have thought, why they have thought this way, what they have learned etc. In this section, students have to describe over the phone what they have learned to another schoolmate, who was absent from class. Moreover, students have to construct a problem based on the knowledge that they have gained. This section constitutes of the fifth phase of van Hiele's model (Integration) with the method of Reflection of the Cognitive Apprenticeship model.

3.1.1.2. RECOMPP

According to Dimakos and Nikoloudakis (2007) RECOMPP is a reusable matrix pattern that helps students produce reasoning production. Its layout and its filling technique are predefined. In more detail, it consists of six discrete sections and its layout consists of rows, columns, and cells that may contain figures, hypotheses or conclusions, proofs, and partial proofs (see Figure 6). Furthermore, when filling RECOMPP, a student follows two basic rules: that of horizontal transit, and that of transfer. These rules will be described in more detail later in this article. RECOMPP can be used in every chapter of geometry content because it is a reusable pattern of reasoning production. The advantage RECOMPP offers, when employed by novice lyceum geometry students attempting to prove a proposition, is that it can help them produce and control their reasoning in a more effective way so as to successfully write the given proof (Dimakos, et. al, 2007).

As shown in figure 6, RECOMPP consists of six discrete sections. What follows is a detailed description of each section:

-*Section 1,* is where the formulation of the problem is given. Here, in a textbox, the student can read the full description of the problem, before he/she moves on to the proving procedure.

-*Section 2,* is where the hypotheses, and the conclusions of the problem must be written. Here, the student is given a table (consisting of two rows and two columns), where he/she must write down, in two separate lines, the hypotheses, and the conclusions of the problem, respectively. Of course, students must have very carefully read the description of the problem, that can be found in Section 1, very carefully, before they are able to find, discriminate, and record the hypotheses.

-*Section 3,* is where the sketch of the problem must be prepared by the student. Here, based on the description of the problem they have read in Section 1, and according to the hypotheses, and the conclusions that they have written down in Section 2, student progress to draw the sketch of the problem in a blank field. Students will use this, as a visual aid, to write the proof.

-*Section 4,* is where the teacher may offer scaffolding to student. Here, in order to offer students contextual, and on the spot help, the teacher can provide them with a list of hints.

-*Section 5*, is where the student is motivated to reason, collect, and write those statements and relationships between the elements of the sketch, prepared beforehand, which will lead him/her to the successful writing of the proof. Here, the student is given a table (consisting of just two columns and several rows). In this table, in the first column, the student must write a statement e.g "Statement A", that needs to be proved, labeled "To prove that…". …". In the second column, the student must write a statement e.g "Statement B", that is necessary in order to prove "Statement A", labeled "It

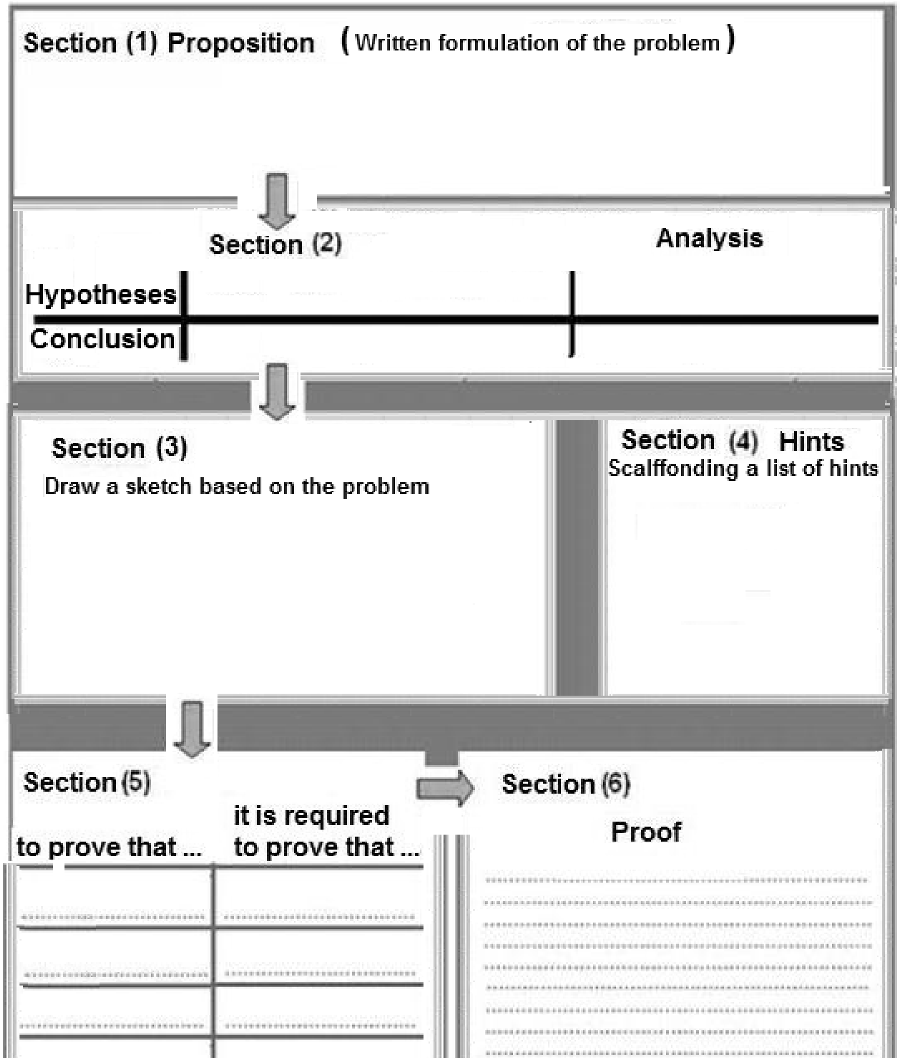

Figure 6

is required to prove that”. The student must set some goals, in order to move horizontally and proceed from the left column to the right one. Thus, the left column represents the earliest stage of the student’s reasoning and the right column represents the latest stage of the student’s reasoning. Section 5 of RECOMPP must be filled according to the following three rules: (See Figure 7)

(a) the rule of horizontal movement from left to right, i.e the student first fills the left column, labeled “to prove that”, and then continues to fill the right one, labeled “it is required to prove that”. The most significant contribution of this rule of horizontal movement from left to right is that it is demanded from the student to produce reasoning. This process is repeated in every row of the RECOMPP.

(b) the rule of how to fill the first cell of the left column, labeled “to prove that”. According to this rule, the student must always fill the first cell of the left column with the conclusion from Section 2. This is especially important for the student, because, it shows him/her, where to start the proving procedure from.

(c) the rule of reassignment of produced reasoning. According to this rule, the content of the right column in each row (produced reasoning), is reassigned to the left column of the row below.

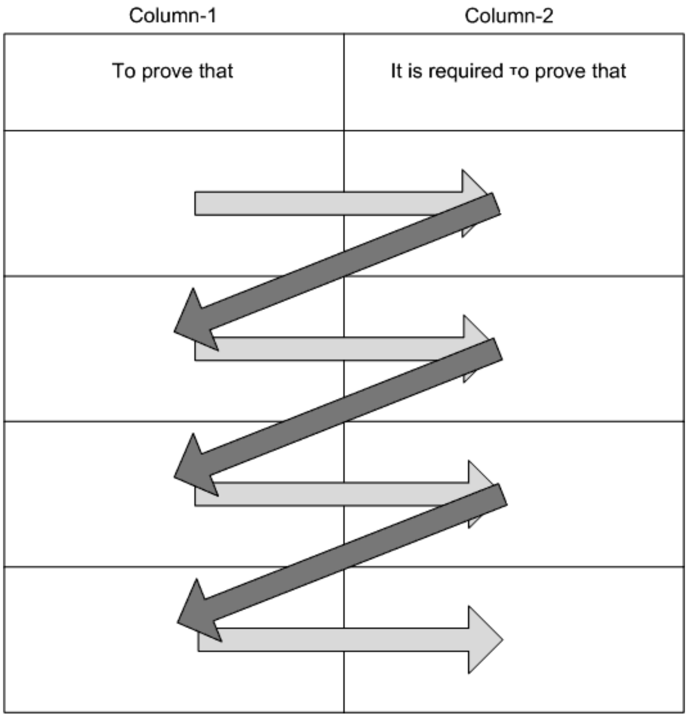

Figure 7

-*Section 6,* is where the proof must be written by the student. Here the student must write the proof in a textfield.

3.1.2. The periods of instruction in model of m-p Combinations

According to this model the instruction takes place in five periods (see Figure 8).

In the 1st period, students relate the visual geometric shapes and their appearance with their names for every cognitive subject, e.g. of all kinds of parallelograms and their appearance with their names. Moreover, the teacher demonstrates shapes that are gradually increasing in complexity. The students are acquainted with more complex shapes and their components.

In the 2nd period, students are taught the attributes and the relative theorems of the cognitive subject (parallelograms), without their proofs. Students confirm the validity of theorems in an experimental way, using computers. It must be noted that dynamic representations that result from dynamic geometry software (DGS) environments (particularly Geometer's Sketchpad and Cabri II+) play a semantic role as they aim to develop spatial sense and geometric reasoning (Kalavassis, Meimaris, 1996; Mariotti, 2003).

In the 3rd period, students classify the shapes (all kinds of parallelograms) and expand the properties of the shapes. For example, the properties of parallelogram are inherited to rectangle and rhombus and from them to the square.

In the 4th period, students deal with simple geometric propositions and use RE.CO.M.P.P to write the proof.

In the 5th period, students learn the proofs of all theorems (of parallelograms).

In every period (see Figure 9) students use the special worksheet mentioned before, named "Structured Form Worksheet" (SFW) which the teacher has prepared beforehand to teach a cognitive object (see appendix).

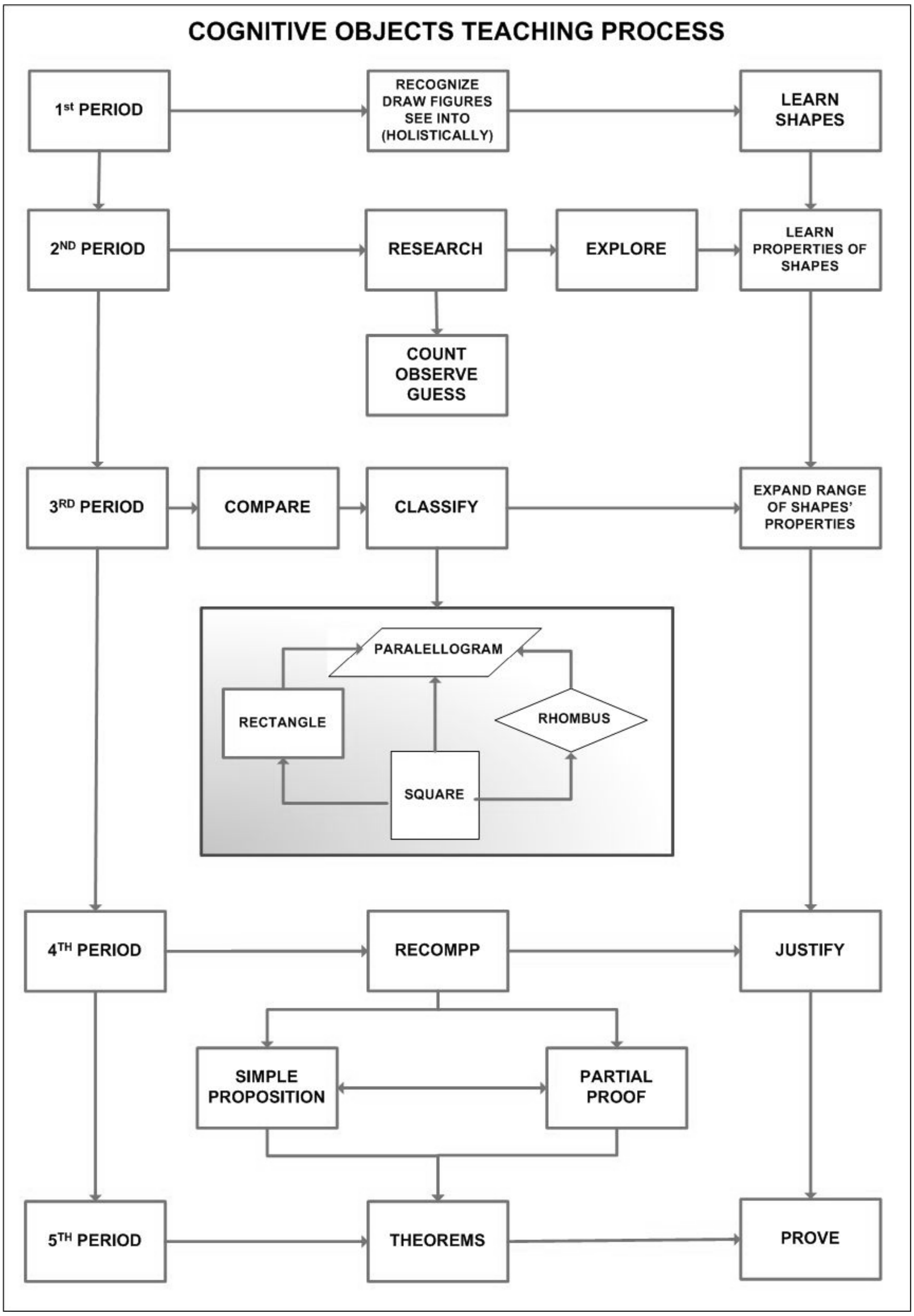
COGNITIVE OBJECTS TEACHING PROCESS
1st PERIOD
RECOGNIZE
DRAW FIGURES
SEE INTO
(HOLISTICALLY)
LEARN SHAPES
2ND PERIOD
RESEARCH
EXPLORE
LEARN PROPERTIES OF SHAPES
COUNT
OBSERVE
GUESS
3RD PERIOD
COMPARE
CLASSIFY
EXPAND RANGE OF SHAPES' PROPERTIES
PARALELLOGRAM
RECTANGLE
RHOMBUS
SQUARE
4TH PERIOD
RECOMPP
JUSTIFY
SIMPLE PROPOSITION
PARTIAL PROOF
5TH PERIOD
THEOREMS
PROVE

Figure - 8

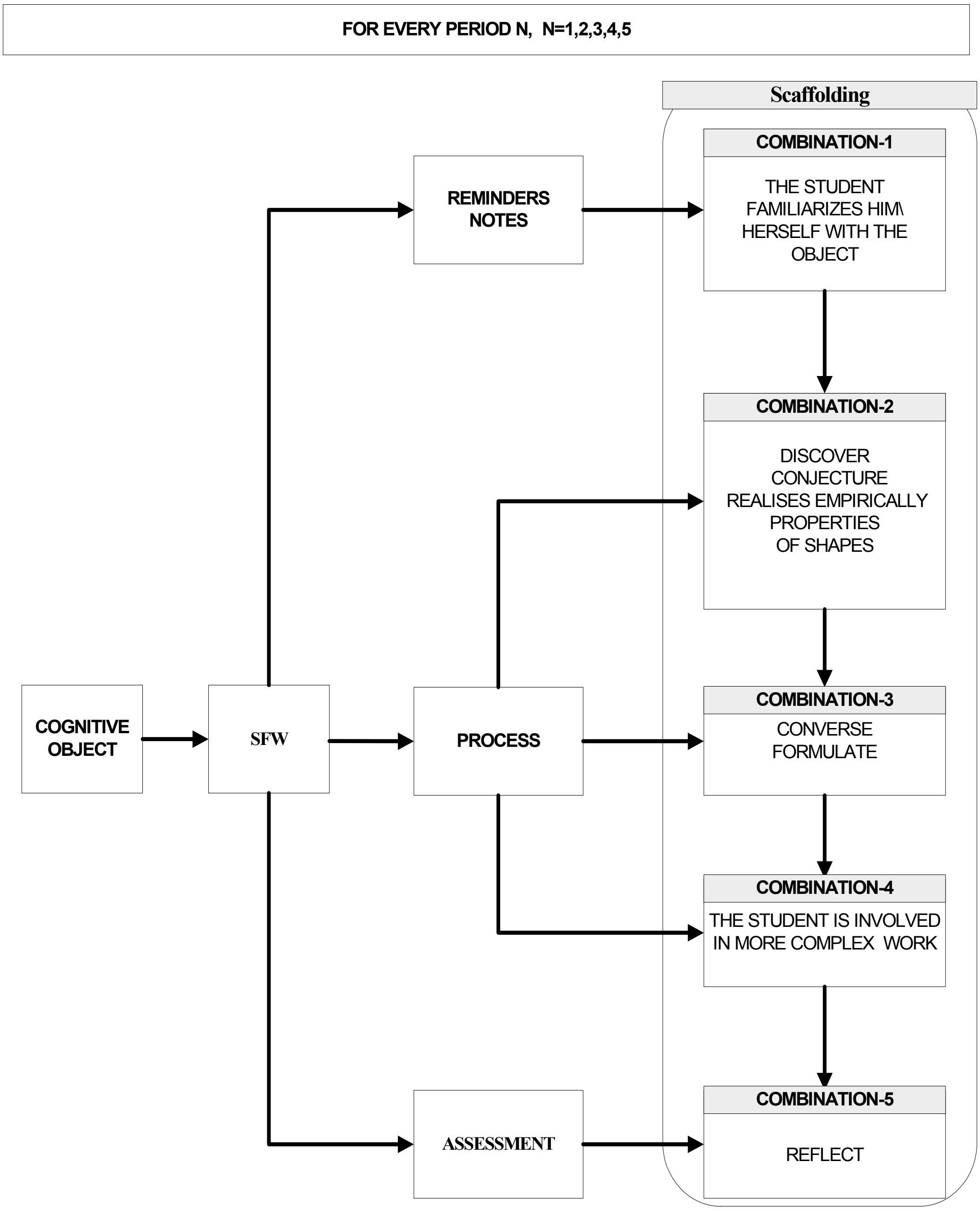
FOR EVERY PERIOD N, N=1,2,3,4,5
Scaffolding
COGNITIVE OBJECT
SFW
REMINDERS NOTES
PROCESS
ASSESSMENT
COMBINATION-1
THE STUDENT FAMILIARIZES HIM\ HERSELF WITH THE OBJECT
COMBINATION-2
DISCOVER CONJECTURE REALISES EMPIRICALLY PROPERTIES OF SHAPES
COMBINATION-3
CONVERSE FORMULATE
COMBINATION-4
THE STUDENT IS INVOLVED IN MORE COMPLEX WORK
COMBINATION-5
REFLECT

Figure - 9

4. Method

4.1. Participants and Procedure

The participants in the research were students, who were studying at the first class of High School. The method of random sampling was used to constitute the sample of the research. Overall, 250 students participated in the research. They came from five public and one private school. The majority of students originated from families of low socio-economical status. These students had never been taught the course of geometry in a theoretical context before. Therefore they were considered to be novice in theoretical geometry. Most of the students had some experience in the use of computers. Students participated in the research as whole classes, based on the distribution of classes already made by the school principals. 138 students formed the experimental group and 112 students formed the control group.

4.2. The instruction

Unit 5 from the textbook, referring to parallelograms, was taught. The instruction took place in five periods that lasted two-months period. In the 1st period, students related the visual geometric shape of all kinds of parallelograms and its appearance to their names. In the 2nd period, students were taught the attributes and the relative theorems of parallelograms, without their proofs. Students confirmed the validity of theorems in an experimental way, using computers. In the 3rd period, students classified the shapes of all kinds of parallelograms. In the 4th period, students argued simple geometric propositions and used RE.CO.M.P.P to write a proof. In the 5th period, students learned the proofs of all theorems of parallelograms.

4.3. Instruments

4.3.1. SFW

The students were taught the unit 5 from the textbook that refers to parallelograms. As we mentioned above the students used the *Reasoning Control Matrix for the Proving Process*. The RE.CO.M.P.P was employed for the proof of propositions that were assigned to the students, since a theoretical document was required.

4.3.2. Proof-writing evaluation exercises

Two pairs of exercises, that were similar to those of the textbook used in class, were given successively during the pre-test and the post-test. (Totally, 4 exercises per test). The first exercise from each pair corresponded to a simple proof and the second one corresponded to a complex proof, which was an extension or a slight modification of the first exercise. We asked students to prove another one result, whose proof was based on the result of the first exercise.

The following exercises comprised the pre-test:

Exercise 1: In an isosceles triangle ABC, with AB = AC, the points M, N lie on the line segments AB, and AC respectively, so that M is the mid-point of AB, and N is the mid-point of AC. We equally extend the base BC of the triangle by the line segments BD, CE, so that BD=CE. Prove that DM=EN (See Figure 10).

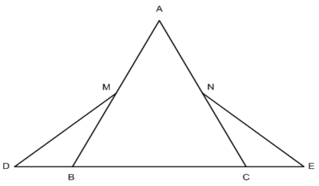

Figure 10

Exercise 2: In an isosceles triangle ABC, with AB = AC, the points M, N lie on the line segments AB, and AC respectively, so that M is the mid-point of AB, and N is the mid-point of AC. We extend the base BC of the triangle through B and C, respectively, to points D and E, so that BD = CE. Segments DM, and EN intersect at point I. Prove that segment AI is a bisector of angle A (See Figure 11).

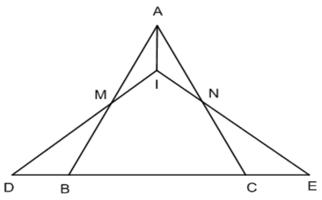

Figure 11

Exercise 3: Prove that the common external tangents BC and B´C´, of two externally tangential circles K and O are equal (See Figure 12).

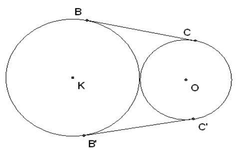

Figure 12

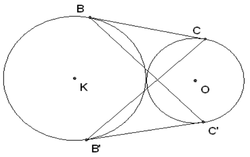

Figure 13

Exercise 4: Given that BC and B′C′ are common external tangents of two externally tangential circles K and O, prove that B′C = BC′ (See Figure 13).

The following exercises comprised the post-test:

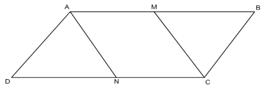

Figure 14

Exercise 1: In parallelogram ABCD, the points M, N lie on the line segments AB and CD, respectively, so that M is the midpoint of AB, and N is the midpoint of CD. Prove that ANCM is a parallelogram (See Figure 14).

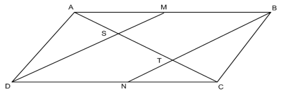

Figure 15

Exercise 2: In parallelogram ABCD, the points M, N lie on the line segments AB and CD, respectively, so that M is the midpoint of AB, and N is the midpoint of CD. Prove that the diagonal AC is trisected by the segments MD and NB (See Figure 15).

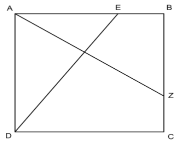

Figure 16

Exercise 3: In square ABCD, the point E lies on the segment AB. From point A we draw a perpendicular to DE, which intersects BC at point Z. Prove that DE = AZ (See Figure 16).

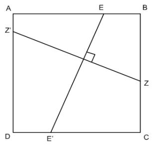

Figure 17

Exercise 4: In square ABCD, two vertically intersected line segments intersect sides AB, BC, CD, and DA of the given square at points E, Z, E′ and Z′, respectively. Prove that EE′ = ZZ′ (See Figure 17).

Each exercise in the pre-test and the post-test was graded in a scale between 0-and 5 grades as follows:

Grade	**Analytical explanation of student's actions**
0 -	The student had not written anything or had not made any valid simple propositions
1 -	The student had written the given input and the objects of the exercise and had also drawn the shape in the corresponding sections of RECOMPP
2 -	The student had written at least a simple proposition along with

its justification

3 - The student had written some thoughts in the “reasoning development” section of RECOMPP

4 - The student had written at least a partial proof

5 - The student had written a complete proof

4.4 Results

Initially, an independent-samples t-test among the experimental and the control group was conducted. The statistical test was conducted to check the existence of statistically significant differences in the performance between the pre-test and the post-test. No statistically significant difference was found in the performance in proof-writing among control group and experimental group, between pretest and post-test (t = -1,896 df = 248, p > 0,05).

Also, a paired-samples t-test among students in the control group was conducted between pre-test and post-test. This statistical test was conducted to check the existence of statistically significant improvement in the proof-writing performance of these students. No statistically significant difference was found in the proof-writing performance among students of control group between pre-test and post-test. (t = 0,711 df = 111, p > 0,05).

Finally, a paired-samples t-test among students in the experimental group was conducted, between pre-test and post-test. This statistical test was conducted to check the existence of statistically significant improvement in the proof-writing performance of these students. It was found that there is a statistically significant difference in the proof-writing performance among students of the control group between pre-test and post-test

(t = -48,271 df = 137, p < 0,05).

Discussion

The findings suggest that students of the experimental group, who had employed the Structured Form Worksheet that contains RE.CO.M.P.P and had initially dealt with simple geometric propositions had significantly improved their ability in writing formal geometry proofs, compared to the students of the control group who had been taught in a traditional method. The exercises given to the students were common exercises, that is, exercises taken from the textbook. These exercises were given in pairs. The second exercise represented an extension of the first one. Thus, the proof of the second exercise was an extension of the proof of the first exercise. We intentionally left the students uninformed of this property between the two exercises. We did so, because we wanted to examine either, whether a student, who had already solved the first exercise had the ability to solve the second exercise too, or whether a student had the ability to solve only the first exercise and not the second one. We found that most of the students in the control group could not solve the second exercise, while most of the students in the experimental group managed to solve both exercises. The above findings allow us to claim that students should be able to write simple and partial proofs before they are taught how to write formal proofs.

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Appendix

Structured Form Worksheet

Instructor …………………………

Students' full names

____

____

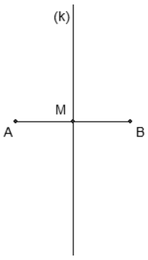
____

Class …………………………………
School ……………………………..
Date …………………………

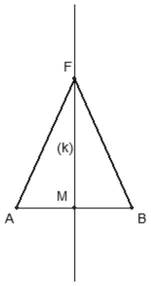
Lesson topic: **The diagonals of a rectangle**

1. The Reminders Notes.

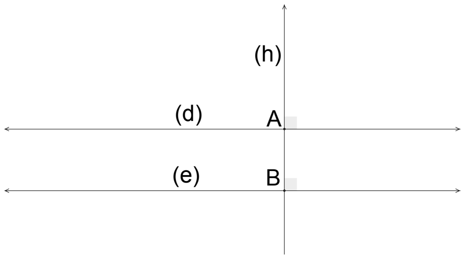
1.1. The perpendicular bisector (k) of a line segment AB is a line that is perpendicular to AB and passes through the midpoint M of segment AB.



1.2. The perpendicular bisector of a line segment is the locus of all points that are equidistant from its endpoints, i.e. GA = GB for every point F of the perpendicular bisector.

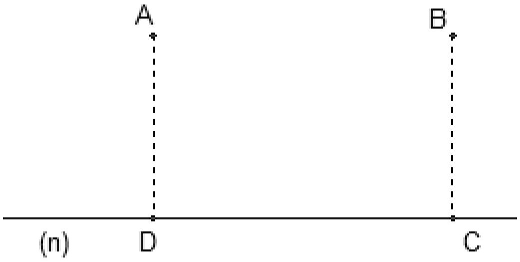


1.3. If a straight line (h) is perpendicular to one of two parallel lines (d), then it is also perpendicular to the other line (e).



If A = 90^o then B = 90^o

1.4. **Problem:** Two cities A and B are equidistant from points D and C of the national road n respectively. Where should a station S be built so that the points A, B, C and D are equidistant from S?

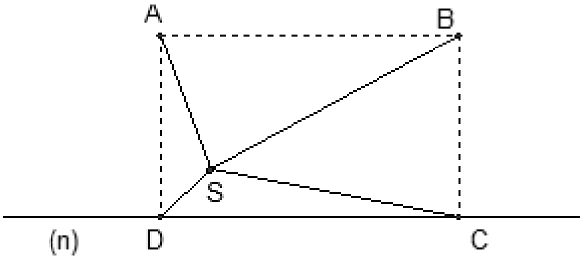


2. Process

(The students work in the computer. The students with the help of command measurement fill a table)

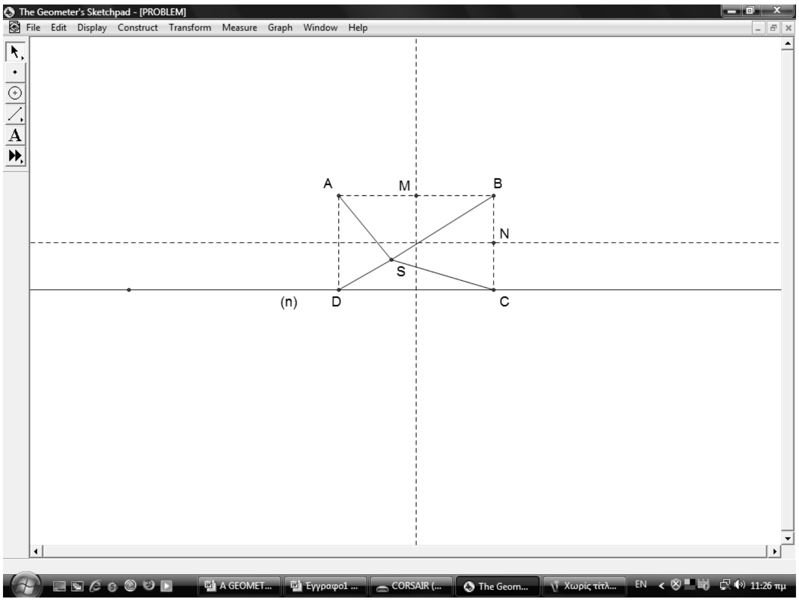
2.1. What shape is the ABCD

2.2. Measure the length of every segment SA, SB, SC, SE. Drug the point S, repeat the measurement and fill the following table.

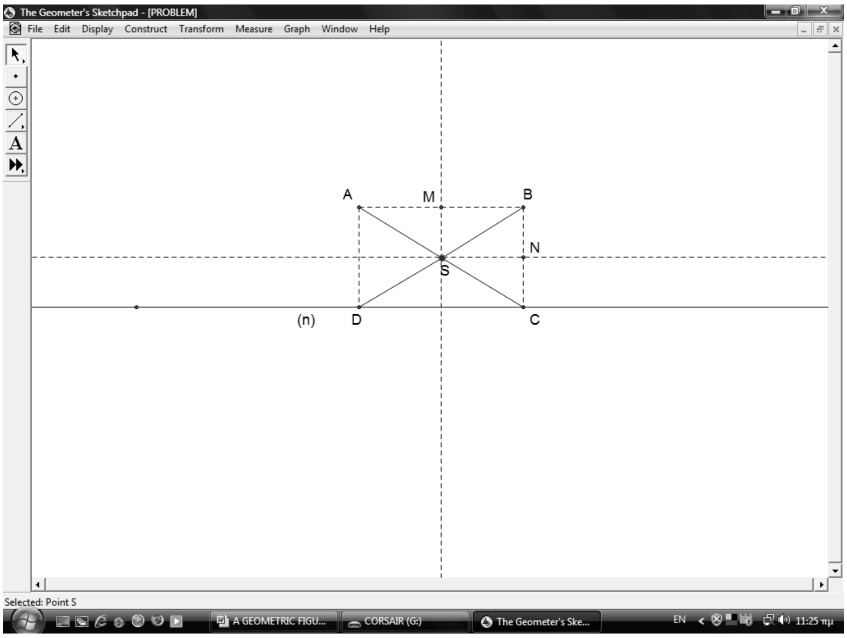


	EA	EB	EC	ED

2.3. Draw “all points” that are equidistant from A and B, and then draw all points that are equidistant from A and D **(The students draw the follow shape)**



2.4. Can you guess where point S lies?



2.5. Write down your observation **(The students discover where the point S lies)**

2.6. Prove that points A, S, C are on the same line and then prove that B, S, D are on the same line too. **(The students employed the RE.CO.M.P.P)**

2.7. Prove that AC = BD. **(The students employed the RE.CO.M.P.P)**

2.8. Formulate the relation. Then write it in words:

(The students discuss the theorem and then write it down)

3. Assessment

(The students replay the following questions)

3.1. What have you learnt? Draw the figure and write the:

THEOREM

……………………………………

……………………………………

……………………………………

……………………………………

……………………………………

……………………………………

……………………………………

3.2. Describe over the phone to another schoolmate, who was absent from class, what you learned.

3.3. Write a problem based on the theorem you learnt

Emmanouil Nikoloudakis
Lecturer (P.D. 407/80)
Faculty of Primary Education, University of Athens
e-mail: enikoloud@primedu.uoa.gr

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Factors that Influence the Development of Students' Regulating Activities as they Collaborate in Mathematics

Sonia Kafoussi, Petros Chaviaris & Rijkje Dekker

Abstract

This paper is focused on the investigation of factors that influence students' self-regulation as they try to develop regulating activities when they collaborate in pairs in mathematics. We investigated the issue of how 10-11 year old students regulated their behavior during their mathematical activity as they reflected on their small-group interaction by observing and discussing their video-recorded collaboration. We studied the collaboration and the metadiscursive reflection of different cases of pairs. The results showed that the students' development of their self-regulation is a complex process, as it is influenced by their beliefs about the role of the others, their beliefs about the role of collaboration in mathematical learning, the occurrence and the treatment of errors and disagreements as well as the difference of students' cognitive levels.

Keywords: *collaborative learning; mathematics; metadiscursive reflection; primary education; regulating activities*

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INTRODUCTION

Recently, the role of students' self-regulation as they are engaged in mathematical activities has begun to gain a lot of interest (cf. Marcou & Lerman, 2006), although self-regulation has a long history in educational psychology (Fox & Riconscente, 2008). Self-regulation has been connected with metacognition, but nowadays researchers are trying to define these constructs more precisely (Dinsmore et al., 2008). Metacognition is described as cognition about cognition, that is, it concerns the awareness of an individual about his/her knowledge and the regulation of his/her cognitive activities in learning processes (Veenman et al., 2006). On the other hand, self-regulation concerns a "systematic process of human behavior that involves setting personal goals and steering behavior towards the achievement of established goals" (Zeidner et al., 2000, p.749, cf. Dekker et al., 2006) and it does not necessarily require that individuals are aware of the processes involved, as it involves motivational and socio-emotional processes. Zimmerman (1995) has mentioned that the interpretation of students' self-regulation has to be treated as a complex interactive process influenced by students' self beliefs-system and "SRL (Self-Regulated Learning) involves more than metacognitive knowledge and skills, it involves a sense of personal influence, such as emotional processes as well as behavioural and social-environmental sources of influence" (p. 218).

Researches in mathematics education concerning self-regulation have mainly been developed in social contexts where the students have the opportunity to engage collaboratively in mathematical tasks (collaborative learning, students' social interaction in the class) and discourse, as these socio-cultural settings facilitate the developing of these activities. They have shown that students regulate their own collaborative learning in the classroom setting according to their commitments, conceptualizations and strategies and there is some evidence that students can regulate their own collaborative learning activities (cf. Dekker, Elshout & Wood, 2006). However, more research is needed in order to clarify the conditions that allow the development of self-regulation in collaborative learning in a classroom setting.

The purpose of this research is to investigate factors that influence students' regulating activities when they try to collaborate in pairs. More specifically, we investigated the issue of how 10-11 year old students regulated their behavior during their mathematical activity as they reflected on their small-group interaction by observing and discussing their video-recorded collaboration. Critical moments of students' collaboration in mathematics are discussed and the opportunities that students' meta-discursive reflection offered to their self-regulation are presented. The study is focused on different pairs of students representing different self-belief systems.

THEORETICAL BACKGROUND

As the mathematical activity is considered as a process that takes place in a macro- and a micro-community with concrete socio-cultural characteristics, students' awareness of their actions in mathematics has to be related not only with the reflection on cognitive aspects of their activity but with social and cultural aspects of it, too. Nowadays, the reduction of the student to a "cognitive subject" seems to be replaced by the acknowledgement of the student as a "social subject" that is influenced by his/her history and culture (Lerman, 1998; De Abreu, 2000; Valero, 2004).

Dekker & Elshout-Mohr (1998) have described an effective process model for interaction and mathematical level raising, of students working in small groups. Moreover, it has been mentioned that if the students often follow this model, they attain more mathematical level raising (Dekker & Elshout-Mohr, 2004; Pijls et al., 2007). The main activities described in the model are key and regulating activities. The key activities are crucial for level raising. The regulating activities provoke the key activities and in that way they regulate the process of level raising. We can represent them in the following way (Table 1):

Regulating activities like:	*Provoke key activities like:*
A asks B to show his work	B shows his own work
A asks B to explain his work	B explains his own work
A criticizes B's work	B justifies his own work
	B reconstructs his own work

Table 1. Regulating activities

All the above activities can help children to become aware of their work and enhance their self-regulation. However, a participant of an interaction in the mathematics classroom "monitors his or her action in accordance with what he assumes to be the other participants background, understandings, expectations… At the same time the other participants make sense of the action by adopting what they believe to be the actor's background, understandings, and intentions." (Voigt, 1995, p. 169). As a consequence, every member of a mathematics class tacitly participates in his/her own way in interacting with others according to his/her personality (interests, expectations, intentions, and beliefs).

Moreover, as Sfard (2001) has mentioned students' initiation to mathematical discourse depends on the "meta-discursive rules that regulate the communicative effort" (p.28). These rules are considered as the implicit regulators of interpersonal and intra-personal communication, as they determine the choices of the participants when they act and they embed their values and beliefs. She has emphasized the role of the interlocutors' intentions in a mathematical discussion using the term "meta-discursive intentions" in order to describe the "interlocutors' concerns about the way the interaction is being managed and the issues of the relationship between interlocutors" (p. 39).

In mathematics education, many researches investigating social interaction in classroom have been focused on cooperative learning contexts concerning small groups of students. These researches have revealed a lot of factors that influence students' mathematical learning like group composition, students' beliefs about their cooperation in mathematics, students' achievement in mathematics, the quality of mathematical activities (e.g. Good et al., 1992; Edwards, 2002; Kieran, 2001; Webb, 1989).

In our research we assumed that students can give their own explanations about their behavior as they try to collaborate in small groups in a mathematics classroom. In most researches the teacher has played a significant role in establishing the social and socio-mathematical norms of students' collaboration (McClain & Cobb, 2001; Dekker & Elshout-Mohr, 2004). On the other hand, the teacher can not really know what the children, as they work in pairs, may discuss. Furthermore, if the students do not collaborate systematically during their mathematical activities, can they develop their regulating activities and take responsibility for the quality of their shared activities? How do the children think on their own about their collaborative learning?

Towards this effort we used the term "metadiscursive reflection" in order to describe this kind of students' reflection that is related with the consciousness of relationships among cognitive, social and emotional components of their mathematical discourse. Metadiscursive reflection concerns students' reflection on their own and their interlocutors' beliefs and intentions about their social interaction and it is revealed through their explanations and justifications about their behavior. Our questions in this research are:

a) What factors influence the occurrence of each regulating activity as students try to collaborate in mathematics?

b) What are the critical situations of students' metadiscursive reflection that allowed the development of these regulating activities for both partners?

METHOD

The research program took place in a fifth grade of a typical public school of Athens in Greece, in 2003–2004 and it lasted six months. The participants were 18 students (9 boys and 9 girls) that worked in pairs, 4 times per week during math class teaching. The mathematical topic, in which the students were engaged during the research program, concerned the concept of fractions (equivalence, comparison and the four operations). The activities about fractions have been given by the researchers in order to be meaningful for the students according to the related literature (Kieren,1992; Streefland,1991) and the students' initial knowledge of fractions. The research program was developed in three phases.

Initially, we studied the students' profiles in order to organize them in pairs. All the students were interviewed by the researcher about their beliefs of their own participation and the others' participation during the classroom mathematical activity as well as about the nature and the goals of mathematical activity (e.g. When do you feel really pleased in mathematics? How do you feel when you make an error in mathematics? Do your classmates help you in mathematics?). Every interview lasted about one hour. Moreover, we investigated the students' informal knowledge on fractions using a questionnaire with mathematical problems on this topic. Finally, the teacher of the class was asked to assess her students in mathematics based on her personal evaluation by using the criterion of the student's need for help in order to solve a mathematical problem (He/she managed in mathematics – He/she managed in mathematics but sometimes with help – He/she managed in mathematics only with help) as well as to provide the students' grades in mathematics of the previous school year.

We based our research on a patchwork case-studies method (Jensen & Rodgers, 2001), studying our cases horizontally. According to this method a set of multiple cases of the same research entity (in our research we define as entity a pair of students) allows a deeper and more holistic view of the research subject.

The criteria for the organization of the students in pairs are presented in the following table:

Criteria for the organization of pairs	*Symbols*
Negative beliefs about collaboration in mathematics	N
Positive beliefs about collaboration in mathematics	P
He/she managed in mathematics	1
He/she managed in mathematics but sometimes with help	2
He/she managed in mathematics only with help	3

Table 2. Criteria for pairs

On table 2 the notion "negative beliefs" was used to describe the students' responses like: "I would like to solve alone the problems in mathematics" or "I would like the teacher to help me in order to solve the problems".

The different cases of pairs that arised from this class are presented in table 3. As there were similar pairs of students in some cases, in our research we studied one pair of each case. The choice of the pairs was accidental.

Case	*Pairs' profile*	*Number of pairs*
1	N1 - P2	3
2	N1- N2	1
3	N2 - N2	1
4	P1– N3	1
5	P2 - P2	3

Table 3. Pairs' profiles

In the second phase of the program, the students' collaboration was videotaped once a week for every pair in the class by the researcher(R) and then the members of the group participated in a session with him. These meetings with each group took place in the school library, immediately after the lesson in their regular classroom and they lasted about 30 minutes every time. Each group realized six meetings with the researcher. During this session, the students observed and discussed issues concerning their video-recorded collaboration. The researcher had a role of coordinator during the students' discussions. He clarified the context of these discussions by reminding them the special issues that they had to discuss, like the assessment of their collaboration, the significant moments of their work or their desires for the improvement of their interaction. These discussions were tape-recorded. On this meta-discursive level, the tape-recorded students' discussions about their own videotaped collaboration were analyzed according to: (1) the way that the students assessed their collaboration (self-assessment), (2) the moments of their interaction that they considered as critical and (3) the targets and their behavior in their next collaboration (self-regulation). We will base on the protocols of 11 years old students' dialogues (as they observed their videotaped collaboration) as evidence for the development of the regulating activities.

In the third phase, the members of each pair were interviewed again about their beliefs of their own and the others' participation in mathematics as well as the nature and the goals of mathematical activity.

RESULTS

We should note that, according to the data from the initial interviews that were conducted in this class, the students that expressed the wish to collaborate with their classmates, were usually average or low achievers, and the students that expressed the wish to work alone, were usually high achievers (according to the criteria described in the method). Furthermore, the students' justifications about their preferences revealed the following beliefs that prevented the development of collaboration in mathematics: a) mathematical knowledge is acquired with personal effort, b) different ideas in mathematics cause confusion and create difficulties in understanding, c) classmates' errors in mathematics negatively influence pupil's thinking and prevent their learning and d) the exposition of a pupil's thinking to his/her classmates does not protect his/her self-image (Chaviaris et al., 2007). We must mention that the students had not worked in groups in mathematics in previous years and their teacher was following a traditional approach in this subject (cf. Cobb et al., 1992).

In describing our results we will try to separate the three regulating activities in order to clarify the factors that influence the occurrence of each one as well as the students' comments about them. Towards this effort, we will present illustrative episodes from the collaboration and the metadiscursive reflection of different pairs.

The first regulating activity "A asks B to show his/her work to her/his partner"

In order to reveal the factors that influenced the occurrence of students' regulating activity "A asks B to show his/her work to her/his partner" using questions like: *What are you doing?*, we present and analyze two illustrative episodes of different pairs of students as they made their first efforts to collaborate in mathematics and as they reflected on their collaboration.

Episode 1:

Paul(P) and Nikos(N) had expressed negative beliefs about collaboration in mathematics (N1-N2) and Paul seemed to manage better in mathematics than Nikos. During their second mathematical activity, they had to solve the following problem:

Put the fractions 1/3, 2/6, 1 1/2, 3/2 and 5/6 on the following number line.

Figure 1.

Their dialogue as they tried to collaborate was the following:

utterances		writings
1	N: 1/3 will be here? (He is indicating the interval 0-1)	
2^a	P: 1,2,3. (He is dividing the interval 0-1 by his finger)	
2^b	Then, it is here. (He is indicating the first third of the interval 0-1).	
3	P: The 2/6 are equal with the 1/3, then it is at the same point.	[3] He is dividing the interval 0-1 in three parts and he is writing 1/3 and 2/6 on the first point.
4	N: Where?	
5	P: Here, with 1/3.	
6	P: The 1 ½ … in the middle after the point 1.	
7	P: And 3/2 will be in the same point because they are equal.	[7] He is writing 1 ½ without dividing in parts.
8	N: Let me look. (He is looking at Paul's worksheet)	
9	P: Now, 5/6…	
10	N: 5/6 will be somewhere about here? (He is indicating the interval 1-2)	
11	P: They don't pass over the point 1. So will be… 1,2,3,4,5, here.	[11] He is dividing the interval 0-1 in six parts and he is writing 5/6 on the fifth point.

In the above dialogue, Nikos was not sure about the position of the fractions on the number line and he was trying to challenge Paul in order to show him his work (cf. ut. 1, 4, 8, 10). Paul showed his work without any question if Nikos could understand his thoughts. When they observed their video-taped collaboration, the following discussion took place:

[1] R: We'll watch the video with your second collaboration in mathematics and after we'll discuss about it.
[2] R: How was your second collaboration in relation with the first one?
[3] N: We collaborated more. In the first collaboration each one of us solved the problem alone.
[4] P: Yes. It was more collaborative than private.

[5] R: How did you collaborate?
[6] N: We talked to each other and…
[7] P: I think that I spoke more because he had some difficulty and I explained to him.
[8] N: Yes. I asked him because I hadn't understood the number line.
[9] R: What did each of you do in this collaboration?
[10] P: I solved the problem in order to be on time and after I showed the solution to him.
[11] N: Paul knows more mathematics than me and he knew that I would ask him, so he finished quickly and after he showed me.
[12] R: Nikos, how did you feel about Paul's behavior?
[13] N: Good. If I ask for help, it's good. If someone helps me without my request, I become angry.
[14] R: Did you understand the number line after Paul's help?
[15] N: Not enough.
[16] R: Did you do something for this?
[17] N: What could I have done?
[18] P: You did not ask me. I could explain to you more.
[19] R: Ok. What would you like to change about your collaboration in mathematics?
[20] P: To discuss the problem more from the beginning.
[21]N: Yes. It is better to ask each other some explanations when we don't understand the problem.
[22] R: Nice. We will meet again in a few days.

During the discussion Nikos justified his behavior (that is to ask Paul to show him his work) according to his beliefs about his interlocutor's mathematical abilities (cf. phrase 11). Moreover, he seemed to accept that Paul's role was to solve the problem alone and then to show him the solution. On the other hand, Paul has also accepted that role (cf. phrases 7, 10, 18). These perceptions about their roles during the solution of a mathematical problem influenced the occurrence of the first regulating activity from the part of Nikos at the beginning of their interaction.

Episode 2:

Stavroula (S) and Alexia (A) had expressed different beliefs about the role of collaboration (N1-P2). Stavroula considered the collaboration to be an obstacle in the understanding of mathematics, because she believed that *"if someone doesn't work on his own, he cannot understand mathematics"*. On the contrary, Alexia believed that collaboration could help her to control her thoughts before she announced them in the classroom and so that she could "avoid mistakes".

At the beginning of the program, the children had to solve the following problem:

In Alexandra's Avenue, public works are being made by 3 different firms of constructors. The works are being made at three different points. The first firm of constructors makes works at a point corresponding to the 1/3 of the avenue, if we count from its beginning. In the ¾ of the avenue there are works of the second firm of constructors and in the 5/6 of the avenue there are works of the third firm of constructors. Note in the following schema where the works are being made. Use red color for the first point, green for the second one and blue for the third one.

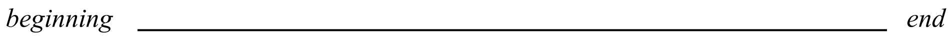


Figure 2.

Utterances		writings
1	A: *(She is reading the problem)*	
2	S: So…	
3	S: 1/3 of the avenue…Which is 1/3?	
4	A: Calm yourself.	
5	A: Do you want to discuss it?	

6	S: We have to count with a ruler.	
7	A: Just a moment, ½ is the half and we have to share it in the middle, like yesterday. 1/3?	
8^a	S: This line is common for all the constructors.	[8b]She notes with the ruler 1 point on the line (distance 1 cm).
8^b	1,2,3…Here is for the first firm.	
8^c	Now, about the second firm with green color, ¾…	
9	S: Did you finish with the first one?	
10	A: I am confused, what did you do?	[11b] She notes with the ruler 3 more points on the line (distance 1 cm).
11^a	S: Do you want me to help you?	
11^b	1 cm for the first, and three more cm for the second …	
12	Now we will see the third one… Ok, 5 more cm for the third.	

Alexia asked her partner to collaborate with her (cf. ut. 5, 10). However, the priority of Stavroula was to solve the problem by her own way. Her reaction "*Do you want me to help you?*" (cf. 11^a) showed that her intention was not to find an acceptable solution by collaborating with her partner.

During their discussion as they reflected on their collaboration they made the following comments:

[1] S: When you find the solution and it is right, you don't have to discuss it with your partner, because she may have a different opinion and she will confuse you.

[2] A: It is better to discuss it, because if it is wrong, you will think: why didn't I ask?

[3] R: So, what happened today?

[4] S: I told her to put centimeters, but Alexia told me to divide the whole in pieces. We made it wrong.

[5] R: The solution on the board with whom solution did it match?

[6] S: With Alexia's.

[7] S: Ok, it is good to collaborate, but if you don't find the solution alone, you

don't understand…You can not do something that the other says, if you don't think alone.

[8] R: How can you be helped to understand the solution of Alexia?

[9] S: …

[10] A: You had to ask me.

[11] S: We have to discuss more our thinking.

As the students reflected on their actions, Alexia justified her behavior (that is, to ask Stavroula to discuss with her about the problem) as she believed that in this way they could prevent mathematical errors (cf. phrase 2). On the other hand, Stavroula explained her behavior according to her beliefs about the negative consequences of collaboration in mathematics (cf. phrase 1).

Discussing on the above episodes we could mention that the occurrence of the first regulating activity (A asks B to show his/her work to her/his partner) was found to be influenced by two factors: a) the established roles that the partners perceived at the beginning of their interaction (who was the helper and who needed help) according to their beliefs about their own and their partner's ability in mathematics (who considered him/herself as a good student in mathematics and who considered the opposite) and b) the students' beliefs about the role of collaboration in mathematics.

The first factor was connected with students' beliefs about their own and their partners' learning ability in mathematics which influenced their behavior during their effort to cooperate. It was easier for the student that had lower self-estimation to perform this regulating activity (c.f the case of Nikos in episode 1). Paul's and Nikos' perceptions about their concrete roles in their interaction in the mathematics classroom provoked relationships of power among them and defined the way of their communication. In the regulating activity *A asks B to show his/her work* the student A seemed to be the one who needed help and the student B the other who was the helper.

The second factor that seemed to influence the occurrence of the first regulating activity was connected with the students' beliefs about the role of collaboration in mathematics. Alexia who wanted to collaborate performed this regulating activity more usually in contrast of her partner who had negative beliefs (c.f episode 2). In the case of school mathematics the willingness to collaborate is important, because of the social dominant belief that learning of mathematics is an individual process and that the social interaction does not play any significant role in it (Chaviaris, 2006). As it was been revealed in students' metadiscursive reflection the conflict of beliefs about the role of collaboration in mathematics defined the way that the student regulated their interaction.

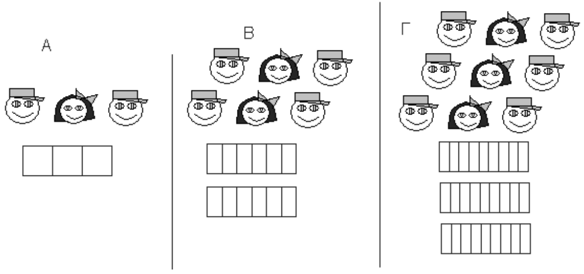
Furthermore, we should note that although the students in the above episodes posed new targets for their next collaboration, they sometimes presented the same behavior during it. That is, they experienced a lot of difficulty in order to regulate themselves and to change their actions. The critical situation that helped the development of the first regulating activity for both students for the different pairs occurred when they experienced the effectiveness of their partner's suggestion, if the solution of the "good" student was incorrect. These moments were discussed during their metadiscursive reflection, like in the dialogue presented above between Stavroula and Alexia. These topics for reflection helped mainly the "good" students to appreciate the efforts of their partner to contribute to the dialogue.

The second regulating activity "A asks B to explain his/her work to his/her partner"

The process of explanation is considered as a significant process for the development of student's mathematical reasoning and has substantial contribution in the development of students' collaboration (e.g. Cobb & Bauersfeld, 1995). In the following we present and analyze two episodes concerning the conditions of the occurrence of the students' request for explanation as they tried to collaborate in mathematics.

Episode 3:

Paul and Nikos (N1-N2) tried to collaborate as they were engaged in the following activity. The details of the students' profile has been presented in the episode 1.



In which group would you like to be if you wanted to eat more cake?
Why?
How much cake will each child eat in every group?

Figure 3.

Utterances	
1	N: (He is reading the problem)
2	N: We will eat the same piece but bigger quantity of these.
3	P: Look here. (He is showing the 2nd group). Each child will eat 2 pieces of these. 1/6 plus 1/6...2/6, that is, the same with 1/3.
4	N: Yes.
5	P: In the third group each child will eat 1,2,3… 3/9 of the cake, that is, the same too.
6	N: With which it is the same?
7	P: Both.
8	P: We will write that all the groups will eat the same.
9	N: Yes.

Although the students tried to communicate their solutions, they didn't manage to give explanations about their thoughts (cf ut. 2, 5-7). When they observed the video of their collaboration, they gave the following justifications about their actions:

[1] P: I did not understand what Nikos said.
[2] N: I said that they will eat more pieces but the same.
[3] P: You didn't say that.
[4] N: But that is what I would like to say.
[5] R: So, Paul what did you do?
[6] P: I showed him the right.
[7] R: Did Nikos have wrong?
[8] P: Now, as he explained it, no.
[9] R: So?
[10] P: I had to ask him again…Nikos has to say what he means so I will not think that he will say something wrong.
[11] N: When you don't understand you have to ask me.

In the above metadiscursive reflection, the interlocutors seemed to become aware that there were misunderstandings during their effort to communicate their mathematical ideas (cf. phrases 1-4). It was this fact that it provoked their reaction to pose questions concerning the second regulating activity, like to ask and to give explanations about their mathematical proposals (cf. phrases 8,10,11).

Episode 4:

The following episode concerns two students with negative beliefs about collaboration and with no differences in their mathematical ability (N2-N2). At the beginning of the program, both students, Apostolos(A) and Elsa(E), expressed their desire to collaborate only with their teacher in mathematics. Elsa justified her view as follows: *"I have to try alone and only the teacher can help me when I have difficulties"*. She declared that she didn't want to help her classmates in mathematics because: *"If I do not know it right, I will say it to the other students in a wrong way"*. On the other hand, Apostolos justified his own view as follows: *"I have the impression that my classmates will think that I do not do well in mathematics and I don't like this"*. Although, both students expressed negative beliefs about collaboration in mathematics, their intentions differed. Apostolos wanted to protect his self-image and Elsa had low self-confidence in mathematics. Moreover, these views were connected with the targets that the two students posed for their mathematical activity. Apostolos declared that he felt happy in mathematics when he could solve a problem that his classmates *"don't manage"*. In contrary Elsa said: *"I feel happy when I answer correctly to my teacher's questions"*. In relation to the students' abilities in mathematics, their teacher commented that both students very often needed help to complete a mathematical activity. In one of their collaborations, they had to solve the following problem:

During an excursion, Helen used 2/4 of her film and Nikos used 2/6 of his film. The films were same. The children discussed about who took more photos. A. Who used more film and how much more? B. How much film was left over for every child?

utterances		writings
1	E: We have to find how much more.	
2	A: How can we find it?	
3	E: We will subtract 2/4 from 2/6 and we will find it.	
4	A: It is 2/2, but it looks weird.	
5	E: I thought something, the denominators are not the same.	
6	A: We can simplify them. 2/4 can be ½ and 2/6 can be 1/3.	
7	E: If we subtract them it is 1/1.	[7]They wrote:
8	A: So, did we find how much more?	1/3-1/2=1/1
9	E: I don't know, maybe.	

During their discussion as they reflected on their collaboration they made the following comments:

[1] E: Apostolos made an error in the subtraction, he subtracted 2/4 from 2/6 and he said it was 2/2.

[2] A: But you accepted it, you didn't ask how I made it.

[3] E: I said that the denominators were not the same.

[4] A: Yes, but then we made the same subtraction again with a simplification and you accepted it.

[5] E: Yes, it seemed right to me.

[6] A: The first time we found 2/2.

[7] E: And finally we found 1/1 again.

[8] A: It looked weird to me from the beginning.

[9] E: I was not sure at the end either. We didn't think why it could not be 1/1.

Through their metadiscursive reflection, Apostolos and Elsa tried to explain their thinking (cf. ut. 2-3, 5-6). Both students tried to shift the responsibility of the error to their partner. However, they discussed on the quality of their explanations for the solution of a mathematical problem (cf. phrases 2, 4), that is if the given mathematical explanation made sense for both partners.

In general, we could mention that the second regulating activity did not easily occur during the collaboration of the groups. The first occurrence of it as a goal in the work of the pairs was observed when the partners did not understand the proposed solutions and they explicitly mentioned this during their metadiscursive reflection (c.f. episode 3). That is, the existence and the acceptance of the difficulties that a member of the pair experienced to understand his/her partner's solution provoked the necessity of an explanation in order to proceed their communication smoothly. Furthermore, the development of the second regulating activity for both partners was connected with the awareness that the existence of explanations during their discussion in mathematics helped them to find a correct solution (cf. episode 4).

However, we have to stress that in the case of the pair of the students where there was a big difference in cognitive level between the partners (case P1-N3), the effort for the occurrence of an explanation, as a regulating activity, was cancelled during their collaboration in mathematics. In that pair, the effort of the "good" student to explain his thought to his partner was continuously ineffective, as there was not a "domain of mathematical communication" between them.

The third regulating activity "A criticizes B's work"

The action to ask your partner to criticize your own proposal and in the same time to be receptive to his assessment consists a high level of communicational behavior. This activity is important in collaborative settings in mathematics classroom according to the process model for interaction (cf. table 1). In the following, we present and analyze concrete episodes of students' collaboration in mathematics in order to study the conditions under which this regulating activity was occurred.

Episode 5:

Paul and Nikos tried to solve the following problem:

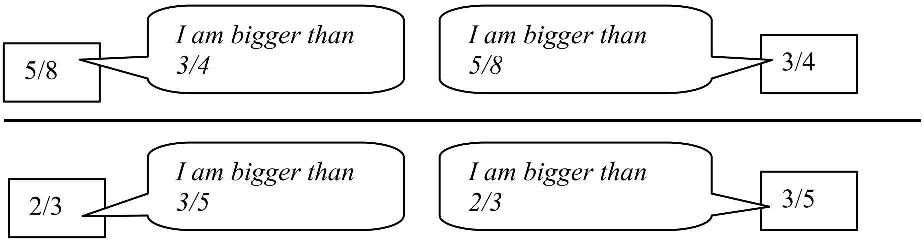


Figure 4.

Facilitate the fractions to find their value. Use the strategy of changing them into fractions with the same denominator.

Utterances		writings
1	N: (He is reading the problem)	
2	P: It suggests changing them into fractions with the same denominator.	
3	N: ¾ will be done 6/8, then it is bigger than 5/8.	
4	P: Yes, I agree.	
5	N: How will we make the second pair?	
6	P: I don't find something that makes them with the same denominator.	
7^a	N: I found it!	
7^b	If we subtract 2 from 3/5, it will be 1/3. So these fractions will have the same denominator and 2/3 is bigger than 3/5.	[7^b] $\frac{3-2}{5-2}=\frac{1}{3}$
7^c	…It isn't right?	
8	P: We can not find fractions with the same denominator by subtraction. We usually use division or multiplication.	
9	N: I know it, but we have to do it in this way because we can make the same denominator.	
10^a	P: 1/3 is not equal to 3/5.	
10^b	If we multiply it by 2, 2/3 will be 4/6.	
11	P: If we divide it by 2… It doesn't work. …. By 3, it doesn't make the denominator 5.	
12	N: We can not by 3 or 2, so what will we do?	
13	P: Oh! I found it.	
14	N: Let me see!	
15	P: 15, 3 times 5… 15. If we multiply it by 3 …and this one by 5 ….it will be here 15 and here 15.	[15] $\frac{2}{3} \underset{5}{\overset{5}{=}} \frac{10}{15}$ $\frac{3}{5} \underset{3}{\overset{3}{=}} \frac{9}{15}$
16	N: Ah! Yes, the first one is bigger.	

The following dialogue took place, as they reflected on their collaboration:

[1] R: This is your fourth collaboration. What targets had you put last time?

[2] P: To discuss the solution of the problem and to ask if we don't understand.

[3] N: Yes, to collaborate.

[4] R: Good, so how do you feel about this collaboration?

[5] P: Very good, we continuously discussed.

[6] N: Very good.

[7] R: Let's watch the second part of your collaboration. Do you want to observe something here?

[8] N: At the beginning it was not so good, we confused.

[9] P: I disagreed with Nikos' solution.

[10] R: What kind of disagreement did you have?

[11] N: I thought that if we subtracted 2 from 3/5 it would be right, because we would have the same denominator. Paul disagreed and he told me that it wasn't right.

[12] P: Yes, I explained to him that it wasn't right and then I thought how we had to do it.

[13] R: Nikos, were you convinced that your idea was wrong?

[14] N: Yes.

[15] R: How?

[16] N: Paul was right, you can not subtract because you don't take equivalent fractions in this way, then he explained to me how he found the correct and I agreed..

[17] N: If Nikos had a correct idea, we would discuss it.

Paul and Nikos had a disagreement about the way of finding fractions with the same denominator (cf. ut. 7-12). This disagreement provoked the occurrence of the third regulating activity, as Nikos asked from Paul to criticize his proposal (cf. ut. 7c) and Paul presented his arguments in order to support his strategy (cf. ut. 8,10a). During their metadiscursive reflection both students had the opportunity to describe how they treated their disagreement (cf. phrases 8-17).

Episode 6:

Towards the end of the program, Elsa and Apostolos (N2-N2) had to engage in the following mathematical problem:

2 children fairly share 5/6 of a pizza. How much pizza will each child get?

4 children fairly share 2/3 of a pizza. How much pizza will each child get?

utterances		writings
1	E: (She is reading the problem)	
2^a	A: 2 children, 5/6.	
2^b	We can multiply 2 by 5/6.	
3	E: Yes.	
4	A: Shall we do it?	
5	E: Just a moment, it don't write each one 5/6, but they share 5/6.	
6^a	A: Oh! Yes.	
6^b	There are 5 pieces. We share them in two, everyone will get 2 whole pieces and it rests one. Right?	
7	E: Yes.	
8	A: We will not divide this one in half?	
9	E: Yes.	[10]They try to draw in a cyclic disk 5/6.
10	A: Can we make a drawing?	
11	E: Yes.	
	…	
12	A: I can not make 6 pieces.	
13	E: I have difficulties too.	
14	A: Can we do operations?	
15	E: Yes.	
16	A: 5/6 will be 10/12, because it holds with 2.	[17] They wrote:
17	E: We will divide it.	↗ 5/12
18	A: Good, 5/12 each one.	10/12
19	E: The next one, 4 children share fairly 2/3 of a pizza.	↘ 5/12

20	A: Now, how can we divide 2/3 in 4?	
21	E: With an equivalent, 2/3 can become 4/6.	

As we can observe in the above episode, both students continuously expressed comments and assessments about their own and their partner's proposals. Apostolos and Elsa successively criticized the expressed proposals and they had an equivalent participation in their collaboration. This was a major advance during their collaboration, because Apostolos and Elsa had both negative beliefs about collaboration in mathematics at the beginning of the program.

Discussing about the conditions that influenced the occurrence of the third regulating activity (A criticizes B's work) we could notice that its spontaneous occurrence was connected with the existence of one student's disagreement during the interaction with his partner (cf. episode 5). However, the critical situation that allowed the occurrence of this regulating activity for both partners was the evaluation of all the different proposals that each partner offered, as they mutually tried to construct a common solution to a mathematical problem (cf. episode 6).

CONCLUSIONS

From the presented results, the students' development of their self-regulation, as they tried to collaborate in pairs in mathematics, using as a theoretical context for our analysis the process model of interaction, is complicated. It is influenced mainly by their beliefs about the role of the others, their beliefs about the role of collaboration in mathematical learning, the occurrence and the treatment of errors and disagreements as well as the difference of students' cognitive level. In our research, we found that the occurrence of each regulating activity was influenced by different factors. More specifically, the students' activity to ask their partner in order to show his/her own work is related to their belief about their own and their partner's learning ability and to their belief about collaboration; the students' activity to ask their partner to explain his/her mathematical solution is related to the existence of misunderstandings; the students' activity to ask their partner to criticizing a solution is related to the disagreement about an explanation.

However, in collaborative settings, these regulating activities have to be developed by both students in order to exist an effective collaboration and this effort needs suitable learning environments. According to our opinion, every mathematical activity can give opportunities to students in order to reflect about their interaction. The students' observations and discussions on their videotaped collaboration allowed them to become aware of multiple aspects of their mathematical activity and to improve their collaboration through the appearance of regulating activities. They had the opportunity to focus on different issues that are connected with the realization of a mathematical activity, like the treatment of an error or the treatment of different solutions. This means that students' self-regulation could be studied through students' metadiscursive reflection as this kind of reflection allows the understanding of factors that influence it. Table 4 summarizes the conclusions of this study.

Regulating activities	*Factors*	*Metadiscursive reflection on*:
A asks B to show his/her work	- Beliefs about their own and their partner's learning ability - Beliefs about collaboration	the effectiveness of their partner's suggestion
A asks B to explain his/her work	- Misunderstanding of a problem solution	the quality of their mathematical explanations
A criticizes B's work	- Disagreement about a mathematical solution.	the evaluation of the proposals offered by each partner

Table 4. Development of regulating activities

The results of this research show the necessity of the development of students' metadiscursive reflection in the mathematics classroom setting. Towards this effort mathematical educators should help teachers to design suitable didactical situations that enhance self-regulated learning. The construction of tools that include metadiscursive self-questions that the students could use as they collaborate in order to solve a mathematical problem is open for future research.

According to our results the organization of mathematics classroom in pairs of students according to their cognitive level is not enough for students' mathematical level raising. Teachers should be aware of their students' self-belief systems in order to appreciate their effort to collaborate in pairs. Maybe many difficulties that children still experience in mathematics, in spite of the progress in didactics of mathematics, are related to an incomplete picture that we have about the interaction between students as human beings and not as mere cognitive subjects.

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Sonia Kafoussi, Associate Professor
University of the Aegean, Greece
e-mail: kafoussi@rhodes.aegean.gr

Petros Chaviaris, Phd
University of the Aegean, Greece
e-mail: chaviaris@rhodes.aegean.gr

Rijkje Dekker, Senior Researcher
University of Amsterdam, Graduate School of Teaching and Learning
The Netherlands
e-mail: r.dekker@uva.nl

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Mathematizing the process of learning a subject matter in the classroom

Michael Gr. Voskoglou
Professor of Mathematical Sciences
Graduate Technological Educational Institute (T.E.I.)
voskoglou@teipat.gr

Abstract

In the present paper we build a Markov model for the description of the process of learning a subject matter by a group of students in the classroom. In this way we succeed to calculate the probabilities for a student to be at any of the major steps of the learning process in each of its phases in the classroom, as well as the probability to pass successfully through all the steps of the learning process in the classroom. Our results are illustrated by a classroom experiment for learning mathematics performed recently at the School of Technological Applications of the Graduate Technological Educational Institute of Patras, in Greece.

Keywords: Learning, mathematical modeling, stochastic models, Markov chains.

Introduction

The concept of learning is fundamental to the study of the human cognitive action. But while everyone knows in general what learning is, the understanding of the nature of this concept has proved to be complicated.

This basically happens because it is very difficult for someone to understand the way in which the human mind works and therefore to describe the mechanisms of the acquisition of knowledge by the individual.

There are many theories and models developed by psychologists and education researchers for the description of the mechanisms of learning. Nowadays it is widely accepted that any instance of learning involves the use of already existing knowledge. Voss (1987) developed an argument that learning consists of successive problem – solving (P-S) activities, in which the already existing knowledge plays the role of the input information, with the solution occurring when the input is appropriately interpreted. The whole process involves the following steps: ***Representation*** of the stimulus input, which is relied upon the individual’s ability to use contents of his (her) memory to find information, which will facilitate a solution development; ***interpretation*** of the input data, through which the new knowledge is obtained; ***generalization*** of the new knowledge to a variety of situations, and ***categorization*** of the generalized knowledge, so that the individual becomes able to relate the new information to his (her) knowledge structures known as schemata, or scripts, or frames.

Studies on P-S show many cases where input information is well interpreted, but no solution is obtained. Therefore the interpretation of the input information must be considered as a necessary only and not as a sufficient condition during the process of learning,

In the area of P-S, which is strictly connected with learning, while early work was focussed on describing the P-S process, more recent investigations have turned the attention on identifying attributes of the problem solver that contribute to P-S success. Carlson and Bloom (2005) drawing from the large amount of literature related to P-S developed a broad taxonomy of such kind of attributes. Then reanalyzing these data they reached to what they called a “Multidimensional P-S Framework” having four phases: Orientation, Planning, Executing and Checking. These are the main actions of the problem solver during the P-S process.

Schoenfeld, who offered a framework for analyzing the P-S process (Schoenfeld, 1980, 1985) has been working in the next 20 years to build a theoretical approach that explains the behaviour of the problem solver. He reached to the conclusion that solving a problem, as well as other human activities like cooking, teaching a lesson and even a brain surgery (!), are all examples of a goal-directed behaviour (Schoenfeld, 2007).

Over the last four decades mathematics education has addressed philosophical and epistemological perspectives with respect to mathematics learning. It has become common to think of mathematics in fallibilistic terms (e.g. Ernest, 1991; Freudental, 1978; Skemp, 1976), to consider learning as a constructive process (e.g. Davis, Maher & Noddings, 1990, Glasersfeld, 1987), to situate knowledge and learning relative to communities of practice (Lave & Wenger, 1991) and to debate the commensurability of constructivist and sociocultural learning theories (Lerman, 1996; Steffe & Thompson, 2000). Theoretical considerations like the nature of mathematical knowledge, what it means to know mathematics and to come to know it, how knowing in mathematics is related to knowing in social settings more widely, have been deeply considered and seriously debated (e.g. Bauersfeld, 1995; Cobb, 1996; Confrey, 1995; Kieran, Forman, & Sfard, 2001). The mathematics education discipline has become mature in such theoretical considerations.

The process of learning a subject matter in the classroom

Our target in the present paper is to build a mathematical model for the description of the process of learning and the model of Voss, described above, is used as a starting framework for this purpose.

Mathematics can usually describe in an explicit and plausible way the structure of a natural object, but things become more complicated when we face situations where the human presence and decisions are involved. In particular, learning is a very complex process that takes place not only in the class, but also between classes, or after a school day is over, or even in unexpected moments (e.g. during sleep). Therefore it is inevitable to put some restrictions in first place and to make some simplifications, in order to obtain a mathematical description of the learning process. This is a standard technique applied frequently during the mathematical modeling process of a real world problem in order to transfer from it to the ***"assumed real system"***, which enables the formulation of the problem in a form ready for mathematical treatment (Voskoglou, 2007; section 1).

Our basic restrictions in this case are that we consider the process of learning a subject matter during the teaching process in the classroom only (and not the process of learning by the individual in general), at the particular moment where the teacher introduces the new topic for the first time. Under these restrictions one must keep in mind that, as it frequently happens, a learner may not be able to pass successfully through all the steps of the learning process in the time available into the classroom. Therefore it is convenient in this case, for purely technical reasons, to include one more state in the sketch of the process (model of Voss) described in the previous section, the state of ***failure to reach categorization.*** We must clarify that the step of categorization could be reached out of the class, or in a next class, but for the particular chronological moment of our study this is counted as a failure.

Under these assumptions we are going to construct a 'flow-diagram' representing the whole process. For this, let us denote by S_i, i=1,2,….,5, the states of representation, interpretation, generalization, categorization, and failure to reach categorization respectively. The starting state is always S_1. From S_1 the learner proceeds to S_2. Facing difficulties there he (she) may return to S_1 to search for more information that will facilitate the interpretation procedure. Then he (she) must go back to S_2 to continue the process. From S_2 the learner is expected to proceed to S_3, unless if he (she) is unable to interpret the input data during the learning process in the classroom. In this case he (she) proceeds directly to S_5, and the process finishes there for him (her). From S_3 the learner, if he (she) has difficulties during the generalization procedure, may return to S_2 for a better understanding of the subject. Then he (she) comes back to S_3, wherefrom he (she) proceeds either to S_4 or to S_5 and the process finishes there in both cases..

According to the above description the flow-diagram of the process of learning a subject matter in the classroom by a group of students is that shown in Figure 1.

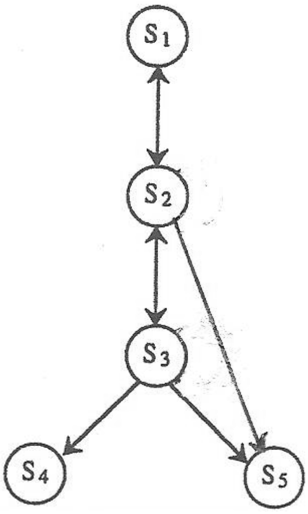

Figure 1: Flow-diagram of the learning process in the classroom

The stochastic (Markov) model

Roughly speaking a ***Markov chain*** is a stochastic process that moves in a sequence of phases through a set of states and has "no memory". This means that the probability of entering a certain state in a certain phase, although it is not necessarily independent of previous phases, depends at most on the state occupied in the previous phase. This is known as the ***Markov property.***

When its set of states is a finite set, then we speak about a ***finite Markov chain.*** For special facts on such type of chains we refer freely to Kemeny and Snell, (1976).

Here we are going to build a Markov chain model for the mathematical description of the process of learning a subject matter in the classroom under the restrictions raised before. For this, assuming that the above process has the Markov property, we introduce a finite Markov chain having as states the five steps of the learning process described in the previous section. This assumption is a simplification made to the real problem through our transfer to the “assumed real system”, which is not far away from the reality (see last paragraph of the section).

Denote by p_{ij} the transition probability from state S_i to S_j, for i,j=1,2,3,4,5, then the matrix A=[p_{ij}] is said to be the ***transition matrix*** of the chain.

According to the flow-diagram of the learning process shown in Figure 1 we find

$$A = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 & p_{25} \\ 0 & p_{32} & 0 & p_{34} & p_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array},$$

where we obviously have that $p_{21}+p_{23}+p_{25}=p_{32}+p_{34}+p_{35}=1$

Further let us denote by $\varphi_0,\varphi_1,\varphi_2,\ldots\ldots$ .. the successive phases of the above chain , and also denote by

$$P_i=[p_1^{(i)}p_2^{(i)}p_3^{(i)}p_4^{(i)}p_5^{(i)}]$$

the row - matrix giving the probabilities $p_j^{(i)}$ for the chain to be in each of the states S_j, j=1,2,3,4,5 in the phase φi, i=0,1,2,.... , where we obviously have that

$$\sum_{j=1}^{5} p_j^{(i)} = 1.$$

The above row-matrix is called the ***probability vector*** of the chain at phase φ_i. From the transition matrix A and the flow diagram of Figure 1 we obtain the “tree of correspondence” among the several phases of the chain and its states shown in Figure 2.

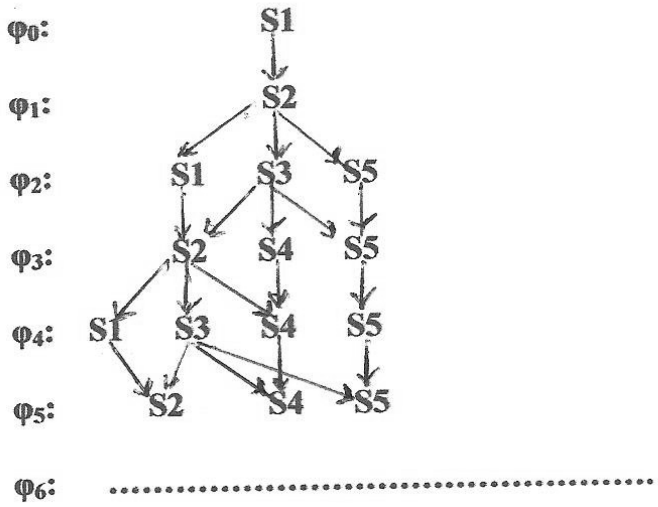

Figure 2: Tree of correspondence among states and phases of the Markov chain

From the above tree becomes evident that $P_0 = [1\ 0\ 0\ 0\ 0]$, $P_1 = [0\ 1\ 0\ 0\ 0]$, and
$P_2=[p_{21}\ 0\ p_{23}\ 0\ p_{25}]$. Further it is well known that

$$P_{i+1} = P_iA, \quad i=0,1,2,\ldots\ .$$

Therefore we find that

$$P_3 = P_2A = [0\ \ p_{21}+p_{23}p_{32}\ \ 0\ \ p_{23}p_{34}\ p_{23}p_{35}+p_{25}\ ] \quad (1),$$

$P_4=P_3A=\ldots\ldots$, and so on.

Observe now that, when the chain reaches either state S_4, or S_5, it is impossible to leave them, because the learning process finishes there. In other words S_4 and S_5 are absorbing states of the chain. Further, from Figure 1 it becomes evident that from every state it is possible to go to an absorbing state (not necessarily in one step). Thus we have an ***absorbing Markov chain***. Applying standard techniques from theory of absorbing chains we bring the transition matrix A to its ***canonical (or standard) form*** A* by listing the absorbing states first and then we make a partition of A* as follows:

$$A^* = \begin{array}{c|cccccc|} & S_4 & S_5 & & S_1 & S_2 & S_3 \\ \hline S_4 & 1 & 0 & | & 0 & 0 & 0 \\ S_5 & 0 & 1 & | & 0 & 0 & 0 \\ & - & - & | & - & - & - \\ S_1 & 0 & 0 & | & 0 & 1 & 0 \\ S_2 & 0 & p_{25} & | & p_{21} & 0 & p_{23} \\ S_3 & p_{34} & p_{35} & | & 0 & p_{32} & 0 \end{array}.$$

Symbolically we can write

$$A^* = \begin{bmatrix} I & | & 0 \\ - & | & - \\ R & | & Q \end{bmatrix},$$

where Q is the transition matrix of the non absorbing states and R the transition matrix from the non absorbing to the absorbing states.

Next we consider the ***fundamental matrix*** N of the chain, which is given by

$$N = (I_3 - Q)^{-1} = \frac{adj(I_3 - Q)}{D(I_3 - Q)} \quad ,$$

where I_3 denotes the 3X3 unitary matrix, adj(I_3-Q) denotes the adjoin matrix of I_3-Q, and D(I_3-Q) denotes the determinant of I_3-Q. A straightforward calculation gives that

$$N = \frac{1}{1 - p_{23}p_{32} - p_{21}} = \begin{bmatrix} 1 - p_{32}p_{23} & 1 & p_{23} \\ p_{21} & 1 & p_{23} \\ p_{21}p_{32} & p_{32} & 1 - p_{21} \end{bmatrix}.$$

We consider further the 3X2 matrix

$$B = NR = \frac{1}{1 - p_{23}p_{32} - p_{21}} \begin{bmatrix} p_{23}p_{34} & p_{25} + p_{23}p_{35} \\ p_{23}p_{34} & p_{25} + p_{23}p_{35} \\ (1 - p_{21})p_{34} & p_{32}p_{25} + p_{35}(1 - p_{21}) \end{bmatrix}.$$

Symbolically we can write B=[b_{ij}], with i=1,2,3 and j=4,5. It is well known then that b_{ij} gives the probability that, starting at state S_i, the process is absorbed at state S_j. Thus the probability for a learner to pass successfully through all the states of the learning process in the classroom is given by

$$b_{14}=\frac{p_{23}p_{34}}{1-p_{23}p_{32}-p_{21}} \quad (2).$$

The calculation of b_{14} enables the teacher to check the efficiency of his (her) lectures. It also could be used either as a measure of comparison of the efficiencies of the lectures of different teachers, or as a measure of the learning abilities of different groups of students.

We must finally notice that there is always the possibility of existing memory of previous states in the movements from state S_2 back to S_1 and from S_3 to S_2. However, as it is emphasized by certain authors (e.g. Kemeny, Schleifer, Snell & Thompson, 1964; Chapter IV, paragraph 12, p. 193), for possessing the Markov property the probability of an outcome is not necessarily independent of the outcomes of previous states, but depends at most upon the outcome of the previous one. This makes our assumption (simplification) that the learning process in the classroom has the Markov property to be near to the reality.

A classroom experiment for learning mathematics

The following experiment for learning mathematics took place recently at the Graduate Technological Educational Institute of Patras (Greece), when I was teaching to a group of 30 students of the School of Technological Applications (i.e. to future engineers) being in their first semester of studies the use of the derivative for the maximization and minimization of a function. During my two-hour lectures I used the method of rediscovery (Voskoglou, 1997). Thus, after a short introduction to the subject, I left my students to work alone on their papers. I was inspecting their works, and from time to time I was giving them some instructions, or hints. After the basic theoretical conclusions I gave them some exercises to solve first, and at the final step some problems including applications to constructions and economics (see appendix).

During the experiment I found that four students didn't understand the new topic at all. In fact, these students had not acquired the proper mathematical background from school (as it frequently happens with some students of the T.E.I.'s) and they didn't attend my previous lectures on a regular basis. As a result, they didn't consolidate notions like the local min-max values of a function, the decreasing and increasing function and even the derivative (!) of a function and therefore were completely unable to follow my instructions in order to approach the basic ideas of the new topic. This became evident to me by checking their efforts on the paper and by asking them relevant questions. Notice that, due to the limited time available to cover my first semester course in mathematics, I have seldom the opportunity to apply the method of rediscovery in my lectures (only in cases of particular interest). On the contrary, sometimes I am using even the monologue, assuming that students know the basics from school. This is actually one of the basic problems of teaching mathematics at the T.E.I.'s (Voskoglou, 2009)..

Coming back to the experiment, I also found that 10 students had difficulties before understanding the basic ideas; they looked back to their notes of my previous lectures and/or asked for help. This is of course a natural behaviour of someone who is trying to understand a new topic, but it means that these students faced the need to reconsider and to analyze better the already existing knowledge in order to reach their target.

Further I found that five students, although it seemed that they understood the basic theoretical ideas, were unable to apply them for the solution of the given exercises and problems. The remaining 21 students solved the exercises, but eight of them faced difficulties before they came through. At the last step 10 students solved the problems and 11 didn't solve them (or solved a small part of them). The solution of the exercises was an indication for me that the corresponding students became able to generalize the new knowledge to a variety of situations, while the solution of the problems meant that the corresponding students were able to relate the new information to their existing structures of knowledge (categorization). The above assumptions could of course be contested as being too simplistic, but I do believe that they give a satisfactory first approximation of students' behaviour during the learning process. A further qualitative analysis is probably necessary to obtain more detailed conclusions and a supporting simulation model could help towards this direction.

Interpreting the above data with respect to the flow-diagram of Figure 1 I was led to the following conclusions, which are represented in Figure 3.

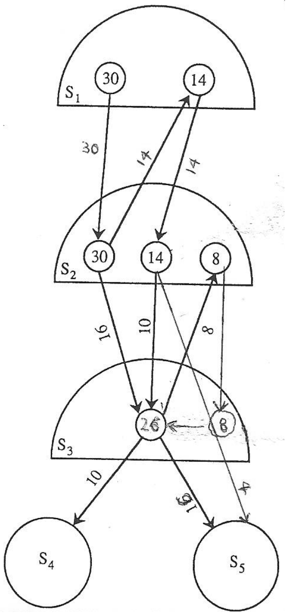

Figure 3: Representation of the experiment's data

- Initially the 30 students proceeded from S_1 to S_2, but 14 of them faced difficulties to interpret the input data. Therefore they returned to S_1 to search for more information that will facilitate the interpretation procedure, wherefrom they came back to S_2. Finally four of them reached directly the absorbing state S_5, because they didn't manage to interpret the new knowledge.

- The remaining 26 students proceeded to S_3, but eight of them faced difficulties to generalize the new knowledge to a variety of situations, ad they returned to S_2 for a better understanding of the new information. Then they came back to S_3.
- At the last step 10 students, who solved the exercises and the problems, completed successfully the learning process in the classroom and therefore they reached the absorbing state S_4. The other 16 students, i.e. five students who didn't manage to solve the exercises and the problems, and 11 who solved the exercises, but not the problems, reached the absorbing state S_5.

Therefore, since we had a total of 52 'arrivals' to S_2, 14 'departures' from S_2 to S_1, 34 'departures' from S_2 to S_3, and four 'departures' from S_2 to S_5, it follows that $p_{21}=\frac{14}{52}$, $p_{23}=\frac{34}{52}$, and $p_{25}=\frac{4}{52}$. In the same way one finds that $p_{32}=\frac{8}{34}$, $p_{34}=\frac{10}{34}$, and $p_{35}=\frac{16}{34}$.

Replacing the values of the p_{ij}'s in equalities (1) and (2) of the previous section we get that $P_3=[0 \ \frac{22}{52} \ 0 \ \frac{10}{52} \ \frac{20}{52}]$ and $b_{14}= \frac{1}{3}$. Interpreting these data with respect to our model we find that the probabilities for a student to be in phase φ_3 of the process of learning in the classroom (i.e. 3 phases after its start) at the steps of representation, interpretation, generalization, categorization, or failure to reach categorization are approximately 0, 42,31%, 0, 19,23% and 38,46% respectively, while the probability to pass successfully through all the steps of the process is approximately 33,33%. Notice that this is an experiment only, performed in order to illustrate the applicability of our model in practice. In order to obtain safer statistical conclusions, one must perform analogous experiments several times, with different student populations, different teachers and different teaching\ conditions. This is proposed as a subject of further applied research.

Remarks and further examples

Most real world problems concerning applications of finite Markov chains can be solved by distinguishing between two types of such chains, the absorbing (e.g. our model in the present paper) and the ergodic ones (Voskoglou, 2006; section 3).

We recall that a Markov chain is said to be an ***ergodic chain***, if it is possible to go between any two states, not necessarily in one step. In this case the corresponding theory enables us to make, apart from the short run forecasts, i.e. calculation of the probabilities for the chain to be in each of its states at a certain phase of the process (as we have done in the classroom experiment of the previous section), and long run forecasts (when the chain reaches its equilibrium situation, as the number of its successive phases tends to infinity) for the evolution of various phenomena. For example, in Voskoglou (1996) an ergodic chain is introduced for the study of the analogical problem-solving process in the classroom, while in Voskoglou and Perdikaris (1991) the problem-solving process (in general) is described through the introduction of an absorbing Markov chain to the main steps of the process (Schoenfeld, 1980; expert performance model)

Further, in Voskoglou (1994) an absorbing Markov chain is introduced to the major steps through which one would proceed in order to effect the study of a real system (modelling process). The stochastic model obtained gives an important theoretical framework for the study of the modelling process. An alternative form of the above model is introduced in Voskoglou (2007) for the description of the mathematical modelling process in the classroom, when the teacher gives such kind of problems for solution to students. In this case it is assumed that after the completion of the solution process of each problem a new problem is given from the teacher to the class and therefore the process starts from the beginning again. Thus the resulting Markov chain is an ergodic one.

In Voskoglou (2000) an absorbing Markov chain is introduced to the main steps of the decision making process performed in order to choose the best among the existing solutions of a given problem, and examples are presented to illustrate the applicability of the model to “real” decision making situations.

We could mention many other known applications of Markov chains for the solution of real world problems in almost every sector of the human activity, but this is rather out of the scope of the present paper.

Some mathematicians, who studied this paper, suggested the introduction to the model of an initial vector (input) of the form [a_{01}, a_{02}, a_{03}, a_{04}, a_{05}] to proceed with matrix A. If we suppose that the initial vector is the people [30, 0, 0, 0, 0], then one should introduce the parameter t of time to re-enter people in this stage. At any case the parameter of time has already tacitly inserted to the model, since we considered the successive phases of the chain. In fact, the chain moves from each face to the next one through time.

They also suggested that the problem examined in the paper could be faced as a flow network from S_1 to S_5. In this case one has to define the initial number of people (30) and assign in each line between two stages the probability of success as line cost. This looks as a very good idea for further future research.

Another interesting approach of the problem is the use of fuzzy logic, to represent the steps of the process of learning as fuzzy sets in the universal set of linguistic labels of a=negligible, b=low, c=intermediate, d=high, and e=complete success respectively of the learner in each step (Voskoglou, 2008). Analogous efforts to use the fuzzy sets logic in the area of student modelling and student diagnosis in particular and in education in general have been attempted by other researchers as well, e.g. Perikaris (1996), Espin and Oliveras (1997), Ma and Zhou (2000), Spagnolo and Gras (2004) etc.

Final conclusions

The theory of Markov chains is a successful combination of Linear

Algebra and Probability, which enables one to make short and long run forecasts for the evolution of various phenomena of the real world.

In the present paper we built, through the introduction of a finite, absorbing Markov chain to the major steps of the process of learning, a stochastic model for the description of the process of learning a subject matter by a group of students in the classroom. Thus, by applying standard results of the corresponding mathematical theory, we succeeded to calculate the probabilities for a student to be at any of the major steps of the learning process in each of its phases in the classroom, as well as the probability to pass successfully through all the steps of the learning process in the classroom.

These outcomes can help the teacher to check the efficiency of his (her) lectures and the learning abilities of different student groups, or of the same group on different subjects. In this way he (she) could be suitably orientated to change, or adapt better his (her) teaching plans and methods.

Our results are illustrated by a classroom experiment for learning mathematics, performed at the School of Technological Applications of the Graduate Technological Educational Institute of Patras, in Greece.

APPENDIX: List of the exercises and problems given for solution in the classroom experiment

A) EXERCISES

Find the min-max values of the following functions:

a) $f(x)=x^2-4x+3$, b) $g(x)=x^3+2x^2+x+7$, c) $f(x)=x^a e^{-x}$, $x>0$, $a\in IR$.

B) PROBLEMS

1) The profit from the sale of a good is given by $K(q)=5q^2+30$, where q is the quantity of the good, while the price of its sale is $P(q)=7q+10$. Which must be the daily production of the good in order to achieve the minimal production cost?

2) Among all the cylindrical buildings having a total surface of 180π m^2,

which one has the maximal volume?

3) The cost of the fuel for the motion of a train is analogous to the square of its speed and it is equal to 250 euros per hour for a constant speed of 25 km per hour. The rest of the expenses for the motion of the train are 100 euros per hour regardless to its speed. Find the speed of the train for which we have the minimal cost per km for its motion.

4) The wall of a building has to be supported by a beam, which must pass over a parallel wall (touching it), whose height is 10 m. The distance of the parallel wall from the building is 8 m. Find the minimal length of the beam, which can be used for this purpose.

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