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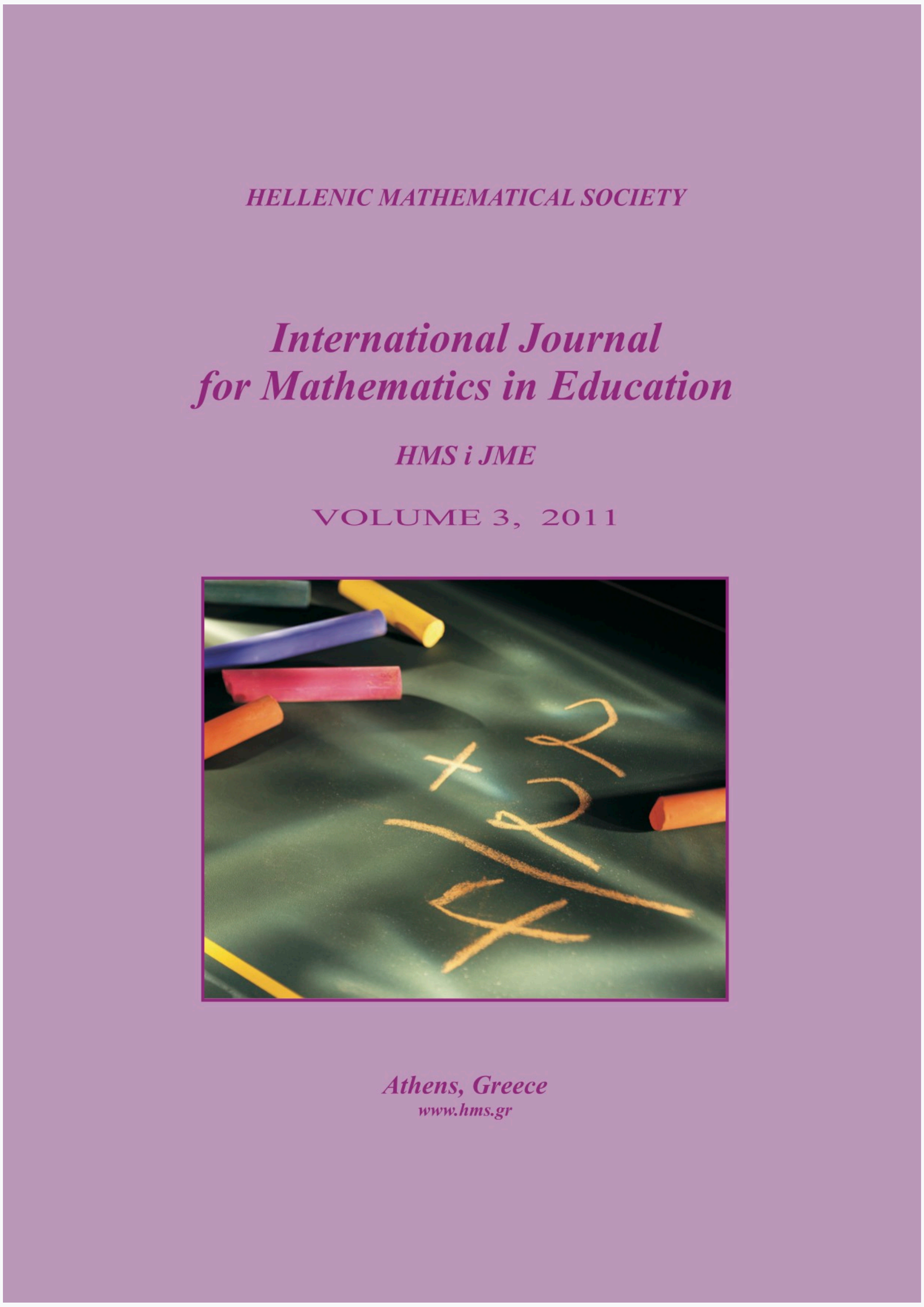

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Hellenic Mathematical Society
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Introduction

The third volume of the HMS International Journal for Mathematics in Education includes five research papers.

The first paper written by Ioannis Karantzis **“***Mental arithmetic calculation in the addition and subtraction of two-digit numbers: The case of third and fourth grade elementary school pupils”* investigates pupils’ ability to add and subtract two digit numbers mentally as well as the effect of teaching based on the new mathematics curriculum. One hundred and twenty pupils of grade 3 and 4 took part in this research. The results showed that the pupils encounter difficulties in operations with a digit carried over, that subtraction

is more difficult than addition and finally, that both the positive effect of contemporary approaches to teaching (when the pupils had followed the new maths curriculum from the first grade), and the negative influence of the teaching of written operations on pupils’ spontaneous mental strategies, are clear.

The second paper written by Michael Gr. Voskoglou *“Mathematical models for the problem-solving process”* corresponds to each of the main stages of the problem-solving process a fuzzy subset U of the set of the linguistic labels of negligible, low intermediate, high and complete success respectively by students at this stage and uses the total possibilistic uncertainty as a measure of students’ problem-solving capacities. Two classroom experiments are presented, illustrating the use of the results in practice.

The purpose of the third paper written by Andreas Moutsios-Rentzos & Adrian Simpson *“University mathematics students and exam-style proving questions: The A-B-Δ strategy classification scheme”* presents a classification scheme for the strategies that university mathematics students employ when they deal with exam-style proving questions, which the authors call the A-B-Δ classification scheme. The authors explore the conceptual validity of this construct and claim that the classification provides a framework for differentiating strategies which might otherwise be conflated. Empirical data supporting this claim are presented.

The fourth paper written by Charoula Stathopoulou *“THE USE OF DIDACTICAL GAME ON MATHEMATICS TEACHING: Young children construct mathematical notions playing naval-battle”* presents the results of a qualitative research that aimed to investigate whether primary school children while playing the game are able to conquer mathematical meanings that in the formal education are taught in higher level. Four children (3 girls about 6 years old, 1 boy about 8) collaborated playing the game of naval battle in an out of school milieu. The young children constructed, through their involvement in this game, mathematical notions such as: ordered pair, Cartesian system of axes, coordinates, and negative numbers.

Finally, E. Stefos, E. Athanasiadis, V. Gialamas and C. Tsolakidis by their article
"The Use of New Technologies and the Project Method in Teaching Statistics: A Case Study in Higher Education" show the importance of the use of New Technologies and the Project Method in the teaching of Statistics. The research was conducted in the Department of Primary Education in the University of the Aegean, Greece, for which the questionnaire SATS of Candace Schau was used. The New Technologies that were used involved the construction and the use of a website and a blog, the use of Projects, teaching in a computer laboratory with statistical software and applications of internet (freeware, applets, etc). The results of the research indicated that by using New Technologies and Project Methods in teaching Statistics the attitudes of students towards Statistics improved.

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Mental arithmetic calculation in the addition and subtraction of two-digit numbers: The case of third and fourth grade elementary school pupils

Ioannis Karantzis

Abstract

The objective of the present study is to investigate pupils' ability to add and subtract two digit numbers mentally, to observe the strategies they use, to investigate the effect of teaching based on the new mathematics curriculum and to ascertain whether there is any notable difference in the performance between pupils in different year groups. One hundred and twenty pupils of grade 3 and 4 took part in this research. The results showed that the pupils encounter difficulties in operations with a digit carried over, that subtraction is more difficult than addition and finally, that both the positive effect of contemporary approaches to teaching (when the pupils had followed the new maths curriculum from the first grade), and the negative influence of the teaching of written operations on pupils' spontaneous mental strategies, are clear.

1. Introduction

Mental calculation is a process in which a numerical calculation can be made quickly and accurately without the aid of external means, for example, manipulatives, pencil and paper, etc, and be done in a conscious way, using some strategy (Maclellan, 2001, 145-146).

According to the international literature, mental calculation holds an important place in the teaching and learning of mathematics. In particular, it develops problem solving skills, provides opportunities for making calculated estimates and contributes to the understanding of the concept of number (McIntosh et al., 1995; Threlfall, 2002). In addition, it encourages children to manipulate numbers with ease, it is the foundation for the development of calculating skills and it aids the understanding and development of written methods of calculation (Varol and Farran, 2007). Finally, Heirdsfield and Cooper (2004) point out that mental calculation helps children understand how numbers ‘work’, and what their structure is, as well as helping them to discover strategies for problem solving while developing skills for making hypothesis and generalizations.

It is known that mental calculations involve a wide range of strategies, unlike traditional problem solving techniques. According to the international bibliography, extensive research has shown the variety of strategies that pupils use when they perform mental calculations involving two-digit numbers, for addition and subtraction (Maclellan, 2001; Threlfall, 2002; Lucangeli et al., 2003; Macintyre and Forrester, 2003; Heirdsfield and Cooper, 2004; Heirdsfield and Lamb, 2005; Lemonidis, 2003; Karantzis & Tollou, 2009).

From all of this research, the strategies which pupils employ in mental calculations for the addition and subtraction of two-digit numbers have been identified and codified, and are presented below:

1.1 Strategies for addition and subtraction

1) *Instant recall (AUTO)*: instant recall of result.

2) *Separation of tens and units or 1010*: With this strategy, first the tens and then the units are calculated (e.g. 54+25: 50+20=70, 4+5=9, 70+9=79, and 68-26: 60-20=40, 8-6=2, 40+2=42), or first the units and then the tens, known as *u-1010* (Heirdsfield and Lamb, 2005) (e.g. 54+25: 4+5=9, 50+20=70, 70+9=59 and 68-26: 8-6=2, 60-20=40, 40+2=42). This category also includes the case which, according to Varol and Farran (2007) is called *10s*, where, for addition, the units of the first addend are added to the sum of the tens, and then the units of the second addend (e.g. 54+25: 50+20=70, 70+4=74, 74+5=79), while for subtraction without carry digit the numbers are analysed in their tens, and then, from the difference between the tens,

the units of the subtrahend are subtracted, and then the units of the minuend are added, (e.g. 68-26: 60-20=40, 40-6=34, 34+8=42). Finally, in subtraction with a carry digit*,* the tens are dealt with first and then the units of the minuend are added to their difference, and from the result, the units of the subtrahend are subtracted (e.g. 84-59: 80-50=30, 30+4=34, 34-9=25). It is worth mentioning that the strategy 1010,u-1010 is characterised as a 'separation' strategy (Heirdsfield and Lamb, 2005).

3) *Calculation based on the first number, or N10*: For addition using this strategy, first the units and then the tens of the second *addend* are added to the first (e.g. 54+25: 54+5=59, 59+20=79), strategy *u-N10* (Heirdsfield and Lamb, 2005), or first the tens and then the units (e.g. 54+25: 54+20=74, 74+5=79). In addition, Lucangeli et al., (2003), separate the second addend in another way, for example: 54+25 = (54+10+10)+5=79. For subtraction, first the units and then the tens of the subtrahend are subtracted from the minuend, strategy *u-N10* (Heirdsfield and Lamb, 2005) (e.g. 68-26: 68-6=62, 62-20=42), or first the tens and then the units (e.g. 68-26: 68-20=48, 48-6=42). This strategy N10, u-N10 is characterized as an 'aggregation' strategy. (Heirdsfield and Lamb, 2005).

4) *Rounding up/rounding down, or N10C*: Here the pupil, in order to calculate the sum or the difference rounds up or down one of the two numbers (e.g. 54+25: 55+25=80, 80-1=79, and 68-26: 70-26=44, 44-2=42 or 34+25:40+19=59, and 48-26: 50-28 = 22). This is known as a 'holistic' strategy.

5) *Mental counting from a specific point (counting on, CON)*: Here, for addition, the units of the second addend are added to the first, one at a time (e.g. 54+25: 54+1+1…= 79) or the tens of the second addend ten by ten, followed by the units (e.g. 54+25: 54+10=64, 64+10=74, 74+5=79). For subtraction, the units of the subtrahend are subtracted one at a time from the minuend (e.g. 68-26: 68-1-1…= 42), or first the tens are subtracted ten by ten and then the units (e.g. 68-26: 68-10=58, 58-10=48, 48-6=42).

6) *Counting:* The counting starts with the units of the one number and continues with the units of the other. The units of the second addend may also be

added to the first addend one at a time, using either the fingers, or sometimes a rhythmic movement of the head (Counting on fingers, COF). Similarly, in subtraction, the counting begins with the objects that the minuend expresses, and then the objects that the subtrahend expresses are subtracted, and finally whatever is left is counted. The units of the subtrahend may also be subtracted from the minuend one at a time, using either the fingers, or sometimes a rhythmic movement of the head. (Counting on fingers, COF).

7) *Strategy C10*: In order to make addition or subtraction easier, rounding up to 10 is used. This strategy is characterized by Varol and Farran (2007) as strategy A10 (e.g. 54+25: 54+6=60, {25-6=19}, 60+19=79 and 74-69: 74-4=70 {69-4=65}, 70-65=5).

8) *Mental-traditional algorithm (MA)*: The traditional algorithm is performed mentally following the algorithm of the vertical addition or subtraction (e.g. 54+25: 5+4=9, 5+2=7,. Sum: 79, and 36-25: 5 from 6 leaves 1, 2 from 3 leaves 1, so the answer is 11).

9) *Completion of the subtrahend*: In this strategy the pupil increases the units of the subtrahend until he reaches the minuend. The number that represents the increase is the correct answer. This strategy is divided into two categories. In the first: when the subtrahend is a small distance from the minuend the pupil adds the units one by one (or all together) to the subtrahend until he reaches the minuend. (e.g. 72-69: 69+3=72, so the correct answer is 3). On the other hand, when the subtrahend is a greater distance from the minuend then the answer is found in stages (making up the tens first, etc). (e.g. 73-36: 36+4=40, 40+30=70, 70+3=73, 4+30+3=37).

10) *Addition of tens until the number is surpassed*: (Van de Walle, 2005). (e.g. 73-46: 46+30=76, 76-3=73, 30-3=27) or *adding on to the minuend* (e.g. 73-46:73+3=76, 76-46=30, 30-3=27).

In conclusion, counting, counting on fingers and mental traditional algorithms are characterized as low level strategies since they are not effective and do not promote mathematical thought in children. On the other hand, high level strategies which contribute to a better understanding of numbers are the following: separation of tens-units (1010), calculation based on the first term (N10), rounding up (N10c), C10, completion of the subtrahend, addition of tens until the amount is surpassed, adding on to the minuend. These strategies are more effective and 'have a tendency to change according to the numbers involved, to make the calculation easier' (Van de Walle, 2007). For example, to calculate the sum '43+26' strategy 1010 may be used quickly and easily, while to calculate the difference '54-18' strategy N10 may be preferred (Blote et al., 2000).

Beliefs relating to the importance of mental calculations in everyday life are supported internationally by many educationalists who recommend the introduction of the systematic teaching of mental arithmetic into the elementary school curriculum. Such efforts were the suggestion of the National Council of Teachers of Mathematics (NCTM) in America (Van de Walle, 2007), and the National Numeracy Strategy which was adopted in England in 1995 (Maclellan, 2001; Macintyre and Forrester, 2003; Lemonidis, 2008).

In Greece, according to the new curriculum for mathematics in elementary school which was introduced in September 2005 with the writing of new books, mental arithmetic calculations occupy an important place in the teaching and learning of mathematics. This is in contrast to the old programme, which did not place enough emphasis on the children's ability to perform operations in their head. So, according to the new curriculum, it is requested that each teacher helps the pupils develop skills which contribute to the adoption of suitable strategies for the mental calculation of numbers, unlike the approach in the old maths books.

From this position, in the research of Karantzis & Tollou (2009), an attempt was made to investigate which strategies are used and to what extent, by third-year elementary school pupils, when they calculate mentally the sum and difference of two-digit numbers. The research took place during the second year of the application of the maths curriculum (November-December 2007, when the pupils of the 3rd grade had followed the new maths curriculum since the 2nd grade) and 90 pupils took part from four (4)

Elementary Schools in the city of Patras. The test criteria consisted of 16 operations, of which 8 were addition and the remaining 8 subtraction. Of the 8 additions, half (4) were operations without a number carried over, while the other half (4), were with a number carried over. The same was true for the subtractions. The operations were given in the following order for all pupils: (Addition: 57+12, 85+14, 68+28, 36+15, 58+31, 27+14, 32+42, 49+33, Subtraction: 87-44, 69-56, 95-19, 51-26, 96-33, 83-48, 78-25, 64-37). The results showed that the children answered correctly to a satisfactory degree (86.11% for addition and 60.3% for subtraction). In addition, a fairly significant percentage (39.6% for addition and 33.07% for subtraction) used low level strategies, mainly the algorithm and the vertical operation, while others (46.25 for addition and 27.23 for subtraction) used high level strategies, mainly strategies 1010 and N10.

1.2 The purpose of study

As we are now in the fourth year of the implementation of the new curriculum for pupils in Greek primary schools, we wanted to re-examine the hypothesis of the aforementioned research, and to extend its scope using fourth year pupils. From this study we may perhaps draw some conclusions about the effect contemporary methods of teaching mathematics will have in the area of mental calculation. In particular, the objective of the present study is to investigate pupils' ability to add and subtract two digit numbers mentally, to observe the strategies they use, to investigate the effect of teaching based on the new mathematics curriculum (when the pupils had followed the new maths curriculum from the first grade) and to ascertain whether there is any notable difference in the performance between pupils in different year groups. With all this in mind, we designed and carried out the present study.

2. Method

2.1 Participants

The choice of the sample for the research came from six (6) Elementary Schools in the city of Patras. The research was conducted during the current school year (November-December 2009) and pupils of grade 3 and 4 of the above mentioned schools participated. In particular, 60 pupils of grade 3 (31 boys and 29 girls) and 62 pupils of grade 4 (31 boys and 31 girls) took part. The schools and the pupils from those schools that participated in the research were selected using the method of simple random sampling (Cohen and Manion, 1994).

2.2 Materials

The test criteria consisted of 16 operations, of which 8 were addition and the remaining 8 subtraction. Of the 8 additions, half (4) were operations without a number carried over, while the other half (4), were with a number carried over. The same was true for the subtractions. Our aim was not to give the children too many tasks, so as not to tire them as this would prove an obstacle to the success of the research. The operations were given in the following order for all pupils: (Addition: 57+12, 85+14, 68+28, 36+15, 58+31, 27+14, 32+42, 49+33, Subtraction: 87-44, 69-56, 95-19, 51-26, 96-33, 83-48, 78-25, 64-37).

The children were given cards on which the additions and subtractions were presented horizontally, as in this way it is less likely that the pupils will be encouraged to use the traditional algorithms of the operations (Van de Walle, 2007).

2.3 Procedure

The participants were tested individually and the tester, having asked each pupil orally, recorded their oral answers. The test lasted about 20 minutes for each pupil and took place in a quiet part of the school, away from the classroom, so that the children could express their thoughts freely and without anxiety. In the beginning, before the procedure started and to create a pleasant, stress-free environment, the pupil was asked about his lessons, his family and so on, and the whole procedure took the form of a game. The tester referred to the conditions, in accordance with which the pupil should calculate mentally the sum of, or the difference between the numbers, explaining his reasoning each time. Before the start of the test proper, there were a few trial runs, until each subject was judged capable of following the test procedure. In other words, the pupil was given two cards with simple additions. He found the answer and then reported which strategy he had used to find the solution. Later (when the additions were finished with), he was given two other cards with simple subtractions and again reported his reasoning.

When the actual test began, the pupil looked at the cards one by one in front of him and used mental calculation to find the answers to the operations. The tester recorded his performance and asked him to justify his answer. In addition, the tester listened carefully to the pupil and from his answers, but also from his behaviour generally while he was performing the mental calculations (observing for example if the pupil used his fingers in order to calculate), recorded the strategy used. Finally, the performance of each pupil in addition and subtraction was calculated for each strategy like this: (number of correct answers/8)* 100. It should be pointed out that in the present study we focus only on the correct answers. The analysis of the incorrect answers will be presented in another study.

Concluding the section on methodology, it should be pointed out that in the present study, as was mentioned above, we only evaluated the strategies that produced correct answers from the pupils. Perhaps, in a future investigation it would be interesting to consider the strategies used in cases where pupils gave a wrong answer, and search for explanations as to why the pupils, despite having chosen a suitable strategy, were still unable to produce the correct answer.

3. Results - discussion

The statistical analysis of the data was accomplished with the aid of the statistical programme SPSS v. 15. In particular, means and standard deviations were calculated for students' performance per class, operation, strategy and the carrying (or not) of a digit. A mixed analysis of variance model was estimated with one between – group factor (class) and two within – group factors (operation and carrying digit), ANOVA 2X2X2 (Snodgrass, 1977).

All factors had two levels. In addition, t-test was applied, for the comparison of the variables "low or high level strategies" within each class, because our sample was random and of a relatively satisfactory size and our variables are evenly spaced. (Snodgrass, 1977).

The mean results of the participants' performances from both classes (Grades 3 and 4) in the mental calculation of addition and subtraction, with or without a carrying digit, regardless of the strategy used, are presented in Table 1, while in Table 3 their performance in each strategy for mental calculation, is presented. In particular, in Table 1 mean (%) and standard deviation of pupils' performance in addition and subtraction with and without carrying digit are presented [e.g. in each case (number of correct answers/4) with and without carrying digit operations)* 100) was calculated]. In Table 3 presented mean (%) and standard deviation of pupils' performance in each strategy of mental calculation for addition and subtraction (e.g., if a student gave the correct sum for 6 additions, applying twice strategy 1010, three times the MA and once the N10 strategy, then this student's score is respectively 25, 37.5, 12.5). In the strategy "Counting on fingers and Mental algorithm (COF+MA)", the pupils use of the two strategies in the same operation should be pointed out.

From a first examination of the results of the present study, it emerges that the pupil, regardless of year group or strategy used, display quite a satisfactory performance in the mental calculation of addition and subtraction with two digit numbers. However, as expected, their performance in addition shows a statistically significant difference to their performance in subtraction, Anova $F(1,120)=94.27$, $p<0.000$. In other words, it seems that pupils encounter more difficulties when dealing with the mental calculation of subtraction, than when dealing with addition.

From these findings we can conclude that addition is easier for the pupils than subtraction. As we know, these operations are 'logical' and their formation is based on certain rules. One of these is the rule of 'inversion', according to which, at any moment a logical operation can be mentally processed in the opposite direction and be cancelled out (Cole and Cole, 2001). In this way, subtraction is considered the opposite-reverse operation of addition (Matthews, 1981), and during mental calculation presents a complexity that creates difficulties, especially for younger pupils (Cebulski & Bucher, 1986; Macintyre & Forrester, 2003; Lemonidis, 2003; Lemonidis & Lygouras, 2008).

Attempting a comparison within each operation and regardless of the year group of the pupils, we notice, as expected, that their performance in mental calculation without a carrying digit display a statistically significant difference to their performance in operations with a carrying digit, Anova ($F(1,120)=103.02$, $p<0.000$). In other words, operations with a carrying digit create significantly greater difficulties for pupils in both year groups. In parallel, however, a statistically significant interaction between the factors 'operation' and 'carrying digit', Anova ($F(1,120)=63.74$, $p<0.000$, Table 1), also emerges. The results are clearly apparent in Table 2. We observe that almost all the pupils, from the two grades, give correct answers for the two sums operations without a carrying digit and in addition with a carrying digit, for a large proportion of the operations they were given. However, for subtraction with carrying digit only about 50% of the pupils answer a large proportion of the operations they were given correctly.

These results are confirmed by the research of Wolters, Beishuizen, Broers & Knoppert (1990), Macintyre & Forrester (2203), Lemonidis (2003). These findings show that the degree of difficulty of the problem, such as the carrying digit, burdens the central executive (unit) of the working memory (Baddeley, 1995) which is responsible for the control of mental operations, the articulatory loop and other probable subsystems of the working memory that retain information temporarily until the solution to the problem is complete. Consequently, the time required to solve the problem increases, and the mistakes multiply (Hitch, 1978; Karantzis, 2004). Naturally, the pupils' flexibility in manipulating the carrying digit during the mental calculation of addition and subtraction may well improve with teaching and maturity. We are led to this conclusion by the statistically significant interaction which was observed between the factors ‘operation’ and ‘carrying digit’ (regardless of the type of operation), Anova $F(1,120)=5.43$, $p<0.02$. In other words, it seems that the performance of grade 4 pupils in the mental calculation of operations with addition and subtraction with a carrying digit is significantly better than that of grade 3 pupils (Table 1).

Another finding of the present study is that, in order to process maths problems mentally, pupils used strategies that they believed would help them to solve the problems successfully. This finding is also supported by the findings of previous researchers (Beishuitzen, 1993; Lemonidis, 2003; Lemonidis & Lygouras, 2008), who observed that the pupils developed skills to interchange their strategies according to the numerical data of each mental exercise. The participants in our research relied for the most part on traditional mental algorithm (MA) for both operations, when they chose low level strategies, and on strategy 1010 and strategy N10 when they used high level strategies.

The results are clearly apparent in Table 4. We observe that approximately 25% of grade 3 pupils and 40% of grade 4 pupils use the strategy traditional mental algorithm (MA), answering more than about half of the operations they given correctly. Most pupils from two grades prefer strategy 1010, with success in more than half of the operations they were given. Finally, fewer pupils from both grades use strategy N10, but more successfully (about 70-75%), in the operations they were given. The greater success, though of fewer pupils, in strategy N10, as compared with strategy 1010 can be explained by the fact that strategy 1010 is a strategy with a number of steps (compared with strategy N10) and as a result, as the number of steps required to solve a problem increases, so the number of mistakes made in the process of solving the problem in creases (Cebulski & Bucher, 1986). Finally, commenting on the fact that in strategy 1010 the performance of the pupils in both grades drops noticeably in operations involving subtraction, this may be due to the fact that this particular strategy causes difficulties for the pupils in subtraction with a carrying digit and leads them to make mistakes such as, for example: 8-4=4 instead of 14-8=6 during the mental calculation of the subtraction 54-18 (Macintyre & Forrester, 2003; Blote et al., 2000; Beishuizen & Anghileri, 1998).

These three strategies, as also emerged from the research of Heirdsfield & Cooper (2004), Askew & Brown (2006) and Varol and Farran (2007), are the strategies most commonly used by children when they perform the mental calculation of addition and subtraction of two digit numbers. It is worth mentioning that pupils in grade 3, as much as pupils in grade 4, choose high level strategies more than low level strategies, and this difference is statistically significant [for grade 3, $t(59)=6.94$, $p<0.000$ for addition and $t(59)=5.11$, $p<0.000$ for subtraction, and for grade 4, $t(61)=4.12$, $p<0.000$ for addition and $t(61)=3.21$, $p<0.002$ for subtraction], in other words, they choose strategies which make an important contribution to the consolidation of the relationships which exist between numbers and which promote mathematical thought.

This finding of the present study (concerning the performance of grade 3 pupils) stands in contrast to the results of the research of Karantzis & Tollou (2009). In this study, it was found that the strategy of traditional mental algorithm (MA) was used by pupils to a greater degree than strategies 1010 and N10. As we mentioned before, it followed the same methodology and used the same materials with the pupils, and took place in the same area (the city of Patras). The only difference was that this research took place in the second year of the application of the new mathematics curriculum (the pupils of the 3rd grade had been following the new maths curriculum since 2nd grade).

Bearing this in mind, a comparison of the results of the two studies would be appropriate. So, the results of the present study permit us, perhaps, to claim that pupils of grade 3 who followed from Year 1 the new maths curriculum, improved their performance quantitatively, but also qualitatively as far as their strategies are concerned, which suggests that the improvement in quality may be a result of a better understanding of the value of the position of the digits in a number. This proposition is supported by results from diachronic research which took place in Greece (Lemonidis, 2003) from which it appeared that the teaching of mathematics that is based on new approaches to the teaching and learning of mathematics, approaches which have been adopted by the new mathematics curriculum in primary schools, made a significant contribution to the quantitative and qualitative improvement in pupils' performance in mental arithmetic calculation. It is worth mentioning however that these conclusions need to the verified in the future with further diachronic research.

The statistical analysis of the data showed that between the year groups and regardless of the type of operation, there doesn't seem to be a statistically significant difference [Anova $F(1,120)=2.65$, $p=0.106$] but especially with low lever strategies [$t(120)=1,70$, $p<0.09$)] and high lever strategies [$t(120)=0,74$, $p<0.46$]. This result poses an interesting question – why wasn't there a statistically significant improvement in the performance of grade 4 pupils as compared with that of grade 3 pupils, in the mental calculation of addition and subtraction of two digit numbers. One explanation could be that in grade 2, and at the beginning of grade 3, according to the new curriculum, pupils spend more time on mental calculation and consequently in grade 3 develop a greater degree of flexibility in the area of operations. For the carrying digit of grade3 and in grade 4 the pupils spend more time on the execution of those operations with larger numbers and with the application of strategies based on the algorithm of vertical operations. As a result grade 4 pupils don't spend as much time on mental calculations and consequently that performance doesn't show a significant improvement from that of grade 3 pupils.

Another reason could be that grade 4 pupils make use of what is for them the 'easy' algorithm of vertical addition and subtraction. This could actually be the case since in the present study we observe an increase in grade 4 pupils' performance in mental calculation using the strategy of vertical addition and subtraction and at the same time a reduction in their performance of subtraction with strategy 1010 (Table 4). This result, according to the research of Cooper et al., (1996), Heirdsfield & Cooper (1996), Lemonidis (2003), could be due to the obvious effect of the teaching of written operations on the children's spontaneous mental strategies. More specifically, prior to teaching, the children presented a variety of effective mental strategies, while after teaching they have a tendency to use a mental strategy, which seems to reflect the written algorithm taught by the teacher. However, further research with a larger number of pupils may provide more certain results and perhaps reveal other factors witch could explain the results of the present study.

4. Conclusions – Implications for Teaching

The main research conclusions from the present study with reference to the performance of grade 3 and 4 primary school pupils in the mental calculation of the addition and subtraction of two digit numbers are the following:

As expected, it was found that both grade 3 and grade 4 primary school pupils performed significantly better on mental addition than on subtraction generally. Also, with reference to the pupils' performance in addition and subtraction with and without a carrying digit, it emerged that operations with a carrying digit present significantly more problems to the pupils than those without, which results in a significant increase in mistakes, which is perhaps due to factors concerning the processing and retention of information in the working memory. Naturally, the pupils' flexibility in manipulating the carrying digit during the mental calculation of addition and subtraction may well improve with teaching and maturity, a fact which is apparent in the significantly superior performances grade 4 pupils as against those in grade 3.

Finally, the participants in the research used, for both kinds of operation, strategies 1010 and N10 much more than other strategies when they used high level strategies, as well as the mental strategy which is based on the traditional algorithm (the vertical operation), when they used low level strategies. It seems that most pupils of both grades prefer strategy 1010 and successfully solved more than half of the operations they were given, using it. Finally, fewer pupils for both grades use strategy N10, but with greater success (approximately 70-75%) in the operations they were given.

Studying the results of the present research, which took place four years after the implementation of the new mathematics curriculum in primary schools, and comparing them with the results of similar research by Karantzis & Tollou, (2009), which took place just two years after the implementation of the new curriculum (the pupils of the 3rd grade had followed the new maths curriculum since the 2nd grade), we can perhaps claim, as we mentioned before, that the quantitative and qualitative improvement of the pupils in the present study is the result of teaching which is based on the new mathematics curriculum and on the teacher himself who better understood the spirit of the new curriculum, adapting the objectives of his teaching accordingly.

Naturally, we should point out that, due to the fact that our research sample is very small, our results require further investigation with studies of diachronic nature, at least as far as the positive effect of teaching based on the principles of the new mathematics curriculum is concerned. However the results of long term research in Greece (Lemonidis, 2003) are encouraging as they show that the teaching of mathematics based on the new mathematics curriculum, which emphasizes mental calculation and actively involves pupils in the learning process with activities that promote mathematical thought, has a positive effect on their performance in mental arithmetic calculations.

Finally, the results of the present research showed that there doesn't seem to be a statistically significant difference between year groups, regardless of the type of operation (addition and subtraction) but particularly in low and high level strategies. The fact that for a large period of time in grade 3, and for the whole of grade 4, the pupils are occupied with the execution of these operations, firstly with larger numbers, and then with the application of strategies based on the algorithm of the vertical operation, mean that grade 4 pupils don't spend as much time on mental calculations and consequently that performance doesn’t improve significantly against that of grade 3 pupils. Another reason could be that the effect of the teaching of written operations on the spontaneous mental calculations of the children is clear, this result is confirmed by the research of Cooper et al., (1996), Heirdsfield & Cooper (1996), Lemonidis (2003). In conclusion, we believe that further research may help us better understand results and perhaps also reveal other factors which could explain our results.

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Table 1
Mean (%) and standard deviation (in parenthesis) of pupils' performance (N=122) in addition and subtraction with or without carrying digit

	Grade 3		*Grade 4*			
	Addition	Subtraction	Addition	Subtraction	*Mean Addition*	*Mean Subtraction*
Without carrying digit	97.50 (7.6)	92.50 (16.8)	97.98 (6.9)	92.34 (20.5)	97.74	92.42
Mean Addit. & Subtrac.	95		95.16		/////////////////	
With Carrying digit	88.75 (23.7)	43.3 (47.6)	91.53 (22.6)	62.5 (46.6)	90.14	52.90
Mean Addit. & Subtract.	66.03		77.02		//////////////////	
General Mean	93.13	67.92	94.76	77.42	/////////////////	

Table 2
Number of pupils (in parenthesis %) and number of operations (%) that they answer correctly in addition and subtraction

		Grade 3		Grade 4	
		Addition	Subtraction	Addition	Subtraction
Without carrying digit	N	60 (100)	60 (100)	62 (100)	61 (98.4)
	Nub. operat.	97.5	92.5	97.9	93.8
With Carrying digit	N	58 (96.7)	29 (48.3)	60 (96.8)	41 (66.1)
	Nub. operat.	91.8	89,7	94.6	94.5

Table 3
Mean (%) and standard deviation (in parentheses) of pupils' performance (N=122) in each strategy of mental calculation for addition and subtraction

Category of Strategies	Strategies of mental calculation	Grade 3		Grade 4	
		Addition	Subtraction	Addition	Subtraction
Low level Strategies	Counting on fingers and Mental algorithm (COF+MA)	1.25 (5.5)	0.63 (2.7)	0.40 (3.2)	0.20 (1.6)
	Mental algorithm (MA)	14.58 (30.7)	13.13 (26.5)	26.01 (39.4)	22.98 (35.2)
	Mean low level Strategies	15.83 (31.6)	13.75 (26.6)	26.41 (39.5)	23.18 (35.2)
High level Strategies	Wholistic (N10C)	0.83 (3.14)	0.00	0.81 (3.1)	0.40 (2.2)
	Strategy 1010	61.25 (44.3)	32.92 (29.6)	46.77 (42.2)	24.80 (33.5)
	Strategy (N10	15.21 (33.6)	20.63 (36.5)	20.77 (37.1	27.22 (40.6)
	Complementing the subtrahend	----	0.00	---	0.20 (1.6)
	Other Means	0.00	0.63 (3.6)	0.00	0.81 (6.4)
	Mean high level Strategies	77.29 (38.1)	54.17 (36.7)	68.35 (41.8)	53.43 (43.1)

Table 4
Number of pupils (in parenthesis %) and number of operations (%) that they answer correctly in MA, 1010, N10 strategies of mental calculation for addition and subtraction

Strategies of mental calculation		Grade 3		Grade 4	
		Addition	Subtraction	Addition	Subtraction
(MA)	N	16 (26.7)	16 (26.7)	24 (38.7)	24 (38.7)
	Numb. operat.	54.7	46.1	67.2	59.9
Strategy (1010)	N	45 (75)	41 (68.3)	44 (71)	26 (41.9)
	Numb. operat.	81.7	48.2	65.9	59.1
Strategy (N10)	N	12 (20)	17 (28.3)	18 (29)	23 (37.1)
	Numb. operat.	76	72.1	71.5	75

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Mathematical models for the problem-solving process

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Abstract

To each of the main stages of the problem-solving process we correspond a fuzzy subset U of the set of the linguistic labels of negligible, low intermediate, high and complete success respectively by students at this stage and we use the total possibilistic uncertainty as a measure of students' problem-solving capacities. Two classroom experiments are also presented, illustrating the use of our results in practice. Our fuzzy model is compared with a stochastic model for the problem-solving process introduced in earlier papers.

Keywords: Problem-solving, Markov chains, Fuzzy relations, Possibility theory, Measures of uncertainty

Introduction

Problem–Solving (P-S) is a principal component of mathematics education. In earlier papers (Voskoglou 2007a, 2008), we have examined the role of problem in learning mathematics and we have attempted a review of the evolution of research on P-S in mathematics education from the time that Polya presented his first ideas on the subject until today. Here is a rough chronology of that progress:

1950's – 1960's: Use of heuristic strategies in P-S (Polya 1945, 1954, 1963, 1962/65, etc)

1970's: Emergency of mathematics education as a self – sufficient science (research methods were almost exclusively statistical). Research on P-S was mainly based on Polya's ideas.
1980's: A framework describing the P-S process, and reasons for success or failure in P-S, e.g. see Schoenfeld (1980, 1985b), Lester, Garofalo & Kroll (1989), etc.
1990's: Models of teaching using P-S, e.g. constructivist view of learning (Voskoglou 2007c and its relevant references), Mathematical modelling and applications (Voskoglou 2006 and its references), etc.
2000's: While early work on P-S focused mainly on analyzing the P-S process and on describing the proper heuristic strategies to be used in each of its stages, more recent investigations have focused mainly on solvers' behaviour and required attributes during the P-S process; e. g. MPS Framework of Carlson and Bloom (2005), Schoenfeld's theory of goal-directed behaviour (2007), etc.

Our purpose in this paper is to develop a fuzzy model for the P-S process. We shall also attempt a comparison of the fuzzy model with a stochastic model that we have presented in earlier papers during the 1990's. For this, we shall start by sketching the stochastic model first.

2. The stochastic model

During the last two decades we have worked on creating mathematical (stochastic and fuzzy) models for the better description and understanding of several processes appearing in the areas of Education, Management and Artificial Intelligence. Among the others we have constructed a stochastic model describing the P-S process. This model is based on Schoenfeld's expert's performance model (Schoenfeld, 1980; section 5), whose stages include: *Analysis (s_1)* of the problem, *design (s_2)* of its solution, *exploration (s_3)* for identifying proper heuristic strategies that possibly could be used and *implementation- verification (s_4)* of the solution. The flow-diagram of the P-S process is shown in Figure 1, where, for technical reasons, we have added one more stage, i.e. solver's *final report (s_5)* on the solution of the problem.

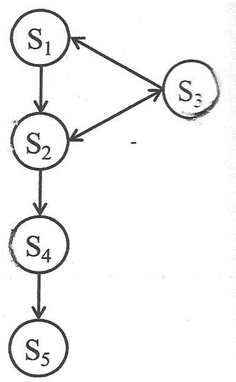

Figure 1: Flow-diagram of the P-S process

In fact, the solver facing some difficulties at the stage of design and in his/her effort to find the proper heuristic strategies for problem's solution is transferred to exploration. From there, if the difficulties are surpassed, he/she returns to design to continue the P-S process (implementation- verification- final report). Otherwise he/she returns to analysis searching for additional information from problem's data that possibly has been elapsed at first glance. We must notice that the two components of stage s_4 (implementation- verification) are presented as two independent stages in Schoenfeld's model. Their amalgamation in one stage, made in order to simplify our stochastic model, does not affect the description of the process of P-S.

In constructing the *assumed real system*, which is a basic step of abstraction in representing the real system through the model (Voskoglou 2007b; section 1), we make the conventional assumption that the P-S process satisfies the *Markov property.* This means that the above process has *"no memory"*, i.e. that its transition to one of its stages at a certain phase depends (mainly) on the stage occupied in the previous phase and not in older ones. Notice that a number of authors, in order to permit the use of Markov property in describing as many real situations as possible, and based on the notion of the assumed real system, they emphasize that the possession of this property does not exclude the existence of some memory from older phases (e.g. Kemeny et al. 1964; chapter IV, section 12, p. 193), as it really happens with the P-S process in practice.

Markov property allows the introduction of a *finite absorbing Markov chain* (cf, Kemeny& Snell 1976 where we refer freely for the relevant theory) having as states the corresponding stages of the P-S process. In this chain s_1 is always the initial state (starting point of the P-S process), while s_5 is its unique *absorbing state*, since, when chain arrives there, it is impossible to leave it (end of the P-S process). Denote by p_{ij} the *transition probability* from s_i to s_j, i,j=1,2,3,4 and form the *transition matrix* Q=[p_{ij}], among the non absorbing states of the chain. Based on the flow-diagram of Figure 1 one finds that

$$Q=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_{23} & p_{24} \\ p_{31} & p_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where we obviously have that $p_{23}+p_{24}=p_{31}+p_{32}=1$ (probability of the certain event).

Then the *fundamental matrix* N=[n_{ij}], i,j=1,2,3,4 , of the chain is given by $N = (I_4-Q)^{-1}$, where I_4 denotes the unitary 4X4 matrix. It is well known that the element n_{ij} of N gives the mean number of times in state s_j (before absorption), when the chain starts from s_t. In our case, since the chain starts always from s_1, it becomes evident that the sum $t=\sum_{j=1}^{4} n_{1j}$ gives the mean number of steps (phases) of the chain before absorption. Making the corresponding calculations we find that $t=\frac{3-p_{23}p_{32}}{p_{24}}$.

The more are the difficulties that a group of solvers has during the P-S process, the bigger is t. Therefore the value of t can be considered as a measure of solvers' abilities in P-S that can be used in comparing either the capacities of different groups of solvers for the solution of the same problems, or the difficulties of the same group of solvers during the solution of different problems. It is important to notice that the above measure, as it has been designed, it is based on all solvers' efforts (successful and unsuccessful) and not on their final outcomes only.

Furthermore, by applying other results from theory of Markov chains, one can make forecasts about the evolution of the P-S process by calculating the probabilities for a solver to be at the several states of the chain on a certain phase. For more details about the stochastic model and examples of its application in classroom look at Voskoglou and Perdikaris 1991, 1993, 1994 and at Voskoglou 1997 for an improved version of the model.

Similar applications of stochastic models in past include Mathematical modelling (Voskoglou 2007b), Analogical transfer of knowledge (Voskoglou 1996), the process of Learning (Voskoglou 2009/10), as well as Decision-making (Voskoglou 2000), Management and Economics (Voskoglou 2009b), Case-Based Reasoning (Voskoglou 2010b), etc.

The fuzzy model

Carlson and Bloom (2005) drawing from the large amount of literature related to P-S developed a broad taxonomy to characterize major P-S attributes that have been identifying as relevant to P-S success. This taxonomy gave genesis to their *"Multidimensional Problem-Solving Framework" (MPSF),* which includes four phases: *Orientation, Planning, Executing* and *Checking*. It has been observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated through out the remainder of the solution process; only in a few cases a solver obtained linearly the solution of a problem (i.e. he/she made this cycle only once). Thus embedded in the framework are two cycles (one cycling back and one cycling forward), each of which includes the three out of the four phases, that is planning, executing and checking. It has been also observed that, when contemplating various solution approaches during the planning phase of the P-S process, the solvers were at times engaged in a *conjecture-imagine-evaluate (accept/reject)* sub-cycle. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected to the phase of planning (Carlson and Bloom 2005; Figure 1).

There are many similarities among the five stages of Schoenfeld's expert performance model and the four phases of MPSF. In fact, the stage of analysis of the problem corresponds to the phase of orientation, the stage of design corresponds to the phase of planning, the stage of exploration corresponds to the conjecture-imagine-evaluate sub-cycle connected to the phase of planning, the implementation of the solution corresponds to the phase of executing and finally the stage of verification corresponds to the phase of checking. The qualitative difference between these two models is actually that, while the former focuses on the description of the P-S process and of the proper heuristic strategies to be used in each of its stages, the latter focuses on solver's behaviour and required attributes during the P-S process (Voskoglou 2008; section 4).

All the above models for the P-S process, including our stochastic one, are helpful in understanding the ideal behaviour of the problem-solver. However life in the classroom is not like that. Recent research on problems of mathematical modelling reports that students in school take *individual routes* when tackling these problems, associated with their individual learning styles (Borroneo Ferri 2007, Doer 2007, Galbraith and Stillman 2001, etc). Students' cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from teacher's point of view there usually exists vagueness about the degree of success of students in each of the stages of the P-S process. All these gave us the impulsion to introduce principles from *fuzzy sets* and of *uncertainty theory* in order to describe in a more effective way the P-S process in classroom.

The notion of fuzzy sets, initiated by Zadeh (1965), was created in response to have a mathematical representation of situations in everyday life in which definitions have not clear boundaries; e.g. this happens when we speak about the "high mountains" of a country, the "good players" of a football team, etc.

Let U denote the universal set. Then a *fuzzy subset* A of U (called often for simplicity a fuzzy set in U), is defined in terms of the *membership function* m_A that assigns to each element of U a real value from the interval [0,1]. In more specific terms $A = \{(x, m_A(x)) : x \in U\}$, where $m_A : U \to [0,1]$. The value $m_A(x)$, called the *membership degree (or grade) of x in A*, expresses the degree to which x verifies the characteristic property of A. Thus, the nearer is the value $m_A(x)$ to 1, the higher is the membership degree of x in A.

Despite to the fact that they can take on similar values, it is important to realize that membership degrees are not probabilities. One immediately apparent difference is that the summation of probabilities of singleton events (sets) on a universal set must equal 1, while there is no such requirement for membership degrees. The methods of choosing the suitable membership function for each particular case are usually empiric, based on experiments made on a sample of the population that we study.

Obviously each classical (crisp) subset A of U may be considered as a fuzzy subset of U, with $m_A(x)=1$ if $x \in U$ and $m_A(x)=0$ if $x \notin U$. Most of the concepts of classical (crisp) sets can be extended in terms of the above definition to fuzzy sets. For example, if A and B are fuzzy sets in U, then A is called a *subset* of B if $m_A(x) \leq m_B(x)$ for each x in U, while the *intersection* $A \cap B$ is a fuzzy subset of U with membership function $m_{A \cap B}(x)\ min\ \{m_A(x), m_B(x)\}$. Further, given a positive integer, say n, a fuzzy subset of the cartesian product U^n is called a *fuzzy relation* in U, etc. Fuzzy sets theory was expanded rapidly, to cover almost all sectors of human activities. Today one can see fuzzy sets both as a formal theory which embraced classical mathematical areas such as algebra, graph theory, topology, etc and as a powerful modelling tool that can cope with a large function of uncertainties in real life situations.

The concept of uncertainty, which emerges naturally within the broad framework of fuzzy sets theory, is involved in any problem-solving situation, especially when dealing with real-world problems. Uncertainty is a result of some information deficiency. In fact, information pertaining to the model within which a real situation is conceptualized may be incomplete, fragmentary, not full reliable, vague, contradictory, or deficient in some other way. Thus the amount of information obtained by an action can be measured in general by the reduction of uncertainty resulting from the action. In other words the amount of uncertainty regarding some situation represents the total amount of potential information in this situation. For general facts on fuzzy sets and uncertainty theory we refer freely to Klir and Folger (1988).

Let us consider now a group of *n* students, $n \geq 2$, during the P-S process in classroom. Denote by S_i i=1,2,3, the stages of planning, executing and checking of the MPSF and by *a, b, c, d,* and *e* the linguistic labels of negligible, low, intermediate, high and complete success respectively of a student in each of the S_i's. Set $U=\{a, b, ,c, d, e\}$. We are going to correspond to each stage S_i a fuzzy subset, A_i of U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of students that faced negligible, low, intermediate, high and complete success at stage S_i respectively, i=1,2,3, we define the membership function m_{Ai} for each x in U, as follows:

$$m_{A_i}(x) = \begin{cases} 1, & \text{if } \frac{4n}{5} < n_{ix} \leq \text{n} \\ 0{,}75, & \text{if } \frac{3n}{5} < n_{ix} \leq \frac{4n}{5} \\ 0{,}5\,, & \text{if } \frac{2n}{5} < n_{ix} \leq \frac{3n}{5} \\ 0{,}25\,, & \text{if } \frac{n}{5} < n_{ix} \leq \frac{2n}{5} \\ 0\,, & \text{if } 0 \leq \text{n}_{\text{ix}} \leq \frac{n}{5} \end{cases}$$

Then the fuzzy subset A_i of U corresponding to S_i has the form:

$$A_i = \{(x, m_{Ai}(x)):\ x \in U\},\ i=1,2,3.$$

In the same way we could also correspond to the stage of orientation a fuzzy subset of U. Nevertheless this makes the presentation of our fuzzy model technically much more complicated and therefore we will not attempt it. After all orientation, although it needs some attention, is actually the preliminary stage of the P-S process.

In order to represent all possible student *profiles (overall states)* during the P-S process we consider a fuzzy relation, say R, in U^3 of the form

$$R= \{(s, m_R(s)): s=(x, y ,z) \in U^3\}.$$

To determine properly the membership function m_R we give the following definition:

DEFINITION: A profile s=(x,y,z), with x,y,z in U, is said to be *well ordered* if x corresponds to a degree of success equal or greater than y, and y corresponds to a degree of success equal or greater than z.

For example, (c, c, a) is a well ordered profile, while (b, a, c) is not.

We define now the membership degree of a profile s to be $m_R(s)=m_{A_1}(x)m_{A_2}(y)m_{A_3}(z)$, if s is well ordered, and zero otherwise. In fact, if for example profile (b, a, c) possessed a nonzero membership degree, how it could be possible for a student, who has failed during the executing stage, to check satisfactorily the solution obtained?

In the next, for reasons of brevity, we shall write m_s instead of $m_R(s)$. Then the *possibility* r_s of profile s is defined by $r_s=\frac{m_s}{\max\{m_s\}}$, where $\max\{m_s\}$ denotes the maximal value of m_s, for all s in U^3. In other words r_s expresses the "relative membership degree" of s with respect to max $\{m_s\}$.

As we have seen above, the amount of information obtained by an action can be measured by the reduction of uncertainty resulting from the action. Accordingly students' uncertainty during the P-S process is connected to students' capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of students P-S capacities. Within the domain of possibility theory uncertainty consists of *strife (or discord)*, which expresses conflicts among the various sets of alternatives, and *non-specificity (or imprecision)*, which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives (Klir 1995; p.28). Strife is measured by the function ST(r) on the ordered possibility distribution *r:*

$r_1=1\geq r_2 \geq \ldots\ldots \geq r_n \geq r_{n+1}$ of the student group defined by

$\mathrm{ST}(r)=\frac{1}{\log 2}[\sum_{i=2}^{n}(r_i-r_{i+1})\log\frac{i}{\sum_{j=1}^{i}r_j}]$, while non-specificity is measured by

$\mathrm{N}(r) = \frac{1}{\log 2}[\sum_{i=2}^{n}(r_i-r_{i+1})\log i]$. The sum $\mathrm{T}(r)=\mathrm{ST}(r)+\mathrm{N}(r)$ is a measure of the *total possibilistic uncertainty* for ordered possibility distributions. Therefore the total possibilistic uncertainty of the student group during the P-S process can be adopted as a measure for students' P-S capacities. This is reinforced by Shackle (1961), who argues that human reasoning can be formalized more adequately by possibility theory rather, than by probability theory. The lower is the value of T(r) (which means greater reduction of the initially existing uncertainty), the better the performance of the student group during the P-S process.

Assume finally that one wants to study the combined results of behaviour of k different student groups, $k \geq 2$, during the solution of the same problems. For this we introduce the *fuzzy variables* A_1 (t), A_2 (t) and A_3 (t) with t=1, 2,…, k. The values of these variables represent fuzzy subsets of U corresponding to the stages of the P-S for each of the k student groups; e.g. A_1 (2) represents the fuzzy subset of U corresponding to the stage of planning for the second group (t=2). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the possibility r(s) of each student profile s with respect to the membership degrees of s for all student groups. For this reason we introduce the *pseudo-frequencies* $f(s) = \sum_{t=1}^{k} m_s(t)$ and we define $r(s) = \frac{f(s)}{\max\{f(s)\}}$, where *max {f(s)}* denotes the maximal pseudo-frequency. Obviously the same method could be applied when one wants to study the behaviour of a student group during the solution of k different problems.

Similar applications of fuzzy models in past include the process of learning (Voskoglou 2009a), Case-Based Reasoning (Voskoglou 2009c), Mathematical Modelling ((Voskoglou 2010a), as well as in problems appearing in Management and Economics (Voskoglou 2003).

4. Applications of the fuzzy model in classroom

The following two experiments performed recently at the Graduate Technological Educational Institute (T.E.I.) of Patras, Greece. In the first of them our subjects were 35 students of the School of Technological Applications, i.e. future engineers, and our basic tool was a list of 10 problems given to students for solution (time allowed 3 hours). Before starting the experiment we gave the proper instructions to students emphasizing among the others that we are interested for all their efforts (successful or not) during the P-S process, and therefore they must keep records on their papers for all of them, at all stages of the P-S process. This manipulation enabled us in obtaining realistic data from our experiment for each stage of the P-S process and not only those based on students' final results that could be obtained in the usual way of graduating their papers.

Our characterizations of students' performance at each stage of the P-S process involved:

- Negligible success, if they obtained (at the particular stage) positive results for less than 2 problems.
- Low success, if they obtained positive results for 2, 3, or 4 problems.
- Intermediate success, if they obtained positive results for 5, 6, or 7 problems.
- High success, if they obtained positive results for 8, or 9 problems.
- Complete success, if they obtained positive results for all problems.

Examining students' papers we found that 15, 12 and 8 students had intermediate, high and complete success respectively at stage of planning. Therefore we obtained that $n_{1a}=n_{1b}=0$, $n_{1c}=15$, $n_{1d}=12$ and $n_{1e}=8$. Thus, by the definition of $m_{A_i}(x)$, planning corresponds to a fuzzy subset of U of the form: $A_1 = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e, 0.25)\}$.

In the same way we represented the stages of executing and checking as fuzzy sets in U by $A_2 = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e,0)\}$

and $A_3 = \{(a, 0.25),(b, 0.25),(c, 0.25),(d,0),(e,0)\}$ respectively.

Using the definition given in the previous section we calculated the membership degrees of the 5^3 (ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles (see column of $m_s(1)$ in Table 1). For example, for $s=(c, c, a)$ one finds that

$$m_s = m_{A_1}(c).\ m_{A_2}(c).\ m_{A_3}(a) = (0.5).(0.5).(0.25)=0.06225.$$

It turned out that *(c, c, a)* was one of the profiles of maximal membership degree and therefore the possibility of each s in U^3 is given by $r_s = \frac{m_s}{0{,}06225}$.

Calculating the possibilities of all profiles (see column of $r_s(1)$ in Table 1) one finds that the ordered possibility distribution for the student group is:

$$r_1=r_2=1, r_3=r_4=r_5=r_6=r_7=r_8=0{,}5, r_9=r_{10}=r_{11}=r_{12}=r_{13}=r_{14}=0{,}258,$$
$$r_{15}=r_{16}=\ldots\ldots=r_{125}=0.$$

Thus with the help of a calculator we found that $\mathrm{ST}(r)=\frac{1}{\log 2}[\sum_{i=2}^{14}(r_i-r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}] \approx \frac{1}{0.301}[0.5\log\frac{2}{2}+0.242\log\frac{8}{5}+0.258\log\frac{14}{6.548}]$

$\approx 3.32[(0{,}242).(0.204)+(0.258).(0.33)] \approx 0.445$ and

$\mathrm{N}(r)=\frac{1}{\log 2}[\sum_{i=2}^{n}(r_i-r_{i+1})\log i]=\frac{1}{\log 2}[0.5\log 2+0.242\log 8+0.258\log 14]$

$\approx 0.5+3.(0.242)+(0.857).1.146 \approx 2.208$. Therefore we finally obtained that $\mathrm{T(r)} \approx 2.653$.

A few days later we performed the same experiment with a group of 30 students of the School of Management and Economics. Working as above we found that $A_1=\{(a,0),(b, 0.25),(c, 0.5),(d, 0.25),(e,0)\}$, $A_2=\{(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e,0)\}$ and $A_3=\{(a, 0.25),(b, 0.25),(c,0.25),(d,0),(e,0)\}$.

Then we calculated the membership degrees of all possible profiles of the student group (see column of m_s (2) in Table 1). It turned out that the maximal membership degree was again 0.06225, therefore the possibility of each s is given by the same formula as for the first group. Calculating the possibilities of all profiles (see column of $r_s(2)$ in Table 1) we found that the ordered possibility distribution of the second group is:

$$r_1=r_2=1,\ r_3=r_4=r_5=r_6=r_7=r_8=0{,}5,\ r_9=r_{10}=r_{11}=r_{12}=r_{13}=0{,}258,$$
$$r_{14}=r_{15}=\ldots=r_{125}=0$$

Finally, working in the same way as above we found that $\mathrm{T}(r)=0.432+2.179=2.611$.

Thus, since 2.611<2.653, it turns out that the second group had in general a slightly better performance than the first one.

Next, in order to study the combined results of behaviours of the two groups, we introduced the fuzzy variables A_i (t), i=1, 2, 3 and t=1, 2, as we have described in the previous section. Then the pseudo-frequency of each student profile s is given by $f(s) = m_s(1) + m_s(2)$ (see corresponding column in Table 1). It turns out that the highest pseudo-frequency is 0.124 and therefore the possibility of each student's profile is given by $r(s) = \frac{f(s)}{0.124}$. The possibilities of all profiles having non-zero pseudo-frequencies are presented in the last column of Table 1.

Table 1: Profiles with non zero pseudo-frequencies

(The outcomes are with accuracy up to the third decimal point)

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	f(s)	r(s)
b	b	b	0	0	0.016	0.258	0.016	0.129
b	b	a	0	0	0.016	0.258	0.016	0.129
b	a	a	0	0	0.016	0.258	0.016	0.129
c	c	c	0.062	1	0.062	1	0.124	1
c	c	a	0.062	1	0.062	1	0.124	1
c	c	b	0	0	0.031	0.5	0.031	0.25
c	a	a	0	0	0,031	0.5	0.031	0.25
c	b	a	0	0	0.031	0.5	0.031	0.25
c	b	b	0	0	0.031	0.5	0.031	0.25
d	d	a	0.016	0.258	0	0	0.016	0.129
d	d	b	0.016	0.258	0	0	0.016	0.129
d	d	c	0.016	0.258	0	0	0.016	0.129
d	a	a	0	0	0.016	0.258	0.016	0.129
d	b	a	0	0	0.016	0.258	0.016	0.129
d	b	b	0	0	0.016	0.258	0.016	0.129
d	c	a	0.031	0.5	0.031	0.5	0.062	0.5
d	c	b	0.031	0.5	0.031	0.5	0.062	0.5
d	c	c	0.031	0.5	0.031	05	0.062	0.

e	c	a	0.031	0.5	0	0	0.031	0.25
e	c	b	0.031	0.5	0	0	0.031	0.25
e	c	c	0.031	0.5	0	0	0.031	0.25
e	d	a	0.016	0.258	0	0	0.016	0.129
e	d	b	0.016	0.258	0	0	0.016	0.129
e	d	c	0.016	0.258	0	0	0.016	0.129

5. Conclusions

In the present article we constructed a fuzzy model representing the P-S process in mathematics and we used the total possibilistic uncertaintyT(r) as a tool of measuring P-S abilities. This model, apart from quantitative information (possibilities of solvers' profiles, value of T(r), etc), gives also a qualitative view of students' performance in classroom through the study of all their possible profiles during the P-S process. Another advantage of our fuzzy model is that it gives the opportunity of studying the combined results of performance of two or more groups of students in solving the same problems, or alternatively the performance of the same group in solving different problems.

On the contrary, the stochastic (Markov chain) model for the P-S process that we have developed in earlier papers, although its application by a non expert (e.g. the teacher of mathematics) in classroom looks rather easier, describes simply the ideal behaviour of a group of solvers in classroom and it is self- restricted to provide quantitative information only, i.e. a measure of students' P-S capacities and certain forecasts (probabilities) about the evolution of the P-S process. Therefore one could claim that the fuzzy model is more useful for a deeper study and understanding of the P-S process.

Similar models (stochastic and fuzzy) have been constructed by the author during the last twenty years for the description of the processes of Mathematical Modelling, of Learning, of Analogical Transfer of Knowledge, of Case-Based Reasoning, of Decision Making and of several other problems appearing in the area of Management and Economics.

Two classroom experiments, performed recently at the graduate Technological Educational Institute of Patras in Greece, were also presented in this paper illustrating the use of our fuzzy model for P-S in practice. Nevertheless further research is needed for the P-S process. In fact, as a general conclusion of all findings from research studies on P-S it turns out that success in P-S appears to stem from the ability to draw on a large reservoir of well-connected knowledge, heuristics and facts, from the ability to manage the emotional responses, as well as from an adequate degree of practice (Voskoglou 2007a, 2008). However, although many studies have investigated and compared the characteristics of novice and expert problem solvers (Lesh and Akerstrom 1982, Schoenfeld 1985b, Stillman and Galbraith 1998, etc), many of the qualitative differences appearing among them still do not seem to be completely understood. It is hopped therefore that the use of our fuzzy model as a tool in future research on P-S could lead to practical ways of restoring the weaknesses appearing to novices with respect to the expert problem solvers

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Appendix: List of the problems given to students for solution in our classroom experiments

Problem 1: We want to construct a channel to run water by folding across its longer side the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water?

Problem 2: Given the matrix $A=\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and a positive integer n, find the matrix A^n.

Problem 3: Calculate the integral $\int \frac{x}{x^2+4}dx$.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet (A=1, B=2, C=3 etc). Let us correspond also to each word consisting of 4 letters a 2X2 matrix in the obvious way; e.g. the matrix $\begin{bmatrix} 19 & 15 \\ 13 & 5 \end{bmatrix}$ corresponds to the word SOME. Using the matrix $E=\begin{bmatrix} 8 & 5 \\ 11 & 7 \end{bmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how he (she) could decode your message?

Problem 5: The demand function $P(Q_d)=25-Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s)=2Q_s+1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the market's equilibrium occurs at (Q_0, P_0), producers who would supply at lower price than P_0 benefit. Find the total gain to producers'.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.

Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?

Problem 8: The rate of increase of the population of a country is analogous to the number of its inhabitants. If the population is doubled in 50 years, in how many years it will be tripled? (ANSWER: In $50\frac{\ln 3}{\ln 2} \approx 79$ years).

Problem 9: A company circulates for first time in market a new product, say K. Market's research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will turn to K next week.

i) Find the market's share for K two weeks after its first circulation, provided that the market's conditions remain unchanged.

ii) Find the market's share for K in the long run, i.e. when the consumers' preferences will be stabilized.

Problem 10: Among all cylinders having a total surface of 180π m^2, which one has the maximal volume?

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University mathematics students and exam-style proving questions: The A-B-Δ strategy classification scheme

Andreas Moutsios-Rentzos & Adrian Simpson

Abstract

In this study, we present a classification scheme for the strategies that university mathematics students employ when they deal with exam-style proving questions, which we call the A-B-Δ classification scheme. Synthesising the mathematics education literature with that from general educational research, we explore the conceptual validity of this construct and claim that the classification provides a framework for differentiating strategies which might otherwise be conflated. Empirical data supporting this claim are presented, in particular drawing attention to these qualitatively different strategies.

Keywords: university mathematics, proof, strategy, exam-type questions

1. Introduction

Much mathematics research examines how students solve problems and answer mathematical questions. In this paper, we focus particularly on how university mathematics students deal with *exam-style* questions. Such questions are especially important, since closed book examinations are by far the most prevalent mechanism of assessment in university mathematics (Goulding, Hatch & Rodd, 2003). Within this style of assessment, the most common forms of question for university mathematics students are in the format 'Given that…, prove that…'; that is, proving in examinations appears to be the main driving force behind assessment.

Various mathematics educators have categorised students' proving strategies (e.g. Mariotti, 2006). In university mathematics, Weber (2005) provides a proving strategy classification that is based on Skemp's (1976) notions of instrumental and relational understanding. Much work on proving strategies, however, has not specifically addressed the issue of the exam-style question and since there is strong evidence that assessment is a key driver of learning (Boud & Falchikov, 2007) the frameworks developed without such a focus may fail to identify strategies specific to this area. This paper aims to produce a strategy classification that is both empirically and conceptually validated for exam-style questions, addressing the fundamental research question:

What are the proving strategies that university mathematics students employ when they deal with exam-style questions?

2. Proof and proving in university mathematics

The notion of proof is central to mathematics and various functions of proof have been identified by mathematics educators (Mariotti, 2006). In this study, we focus on proof as a means of establishing the truth of a statement: its *verification* function. In university mathematics, the proofs presented often omit a number of steps of the complete proof, leaving the reader to infer them. Hence, the validity of each proof largely depends on what the university mathematical community defines as acceptable, highlighting the *institutional meaning* of proof (Recio & Godino, 2001). Students' difficulties with proof have been identified in numerous studies (e.g. Selden & Selden, 2008), notably demonstrating that students' personal meanings of proof often clash with its institutional meaning; i.e. students may be convinced about the validity of a proof based on different warrants from the ones accepted by mathematicians.

Mathematics educators (notably Harel & Sowder, 1998) acknowledge a contrast between *ascertaining* (convincing oneself) and *persuading* (convincing others). In this study, we will note the different warrants the same students might employ for the ascertaining and persuading aspects of a proof construction.

Weber (2005) identified three types of proof construction that a student may undertake: *procedural*, *syntactic* and *semantic*. In a *procedural proof construction*, the students attempt "to locate a proof of a statement that is similar in form and use this existing proof as a template for producing a new one" (p. 353), whereas in a *syntactic proof construction*, the students begin "with a collection of definitions and assumptions, and then draw inferences about these statements by applying established theorems and logical rules" (*ibid*). A *semantic proof construction* is characterised by the initial exploration of "informal or intuitive representations of relevant concepts to see why the statement to be proven is true" (*ibid*), which students use to inform "their examination of informal representations of mathematical concepts to suggest and guide the formal inferences" (*ibid*) they would draw.

Weber's work resembles Lithner's (2003, 2008) investigations into the students' mathematical reasoning about textbook exercises which differentiated between *imitative reasoning* and *creative mathematically founded reasoning* (akin to 'semantic'). Imitative reasoning is further divided into *memorised* (akin to 'procedural') and *algorithmic* (akin to 'syntactic').

We argue that these classifications may not be sufficient for the purposes of this study. Both Weber and Lithner appear not to fully consider the potential contrast between 'ascertaining' and 'persuading'. It may be that the arguments that ascertain and persuade in a 'semantic' proof construction are essentially the same. On the other hand, a student may be convinced by one argument, but employ a completely different argument in the process of persuading. Though Lithner (2008) noted that creative mathematically founded reasoning might be *partially* employed, the ascertaining/persuading contrast is not explicitly addressed.

Our view is further supported by the ‘approaches to study’ perspective (e.g. Marton & Säljö, 1976). ‘Approaches’ can be differentiated into: *deep* (focussing on the meaning and ideas conveyed in the task), *surface* (focussing on the more superficial characteristics and procedures of the task) and *achieving* (concentrating on attaining better performance in the assessment procedures). This perspective draws attention to the students’ goals, an aspect which might compliment Weber’s and Lithner’s classifications. In this study, we focus on questions that are highly goal-oriented and, hence, we need to develop a strategy classification sensitive to the students’ goals.

Conceptually, a ‘semantic’ strategy may be linked with a ‘deep’ approach and a ‘procedural’ strategy with a ‘surface’ approach. One might question, however, whether a ‘syntactic’ strategy may be linked with the ‘achieving’ approach in the case of mathematical exam-style questions. Moreover, mathematics educators stress the importance of flexibility in the students’ thinking when they deal with mathematics (Gray & Tall, 2007) and so one can further ask whether there are aspects of Weber’s classification which relate to this notion of ‘flexibility’ in the context of exam-style questions. Thus, we conducted an empirical investigation into students’ proving strategies in the context of exam-style questions to see which classification framework best accounts for students’ practice.

3. Method: participants and procedures

The study was conducted with 15 students (8 male and 7 female) who were in their 2nd year of a BSc in mathematics at a large university in Greece. Six mathematical questions (listed in Table 1) were used to identify students’ strategies. These were taken from two areas which are core to their mathematics degree – algebra and analysis – to maximise the students’ familiarity with the mathematical content, and are phrased in an examination style to evoke the most realistic strategies from the participants, using notational conventions with which the students would be familiar.

1st Interview	Let $a, b, c \in Z$ and $(a,b)=1$ and $a \mid bc$. Prove that $a \mid c$. (*a divides c*)
	Let a sequence $(a_n) \in R$, $n \in N$. Prove that if (a_n) is convergent, then (a_n) is bounded. (*convergent-bounded*)
	Let A, B non-empty subsets of the real numbers R and A, B are bounded. Does the $\sup(A \cup B)$ exist? If yes, find it. Justify in full your answer. (*supremum of union*)
2nd Interview	Let $G = \langle a \rangle$, cyclic, finite group, rank n. Prove that a^{κ}, $\kappa \in Z$, is generator of G, if and only if $(\kappa, n) = 1$. (*group generator*)
	Let $a \in N$. Prove that a is divisible by 9, if and only if the sum of its digits is divisible by 9. (*divisible by 9*)
	Let $f: R \rightarrow R$ and f periodic with period $T>0$. If $\lim_{x \to \infty} f(x) = b \in R$, then prove that f is constant. (*periodic-constant*)

Table 1: The interview mathematical questions (code names in brackets).

Each participant attended two interviews (each lasting from 40-60 minutes) one week apart. All interviews were video recorded. The interviewees were asked to 'think aloud' and specifically encouraged to provide 'exam-style answers'. Given that the questions were not answered in examination conditions, we drew on the methods suggested by Weber (2001) and informed the participants that they could use the interviewer as their 'revision memory' – that is, they could ask for pieces of mathematical knowledge, but not solution strategies. This technique shifts the focus from the students' *access* to mathematical knowledge to their *manner of accessing and using* mathematical knowledge – that is, specifically to their proving strategies.

The students' strategies were coded directly from the video. Since the purpose of the study was to explore existing classification frameworks, pre-defined strategy categories were considered in strategy identification.

Importantly, however, the coding technique allowed for new categories to emerge. This built on findings of a pilot study that suggested that the students' strategies differed depending on whether the students explored the 'truth' of the statement they were asked to prove (*truth development*) or whether they effectively ignored its truth-status (*proof development*).

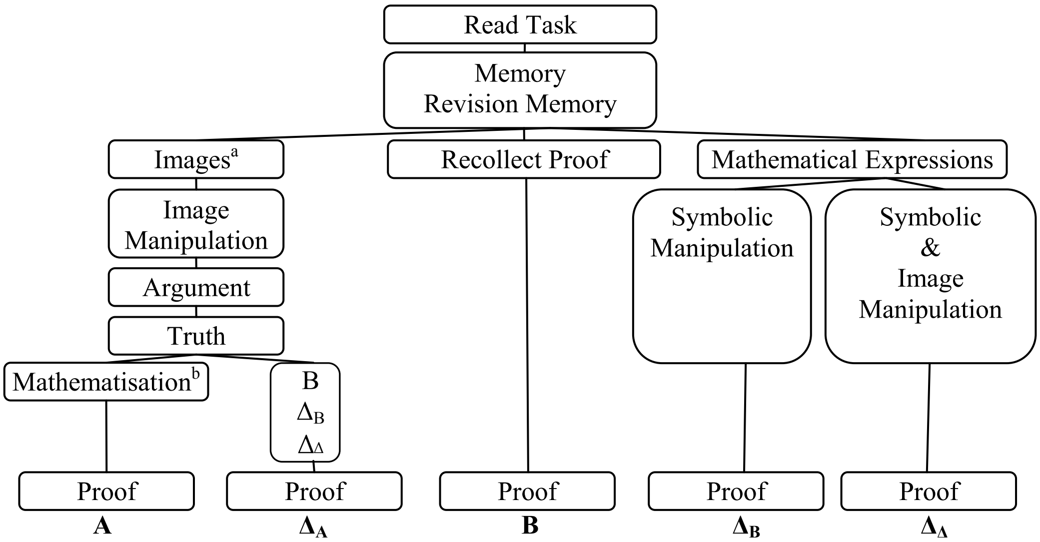
Upon completion of this initial coding process, a number of un-coded strategies remained, suggesting that the pre-existing classification schemes were incomplete. These un-coded strategies were examined to determine any new categories. Samples of the interviews, with the initial coding scheme, were given to a colleague to refine the initial codes (to see if the pre-existing schemes could account for more of the data).

Finally, a large sample of the interviews with the final coding scheme was given to another colleague to obtain *inter-rater reliability*. This analysis suggested an *almost perfect* (Landis & Koch, 1977) consistency between the two raters (Kappa = 0.924, $p < 0.001$). The same strategies were identified in 94.16% of the times.

4. Introducing the A-B-Δ strategy classification: A, Δ_A, B, Δ_B and Δ_Δ strategies

The interview analysis revealed that the students used five main strategies when they tried to answer the sample exam-style questions, which can be coded within a system we call the *A-B-Δ strategy classification* ('Alpha-Beta-Delta classification'). This classification can be viewed as an expansion of Weber's (2005) classification and consists of five strategies: A, Δ_A, B, Δ_B and Δ_Δ. In Figure 1, a diagrammatic outline of the strategy classification is presented and detailed explanations and examples of the five different strategies form the bulk of the remainder of the paper.

As we explore each strategy uncovered in our data, we shall show that A, B and Δ_B are similar to Weber's (2005) 'semantic', 'procedural' and 'syntactic' strategies respectively. However, we also uncover what we believe are two new strategies not present in Weber's or Lithner's classifications: In Δ_A, different arguments are employed for ‘ascertaining’ and ‘persuading’ (as opposed to *A*, in which the same argument is used for both ‘ascertaining’ and ‘persuading’). In contrast, Δ_Δ is a flexible, goal-oriented strategy (akin to the ‘achieving’ approach) involving various cognitive tools (both symbolic and ‘image’ manipulation) to obtain what is ‘wanted’ from what is ‘given’.



[a] : Mental images (pictorial or other) akin to Tall and Vinner’s (1981) notion of ‘concept image’.
[b] : The translation of the arguments used to establish the truth of the statement into mathematical language.
[c] : Including mathematical expressions, formulas and statements.

Figure 1: Diagrammatic outline of the A-B-Δ strategy classification.

4.1. The A strategy

We saw the *A* strategy used in 25 responses (i.e. 20% of the total number of strategies used). In an *A* strategy, the student attempts to identify the definitions of the concepts involved. Subsequently, the student’s concept image of the mathematical objects identified is triggered. These images are manipulated to extract the meaning of the mathematical notions involved, to understand what the question asks and to produce an acceptable argument for *ascertaining*. Once constructed in this way, the argument can be viewed as a warrant of the truth of the statement in the question and the ascertaining phase has been accomplished. Then follows a process of mathematisation of *the same argument*, but now reformulated to fulfil the requirements for *persuasion*. This strategy appears closely aligned to Weber's description of his 'semantic' approach.

For example, in the 'convergent-bounded' question, Kostas first describes the argument that convinces him. There is a continuous feedback between mathematical definitions and his concept image about these mathematical notions [1][1]. Kostas is convinced about the truth of the argument [2] and realises the need to write this argument "in a mathematical way" [3]. Kostas' proof is an effort to write the arguments he previously described in everyday language with mathematically acceptable symbolism; a case of mathematisation [4], typical of an *A* strategy.

Kostas: Since the sequence is increasing and converging, then this means that it increases until it reaches a certain point that the sequence cannot overpass [He also makes gestures to 'show' the way the sequence 'moves' and stops when it reaches that certain point]. [1]

This means that for every *n* the sequence cannot pass this point. This means the sequence is bounded. [2]

Of course now I cannot prove this in a mathematical way. [3]

It just...In the figure ... it is ok ... [He stops for a moment staring at 'infinity' and then continues] ... if we consider a line ... a sequence that increases-increases [He draws an increasing line] [1]

and this is its limit [He draws a line showing this 'limit'] then it cannot overpass this point whatever we do. Therefore it is bounded above. Then the sequence starts from somewhere [He refers to the first term]. Therefore, the sequence is upper-bounded. [2]

Following the same [rationale] we can prove the opposite [in the case the sequence is decreasing]. Now like mathematics ... with formulas ... I don't think I possess the required knowledge. If I had the book of 'Infinitesimal Calculus' with me

Researcher (A): Then what will you do? Will you proceed to the next one? Will you write

[1] In order to make the link between the processes involved in each strategy and the examples used to illustrate them, we use numbers in square brackets.

something?

K: Certainly I would write something ... Well...We have ... We consider the sequence a_n ... which is... We take two cases... Let it be increasing ... Ah! Just a minute... We can say that since the absolute value of a_n - a is smaller than ε for every n less or equal to n_0, then this is [He expands the inequality using the $|x|<\theta$ property of absolute values]. [4]

A: What did you mean when you said “Ah!”? What were you thinking?

K: This is from the properties of absolute values.

A: And why did you seem so pleased?

K: I thought of a solution [He makes gestures showing what he describes]. Because if we subtr ... add ‘a’ [He refers to the double inequality] we would now have a minus ε less than a_n less than a plus ε for every n larger or equal to n_0. If we prove that this holds true for every n, we have shown what we want...because we would have restrained the sequence in between a minimum and a maximum.

4.2. The Δ_A strategy

We saw the Δ_A strategy used in 15 responses (i.e. 12% of the total number of strategies used). Δ_A has similar initial processes to the *A* strategy, differing primarily in the final phase. That is, the students who employ it look for the truth of the argument (the *ascertaining* phase), but when it comes to persuasion, they seem to restart their efforts and develop a potentially different argument. It appears that there is a ‘conceptual chasm’ for them between ascertaining and persuading and, thus, two questions to be answered. Hence, in contrast with *A* and in contrast to the strategies in Weber’s and Lithner’s frameworks, in Δ_A students do not mathematise the arguments developed when ascertaining to produce the *persuading* argument. Instead, in Δ_A, characteristics of *A*, *B* and Δ_B may be discerned as they develop a new argument. Δ_A essentially consists of an *A* strategy for ascertaining and a *B*, Δ_B or Δ_Δ for persuading (these two components differentiate Δ_A from Δ_Δ; see §4.5).

For example, in the ‘divisible by 9’ question, Nikos reads the question and notes that he needs to examine whether or not the statement he is asked to prove is actually 'true'. A specific example that he found appears to be satisfactory for this [1]. When it comes to persuasion, however, Nikos focuses on trying to remember a theorem that may help him to solve it, typical of *B* and Δ_B strategies [2]. Once this fails, he tries to re-examine the example he used for ascertaining with the purpose of identifying an argument that can be later mathematised [3]. Though this might seem like an *A* strategy, in this case, Nikos does not initially try to mathematise his ascertainment argument. Instead he tries to find a solution starting from a completely new path: recollection. Only when this fails, does he try to mathematise the ascertainment argument. When mathematisation fails [3], he resorts to memory aiming to find ways to approach the question [4]. Thus, Nikos uses a more flexible strategy for persuading; a Δ_Δ strategy, rather than just *A* or *B* (or Δ_B).

Nikos: [pause] I am trying to think which way to solve it ... Err ... [He re-reads aloud the question] ... [in a confident manner] the first thing I do is to think of an example ... a 2-digit, a 3-digit number in order to check if it is valid ... which three-digit number I can think of. [1]

A: Do you want me to give you an example?

N: [Without paying attention to me] Say '333' ... right! [1]

[...]

A: What are you thinking?

N: I am trying to remember if there is a 'divisible by 9' criterion, like the one that exists with '2' ... as far as I can remember ... [2]

A: This one [the one he tries to prove]?

N: Something I can use for this one ... for proving this criterion. Maybe there exists a second one. Obviously this is one criterion [for the divisibility by 9] ... This must be the criterion ... right ... therefore I must look for when a natural number is divisible by 9 ... I am trying to make it more clear in my mind, like cases ... maybe if I take cases ... obviously cases like if the number 'ends' in 0,1,2,3 and so on until 9, but [pause] ... take a number say '333' the example [he used before] the sum of its digits is 9, which obviously is divisible by 9 ... how can I ... for this example '333' ... I can show that

333 is actually divisible by 9 ... [pause] ... Therefore I reach to a dead-end ... I cannot find a way of thinking ... [3]

A: What are you missing? Is there anything that you would like to know? [The 'role' of the 'revision memory' included prompts like this to remind him of the existence of this option.]

N: I don't ... I don't know right now... The only thing I am thinking of is that maybe there is a specific symbolism or a specific theorem ... that I miss ... that can help me ... I cannot remember something that I can tell 'this I can use'. [4]

4.3. The B strategy

B strategy was used in 30 responses (i.e. 24% of the total number of strategies used). The students employing the *B strategy* focus on reproducing and on remembering, rather than on generating, the answer. Based on memory and/or 'revision memory', they attempt to identify the definitions of the concepts involved (the concept definition in the sense of Tall & Vinner, 1981). These definitions and, substantially less commonly the students' 'images' (including figures, diagrams, examples), are used to trigger the students' memory, in order to recollect and reproduce the proof. The recollected proof is validated based on memory triggered and memory regulated arguments. This accords with Weber's 'procedural' approach.

In the 'supremum of union' question, Maria concentrates on each mathematical notion with the purpose of recollecting any relevant knowledge ([1], [2]). She manages to recollect the necessary theorems for answering the question and to quickly produce the answer [3].

Maria: [She reads the question loud.] Bounded sets ... [pause] ... union ... [pause] ... bounded [She repeats the mathematical objects of the question making pauses after each one of them] ... If the sets are bounded then their union is also bounded [She looks at me]. [1]

A: The finite union of bounded sets is also bounded. We know that from theory [she described the theorem well enough]. [2]

M: [Pause] ... subsets of the real numbers ... [pause].

A: What are you thin ... ? [She stops me]

M: Firstly ... a set that is non-empty and bounded has a supremum. [1]

A: Every non-empty sub-set of the real numbers has a supremum. [2]

M: [She repeats it] ... therefore ... since A and B [the two sets] are non-empty then the union is non-empty. Therefore the union is a non-empty subset of the real number and it is also bounded ... Therefore it has a supremum ... It follows by the property [She means the completeness theorem of the real numbers] of that ... it has a supremum. [3]

4.4. The Δ_B strategy

Δ_B was used in 37 responses (i.e. 29% of the total number of strategies used). The students employing a Δ_B strategy rely mainly on their memory to generate and manipulate mathematical expressions, but Δ_B differs from *B* in that the memory is used for the purpose of *generating expressions*, rather than for reproducing an already known proof. Imagery is either completely absent from this strategy or has only a minimal presence. Even in the latter case, these images seem to be isolated and separated from the argument generation process, which clearly differentiates Δ_B from Δ_Δ strategy. This strategy appears to fit the 'syntactic' approach described by Weber.

In the 'a divides c' question, after having read it, Stavros asks for the definitions of the mathematical notions involved (revision memory). He relies on his memory [1] and the interviewer as 'revision memory' to produce mathematical expressions, which are the clues that can help him answer the question [2]. Stavros concentrates on recalling the expressions that he can manipulate into the relation he is asked to prove [3].

Stavros: I try ... I am trying to combine my given data [He refers to the definitions he asked from me] and I am trying to remember a little the notions, so that I can combine them together and to find out which are the common elements among the notions. [1]

A: When you say 'I am trying to remember the mathematical notions', what exactly do you mean? The definitions? A specific example? [He stops me]

S: Yeah ... The definition of the greatest common divisor and what these two ... the fact that the greatest common divisor equals to 1 'gives' to me ... that is which is my 'clue' and after that to check the notions to see which 'clues' I can get from these, in order to find a starting point [2]

... [pause]... Ah! ... [pause] ...

A: What did you think?

S: Let me write them down first, so that I can be able to see them. [He writes down the exact definitions in 'draft', with the help of the 'revision memory']

S: We have 'a divides bc'. This means that there exists k, so that bc is k times a. Therefore [pause] ... ok ... Ah! to prove ... we want to reach to ... that a divides c ... that is that there exists ... that there exists a λ which belongs to Z [the integers] so that λa equals to c [At the same time he writes what he utters] ... [He goes on making algebraic manipulations of these relations) [3]

A: Why did you do these [I point to the algebraic manipulations]?

S: So that I can reach a conclusion ... so that I can make substitutions ... [pause] ... I think now that I could start forming the relation I want to prove and er to reach a relation that is valid so that I can say ... and then go back and say that since that is true then this is also true [He writes what he says]. [3]

4.5. The Δ_Δ strategy

The Δ_Δ *strategy* was employed in 19 responses (i.e. 15% of the total number of strategies used). Δ_Δ is characterised by the flexibility of the means that the students coordinate to promote the manipulation of the expressions leading to a proof. Those means might include figures, diagrams, examples, known proofs, common strategies etc. Δ_Δ may also include image manipulations as part of the whole effort to produce mathematical expressions, rather than, for example, of the effort to meet an internally set requirement to examine the 'truth' of the statement (as in *A* strategy). In addition, images are used for generating arguments, rather than for just triggering the students' memory, which differentiates Δ_Δ from Δ_B. Again, we felt that this approach was not adequately covered by the existing frameworks from Weber or Lithner.

In the 'periodic-constant' question, Sofia first reads the questions and writes down the mathematical definitions of all the notions mentioned. For example, she asks: "*Do you have the exact definition of the periodic function?*" or "*For the period [of the periodic function], what exactly does it [the theory] say*?". Subsequently, she seems to concentrate hard on each definition to remember the mathematical relations linked to each notion. Sofia doesn't appear to think of a specific figure and she produces a mathematical expression, which immediately derives from the definitions [1].

Sofia: Its limit is a number ... constant ... and it says that T is greater than zero ... [pause].

A: What are you thinking right now? Of a figure? Of [She interrupts me].

S: No, I am just thinking about them. A function f that is periodic ... that is, it is f(x+T) equals to f(x) ... and its limit is β ... now I thought that the limit of f(x+T) is also β ... [1]

Sofia mentally manipulates her images to answer the given question [2]. Moreover, she tries to extract a mathematical expression through the identification of the property that the graph has [3]. For Sofia these manipulations are not part of a mathematisation process, but just another tool that can help her produce mathematical relations.

S: Now I am thinking of the graph of the periodic ... [2]

A: Fine ... When you say the 'graph' ...

S: Yes. No, I mean which property has ... 'graphically' what property it has. [3]

A: 'Graphically'? [She doesn't pay attention to me]. When you say 'properties' you mean from the graph?

S: Yes , from the graph ... [pause] ... and from the graph we know that this will tend to β ... its limit [She also makes a gesture showing this limit] ... [2]

5. Discussion

We claim that, though the A-B-Δ classification can be conceptually linked with existing mathematics education research, it is a more extensive classification framework than those most commonly used. To explore this claim, we need to provide both conceptual and empirical validation for it. Our classification can be viewed as an expansion of Weber's classification.

The main difference between them is Δ_A, which consists of an A strategy for ‘ascertaining’ and a B, Δ_B or Δ_Δ for ‘persuading’. Δ_A shares conceptual links with Lithner’s ‘local creative reasoning’. In addition, our empirical work identifies a flexible strategy (Δ_Δ) which does not appear to be present in either Lithner’s or Weber’s work (see Table 2).

Our expanded scheme derives from a different choice of perspective. Weber (2009) examines proof production in terms of representational systems: he identifies a semantic proof production with students thinking outside the representational system of proof, whereas in a syntactic proof production the students predominantly work within the representational system of the proof. In contrast, we focussed on the students’ goal framework and thus our classification is student-centred, whereas Weber’s is proof-centred. For example, we clearly differentiate Δ_A from A in the use or not of the same argument for ascertaining and persuading. In Weber’s terms, these strategies would both be classified as being ‘semantic’, thus obscuring the ‘achieving’ part of Δ_A. Moreover, most of the strategies identified as Δ_Δ in our classification would also be classified as ‘semantic’ in Weber’s classification scheme. One can ask whether this ‘refinement’ of ‘semantic’ is a substantial addition to our understanding of proving strategies. However, when the goal framework is a core part of the situation the students have to cope with, it is crucial to have an understanding of what drives the students to choose a strategy. For example, in a study investigating the links between the students’ strategies and their preferred way of thinking (Moutsios-Rentzos, 2009), Weber’s classification would be less discriminating in spotting and delineating the complex phenomena involved.

<table>
<tr><td></td><td>α-type (‘truth’)</td><td>β-type (‘memory’)</td><td>δ-type (‘flexibility’)</td></tr>
<tr><td rowspan="3">A-B-Δ
‘Weber’s classification’
‘Strategy Type’ –
‘Approaches’</td><td>A
‘semantic’[c]
‘truth’-‘deep’</td><td></td><td></td></tr>
<tr><td></td><td>B
‘procedural’
‘memory’-‘surface’</td><td></td></tr>
<tr><td>Δ_A
-
‘truth’-‘achieving’</td><td>Δ_B
‘syntactic’
‘memory’-‘achieving’</td><td>Δ_Δ
-
‘flexible’-‘achieving’</td></tr>
</table>

Table 2: The A-B-Δ strategy classification (comparisons and contrasts).

In our study, the students’ strategies seemed to revolve around *truth*, *memory* and *flexibility*, corresponding to three *strategy types* (see Table 2). The students employing *α-type strategies* (A and Δ_A) appear to examine the *truth* of the statement they want to prove, which, notably, is not necessary for answering an exam-style question. On the other hand, the students employing *β-type strategies* (B and Δ_B) concentrate on *memory* to reproduce a proof or to generate mathematical relations, thus operating only *within* the ‘requirements’ of exam-style questions. The *δ-type strategy* is Δ_Δ, which is highly goal oriented and is characterised by the *flexibility* of resources the students use. Hence, we argue that the strategy types can be linked with the students’ choice to work within the requirements of the exam-style questions or not.

In addition to its links to Weber and Lithner, our classification can be linked with the ‘approaches to study’ body of research. Students employing an A strategy appear to use a deep approach, in the sense that they concentrate on finding the *meaning* of what they are trying to prove. The B strategy can be linked to surface approaches, since in this strategy the focus is on remembering and reproducing. On the other hand, Δ_A, Δ_B and Δ_Δ

strategies appear to share combinations of deep, surface or achieving characteristics. Δ_A appears to be a deep strategy, as far as ascertaining is concerned, and, for persuading, more flexible and goal oriented. Moreover, Δ_B seems to rely on memory and reproduction, which is linked to surface approach, but also combines more flexible tools. Hence, we argue that Δ_A is a deep-achieving strategy, whereas Δ_B is a surface-achieving strategy. Regarding Δ_Δ, the flexibility of the employed tools stresses the goal orientation of this strategy and, thus, it has clear links with an achieving strategy.

Thus, we can argue that the α-type and β-type strategies consist of two parts: one goal oriented (achieving) and one focussed on the special characteristics of each strategy type (deep for α-type and surface for the β-type). On the other hand, δ-type is purely goal-oriented and hence consists of only the achieving aspect. Consequently, the A-B-Δ classification can be linked to the Deep-Surface-Achieving classification 'localised' in university mathematics (see Table 2).

Bergqvist (2007) found that 70% of the tasks that university mathematics majors have to pass could be successfully dealt with imitative reasoning. This implies that β-type strategies, which are memory related, and δ-type strategies, which are goal oriented, should be preferred to α-type strategies. This was confirmed in our data: more than half (53%) of the participants' strategies were β-type, with a substantial minority being α-type strategies (32%).

Our theory would also suggest that a preference for 'achieving' (Δ_A, Δ_B and Δ_Δ) strategies would be dominant in the strategies students used if their initial strategy had been unsuccessful or incomplete, since we expected that the alternative to their first choice strategy would be more focussed on the requirements of the task. Our data support this hypothesis: 78% of these secondary strategies were 'achieving'.

Moreover, though our focus was not on the strategies used separately for each question, the nature of the question appeared to affect strategy selection in the expected ways. For example, we expected the stronger preference for β-type strategies in ‘a divides c’ (a known theorem) and for α-type strategies in ‘periodic-constant’ (taught in courses that favour image manipulations). Our data support these expectations: more than half of the initial strategies used were respectively β-type (67%) and α-type (53%).

Thus, we maintain that this A-B-Δ classification scheme can be viewed as an expansion and refinement of Weber’s differentiation amongst ‘semantic’, syntactic’ and ‘procedural’. Though further research should be conducted investigating the findings of this study, this scheme appears to have conceptually and empirically validated advantages over existing classifications. By taking a goal oriented viewpoint, instead of Weber’s focus on the representational framework of the proof, the A-B-Δ classification explicitly addresses the ascertaining/persuading contrast, and includes a ‘flexible’ strategy which is conceptually linked with the ‘approaches to study’ perspective.

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THE USE O F DIDACTICAL GAME ON MATHEMATICS TEACHING: Young children construct mathematical notions playing naval-battle

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Abstract

Research has evinced that the use of didactical game in teaching mathematics is an ideal framework to activate students. In this paper are presented and analyzed the results of a qualitative research that aimed to investigate whether primary school children while playing the game are able to conquer mathematical meanings that in the formal education are taught in higher level. Four children (3 girls about 6 years old, 1 boy about 8) collaborated playing the game of naval battle in an out of school milieu. The young children constructed, through their involvement in this game, mathematical notions such as: ordered pair, Cartesian system of axes, coordinates, and negative numbers.

1. Introduction

The searching of motivation that can make students become active learners in the learner-based model—that today is the dominant one—has emerged as a major issue in the research concerning mathematics teaching. Mayer notices: *one of the biggest problems in all formal learning is keeping students motivated enough to stick with the learning process to the end of e.g a lesson, course, semester, ...* Motivation refers to "*a student's willingness, need, desire and compulsion to participate in, and be successful in the learning process*" Bomia et al. (1997, p. 1). According to Ames (1992) motivation is connected to the goal and the evaluation of something that is important for the learner—something that is decisive for the learner's involvement. Moreover, research has revealed that in case that the involvement in mathematical tasks is of high probability to be successful, it makes the students' decision for this involvement more possible (Middleton and Spanias, 1999; Dickinson and Butt, 1989).

Research has revealed that games can activate children's interest and motivation since some characteristics of games as: competition, challenge, and fun (Bragg, 2003; Bright, Harvey, & Wheeler, 1985; Ernest, 1986; Bragg, 2007) are very attractive for students. In the corresponding literature a lot of empirical studies that have examined the effects of game based instructional programs on learning are appeared. Garris et al. notice (2002), through the results of two different researches—Whitehall and McDonald (1993) and Ricci et al. (1996)—that instruction incorporating game features was very effective for learning. More concretely, through Whitehall and McDonald research it appears that the way they used the game had as a consequence the improvement of students' performance (Garris et al., 2002, 443).

Vankúš, (2007 p. 66) analyzing data coming from history, present and researches dealing with didactical games in mathematics education concluded that the use of didactical game improves students' mathematics learning and nowadays game is a proper and important teaching method by many educational theories. What it maintains as an open question is the exploration of what games and under which situations could be effective for mathematics learning.

In the project that is presented here we explored the role of a game in the development of mathematical meanings as well as investigated whether young students could grasp notions that are being taught when being much older. We focused on the investigation of which notions could be understandable by children contributing to the discussion regarding the suitable curriculum for young children. The research recently has shown that children develop much more mathematics notions than Piaget supposed. For example, Ginsburg (2002) working with children aged 3-5 resulted that they are more competent than usually are considered. He stressed the importance of knowing what the mathematical knowledge they hold is and incorporating this information in curriculum since research and curriculum are interrelated. He sustains that exciting mathematical environments create new expressions of competence, which is not static and using Vygotsky's words he speaks about the need to study the "…*dynamic mental state, allowing not only for what has been achieved developmentally but also for what is in the* *course of maturing*" with adult assistance (Vygotsky, 1978, p. 87; Ginsburg, 2002, p. 13).

In the project we designed 4 young students are playing the Naval battle game in an out of school milieu, initially by conventional materials and later by means of a dynamic geometric environment. Here is presented the first part of the research. Among other questions the following two were of major importance for the research being conducted:

- Can this specific game activate children in the development of mathematical notions?
- Could these children understand and construct mathematical notions that according to the school syllabus are targeted to much older ones?

2. The Game as a cultural activity

Bishop refers to the playing as one of the 6 global activities –counting (delete: -), measuring, playing, designing, explaining, locating. The game (playing) is an activity that is met almost in every culture and is of interest for mathematics education as most games involve mathematics, in a more or less obvious way. Huizinga, in his classic book, 'Homo Ludens' (1949, p. 173, Bishop, 1988, p. 43), considers that *the spirit of playful competition, is, a social impulse, older than culture itself and pervades all life like a cultural ferment.*

Bishop (1988:44) speaks about the game as a schematization of the procedure of playing that is a form of representation of "playing". Similar to the way that the *counting develops number language, number imagery and number systems of speech, locating develops spatial language, images and coordinate systems….. playing seems to develop the idea of the game*.

Moreover, he considers playing as a crucial activity for mathematics development:

"*There is no doubt in my mind, nor do I hope in the reader's by now, that playing is a crucial activity for mathematical development, and I hope therefore that and the anthropological and cross-cultural data base will be filled out more to enable us to exploit educationally the significance of this universal activity for cultural growth"* Bishop (1988:48).

The game is a form of social activity of different character from any other social interaction. The meeting point between real/non real constitutes a common ground, as well as the agreement that the individuals do not act when playing "in the usual manner". Bishop (Bishop, 1988) states the following rhetoric questions regarding game's role for children's preparation for mathematics knowledge:

- Could the special characteristics of the game be used as roots for the development of hypothetical thinking?
- Could the act of playing depict the first stage, where one can discern one situation from reality, so as to reflect on it in the process and possibly devise modeling?

In the literature a lot of games traditional or not appear with a variety of characteristics. An interesting classification of games is supposed by Roth in his paper "Games, Sports and Amusements" (1902):

- Inventory: games that involve storytelling, legends etc. These games are evaluated on the basis of intelligence and humor.
- Realistic: refers on games from which pleasure is derived from real objects of nature-organic or non- such as playing with domestic animals or the slide.
- Imitational: in this category belong a great number of games and they are of two types. a) Games where aspects and objects of nature become objects of imitation by means of movements, gestures, as well as games with string. B) Games in which children imitate adult activities.
- Insight games: such as "hide and seek" and guessing games.
- Quarreling games: games as to who will prevail.
- Propulsion: games that involve motion (such as balls) and wood tossing.
- Elation games: such as music, singing, dancing (Bishop, 1988:44).

The game used here for didactical purposes is the Naval battle, a game that in a similar type is used by older children just for fun and without their realizing the mathematical notions that are involved in its construction and operation. Although this game is offered in market very often children construct this by themselves and it is traditionally played in classroom invisibly from teachers, in teaching time.

3. Didactical games in teaching of mathematics

Huizinga (Huizinga 1949, Bishop, 1988) makes an important point about the game: the game is not serious in terms of the purpose, but in performance. The performance, that is, plays a dominant role since it is connected to a reward. The game develops rules, procedures, duties and criteria. So the games are appreciated by mathematicians, because they dictate a behavior based on specific rules control- procedure similar to the ones used in mathematics. Furthermore, in many games, the aim is imitation or modeling of reality, elements particularly significant for mathematics education. All the above could justify the introduction of didactical games in the teaching of mathematics.

But what is the difference between game—with the everyday meaning—and didactical game? According to Guy Brousseau (1997) the main differences between them are (Brousseau, 1997; Vankúš. 2007 p. 54):

- normal game is totally free, in didactical game all pupils have to participate,
- didactical game is used to realize educational goals, the main aim of normal game is, just fun and pleasure,
- didactical game has its external management (teacher, rules of game).

For the description of didactical games we adopt the determination of Průcha, Walterová and Mareš (1998, p. 48) as it is appeared on Vankúš (2005, p. 370) paper.

Didactical game: *Analogy of spontaneous children's activity, which realize (for children not every time evidently) educational goals. Can take place in classroom, sport-hall, playground, or in the nature. Each game has its rules, needs continuous management and final assessment. It is suitable for single child either for group of children. Teacher has various roles: from main coordinator to an onlooker. Its advantage is motivational factor: it raises interest, makes higher children's involvement in teaching activities, and encourages children's creativity, spontaneity, co-operation and also competitiveness. Children can use their knowledge, abilities and experience. Some didactical games approach to model situations from real life.*

The value of the game as a means of education has been discussed by many philosophers, psychologists, educators etc from time immemorial.

Here we present some interesting perspectives regarding the use of game/ playing according to philosophers, educators, psychologists based on an interesting and entire historical review of Vankúš (2007).

Plato (Republic and Laws) considers as main educational method for young children the game, since it can prepare children for future job. Moreover, he stresses the role of game in doing the education playful instead of violent.

Aristotle (Politics and Ethica Nicomachea) supported the need of games in childhood, considering that game for children is the more suitable activity.

European didactic, J. A. Comenius considers the game as something very important to children since it adds a playful and joyful dimension as well as contributes to children' social skills such as cooperation.

Montessori, the Italian pedagogue, worked on an educational program for children at the age from 3 to 6 years. In her educational system main focus was on the development of whole children's personality and she used games considering them as proper for children.

Piaget classified four types of playing: exercise game (game just for joy), symbolic game (assimilation's game), game with rules (it emerges the importance of socialization and cooperation), and constructive game (they make children to undertake responsible roles). Directly through Piaget's perspective the game is important for children's socialization, development of cooperation and as a preparation for creative activities and solving of problems and hence a necessary and important part of education.

Vygotsky also stressed the role of game for socialization (Vankúš, 2007, p. 59). He asserts that the effect of the game is enormous for children's development without saying that playing is an activity just for children.

Brousseau, also in his book '*Theory of Didactical Situations in Mathematics'* refers to the notion of "didactical situation" by using the metaphorical concept of game. The didactical situation is simulated in a game determined by specific rules and roles undertaken by both students and teachers.

We close the short review regarding the role of game for mathematics teaching with the reference to The Mathematics Assistance Centre from the University Griffith (Vankúš, 2005, p. 62). This is a current and a very active centre with the researchers studying ways of game's integration to the education for the implementation of constructive way of mathematics' teaching. Through the research about the use of didactical games in mathematics teaching they realized among other things that Didactical games:

- *give for pupils a real context in that they can realize themselves fully. This supports constructive teaching.*
- *enlarge children's subjective valuation of mathematical knowledge, because this knowledge is needed for participation in the game that is wanted activity.*
- *help pupils to construct mathematical concepts by manipulation with objects in the frame of game and by verbalization of pupils' activities, thoughts and attitudes.*
- *demanded respect to rules of game. That is support for on rules based mathematicaldisciplines.*
- *are more effective if were built on mathematical ideas and for the sake of game is needed to understand certain mathematical notions and to posses certain mathematical skills.*

− *support pupils in building of new ideas. These ideas have to be defended against other players*.

The research

a. Methodological issues/ the method of the research

The realization of the complexity of the teaching/learning procedure has leaded the research into using a qualitative method analysis. As Presmeg (2009, p.134) notices the current trend in the research on mathematics teaching focuses on the integrated nature of all the aspects of being human that play out in the learning of mathematics. Burton (2002, p. 8), considers that '*as researchers in the field of mathematics education, we are part of a social science in which we are researching aspects of human experience. Such experience is embedded in social and cultural contexts that cannot be ignored whether the researcher shares, or is alien to, those contexts*.'

Also, Goodchild (2009, p.221), speaks about the researchers' need to be conscious of the paradigm within which they are working. He also notices that human behavior is so complex and dynamic that is simultaneously connected to a range of social, historical, emotional and physical phenomena so that general rules, such as those proposed by quantitative methods (positivists) are not possible to be known even if they exist.

So, in research that wants to explore in depth how the participants think, the relationships formed, and to look for interpretations, a framework of positivist orientation research doesn't provide important information.

In this case, since we wanted to study not only the results of teaching in the group of participating students but the procedure, the discourse and as well the communication that developed, we were led to naturalistic orientation research, according to Lincoln and Guba's (1985, p. 38) categorization.

Such a kind research is characterized by the fact that all the entities are in a state of mutual simultaneous shaping so that it is impossible to distinguish causes from effects.

In particular, we exploited the teaching experiment[i] (Chronaki, 2008) for the design of the activities in combination with ethnographic techniques for the observing and gathering of data. Four activities were designed and they appeared in sheets of graph paper that depicted the Cartesian coordinate system of axes. In each sheet the children had different tasks. Our data selection was based on the observation of the entire procedure and as well as the analysis of the videotaped sessions.

b. The setting of the research

The place and the time: The research conducted in an out of school milieu on April of 2010. The pragmatological material was emerged from 4 sessions, each one of about 30-45 minutes.

The participants: The participants were three girls that attended 1st grade class and a boy of 2nd class. The age of the girls was between 6 years and 2 months and 6 years and 8 months. The boy was less than 8 years old. Children were encouraged to collaborate. All of them come from families of middle economic status, while all parents are graduated. They were attending the public school of their district and they were of high performance in mathematics.

The research was designed from a Ph. D student (he is mentioned in the below dialogues as R2) and me (R1). During the research conducting we were both present. R1 videotaped and sometimes intervened while I posed the activities to children observing and facilitating the procedure. The first objective was to activate children to be involved in playing naval battle through designed activities with clear mathematical content.

The material. In this first phase we gave students sheets of millimetre paper, some sticks, and pawns in order to represent the ships.

c. The implementation of the research

To carry out the research we divided the children in two groups. In one group the two girls Maria (6 years old and 2 months) and Katerina (6 years old and 8 months) and in the other Eleni (6 years old and 6 months) and John (about 8 years old), Katerina's brother. The children sat in groups opposite in a way that prevented the eye contact to the material of the one group to the material of the other group and vice versa.

They were given 4 sheets of graph paper that depicted the Cartesian coordinate system of axes and the activities gradually became more difficult:

1st . The first paper depicted a Cartesian coordinate system of axes limited only to the positive semi axes. In the horizontal axis were plotted letters and in the vertical numbers. The students very quickly grasped the agreement, the notion of ordered pair, and wanting to identify a position mentioned first the letters and after the numbers. In this first step they had to place two ships as follows: two ships either horizontally or vertically so as to have in between them an empty box.

2nd . Here in the axes were numbers and the students had to understand the new agreement: which number to utter first. Now each pair of coordinates corresponded to a point and not to a square and the students had to put 4 boats in a square formation leaving a gap between them. To ensure that the students grasp the notion of square they were asked to form a square with sticks having at their disposal and to try to define it.

3rd . At this stage, the students had to put in a square formation 4 ships without any limitations regarding the arrangement and the number of gaps between them.

4th . In the last stage we also gave a Cartesian coordinate system complete. The ships here are "sailing" in all four quadrants.

Below we will mention characteristic excerpts from those 4 different phases and discuss the analytical categories derived either having being planned or not.

For reasons of ethics the names of the children referred to the following incidents are fictitious.

Incident 1

[1]. *R1:.Now there is no need to think hard. Make a guess. But if you find number 1 will you after have to think?*
[2]. J: *We will have to think to drop off a box and to move on*
[3]. R1: *But now, do we have John to think hard ?*
[4]. *J: No there is no need to do so. We say a number at random*

Incident 2

The new convention is that each ordered pair corresponds to a point, contrary to the first activity where each pair corresponded to a box. As well, in both axes are numbers starting from number 1, while at the origin we have not marked a number.

[5]. R1: *Here where there is no number, what do you think? What number shall we put on this dot which is not numbered?*
[6]. R2.: *0*

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[7]. R1: *John said something different, you say this as (2,3). Any other suggestions?*
[8]. M: *Yes, to move on from number 3 to number 2 and to point to location (3,2).*
[9]. R1: *So we have to decide….What must we decide?*
[10]. J: *to say first the horizontal, the ones (and points) and then the vertical the ones (again points)*
[11]. R1 : *So we have to make a deal , the first number will be from the horizontal and the second from the vertical.*

[12]. R1 : *Eleni did you understand it? If we say (5,2) which is it? (Eleni points correctly to the point*)

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Incident 3

[13]. R1: *Ок, so we can now put the ships only on the dots and we will have to put another cluster, it should be a square,* (the children with the sticks are trying to make a square)

[14]. R1: *? If it is bending like this why can't it be a square? You (Maria and Eleni) put them vertically here. If I change them and put them like this, is it still a square?*

[15]. R2: *Suppose you were not given the sticks and you were asked what a square is…. What could you say?*

[16]. J: *The one that has four angles (etymological definition)* (I show him a rectangle)

[17]. J: *It could be a rectangle. 4 angles with all sides equal to each other.*

[18]. R1 : *Just that? Look at this (a rhombus shaped) .Its sides are equal and it has 4 angles .*

[19]. R1: *is this a square?*

[20]. John: *its… sides but it's not ok …its angles (corrects)*

[21]. R1 : *So, what is the only correct way to be square ?*

[22]. John: *to be vertical (and points to perpendicular lines)*

[23]. R1: *to be vertical, so, to have …….*

[24]. John: *to be vertical and horizontal and equal too because if it is like this* (points to two different lengths) *it would be a rectangle, and if it such as well* (points to two different lengths)

[25]. R1: *So we grasped what a square is. We will put ship in a square cluster . Shall we start? Who will be the first one to say? You will use a new column to check your strikes.*

Incident 4.

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[26]. R1: *Since the one is here (4,4) where is their other one likely to be? Since they must form a square isn't so ? (Looking at their figure and pointing to possible locations)*

[27]. Maria: *(4,6)*

[28]. R1: *Did you strike anything on (4,6)?*

[29]. ……………..
[30]. John: *No , just empty waters*
[31]. R1: *Now it's your turn to strike. Hit it John, you had said (3,3) and were successful* (Eleni pointing to (1,1))
[32]. John: *(1, 1)*
[33]. R1 : *How did you think of (1, 1)?*
[34]. John: *It was Eleni 's thought*
[35]. R1: *How did you think of it?* (Shows with the finger how a square is formed)
[36]. M: *(4, 2)*
[37]. John: *Are you targeting at (4,2)? (4, 2) is correct .*
[38]. Maria: *Yes,* clapping
[39]. John: *(3, 1)*
[40]. John: *Maria correct*
[41]. John: *We won, we located the whole square!*

Incident 5

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[42]. R1: *tell us Maria* (points at first with the pencil the location and then says (1,3))
[43]. Katerina: *Is it correct or not, come on….*
[44]. Eleni: *Correct!* (Cheering)
[45]. R1 : *How many did you hit?*
[46]. Maria: *(2,3)*
[47]. Katerina: *One more and we won…*
[48]. R1: *Do you have any second thoughts on where the other one is?* (Maria initially points to a wrong location)
[49]. R : *Maria be careful, what shape must it be?* (and then pointing to the fourth vertex)
[50]. R2: *Come on pay attention you too, it's your last chance, whose turn is it?*
[51]. (Eleni after considering the figure says (3,2))
[52]. Maria and Katerina (laughing) *wrong Ha, ha*
[53]. R1: *Maybe John you should somehow think in a different way*
[54]. Katerina: *(1,4)*
[55]. John: *Correct* (Maria and Katerina clapping and laughing)
[56]. R1: That was piece of cake (I move towards John's and Eleni's side)
[57]. R1: *Let me have a look John. Did you hit all these locations? Which one is correct up to now?*

[58]. John: *points to (4,2)*
[59]. R1 : *And you were looking the whole time like that? (meaning horizontal) because yours were like that as well , is it not possible to be like this? To be here too?*
[60]. John: *We were looking there as well*
[61]. R1: *Yes, but you were looking like this, at this side the big one and at that side the small one as yours. Could that be the reason?*
[62]. R1: *Did we have such a constraint where to be the small and where the big one?*

Incident 6.

I hand them graph paper with 4 quadrants and ask them how to head to a specific location. Maria puts her hand and initially says (1,1)

[63]. R1: *How shall we say this location Eleni?*
[64]. Eleni: *(2,2)*
[65]. Maria: *No, (2,1)*
[66]. R1: *Right?*
[67]. *Yes* (they all say)
[68]. R1: *But the ship is not restricted by none at this sea, it can go there and there…*
[69]. R1 : *If the ship is at this location (-1, 0) how shall we call that location? Don't say it John, all must think. This is 0 , this is 1 and the ship sails to that location, what must we put here? How shall we call that? Any good ideas?*
[70]. Katerina: *O*
[71]. R1: *O is here, if it sails here we call it 1, if it sails there how do we call it? Don't say it John!*
[72]. John: *I haven' t found it*
[73]. R1: *A*
[74]. R1: *Maria could we say it 4?*
[75]. R1: *4 is this, how shall we make it clear to someone, look here* (I stand up and move right and left). *If I move towards here I say it, If towards there how do I say it?* (Maria mimics my footsteps)
[76]. R1: *Think of a way to describe the numbers that go backwards*. (Maria and Katerina go away to think of it)
[77]. (I show them the graph paper and tell them that there lies their solution to their problem). *How will someone understand that he is at this point?*

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[78]. R2: *Let's say it in another way: At 0 is an airport and these are houses (showing the points) and I must visit this house* (pointing at(-1,0))
[79]. R1 : *How can we describe it to someone to understand it?*
[80]. John: *(1 1,)*
[81]. R2 : *Then, when I arrive at the airport I will go to house 1 and I won't reach here, I also carry a box of sweets and they will spoil…*
[82]. John: *5*
[83]. R2: *If they go at 5, I will count 1 2 3 4 5! (*at the graph paper the x-axis is until 4 , that's perhaps why he said to 5*) and move to 5… the sweets will end to a different owner.*
[84]. John: *But there is not a house as this one is, which has houses till number 4*
[85]. R1: *It goes on , we just didn't plot more numbers , think how to describe to move to this location*
[86]. John: *I thought of it, it was difficult, the young ones won't find it*
[87]. R1: *tell us*
[88]. Katerina: *It is below 0, since here is 0 and we move even more down*
[89]. R1: *How will you say this? 1 below 0?*
[90]. Katerina: *We will say it minus 1*
[91]. R1: *How will you say it?*
[92]. Katerina: *minus* 1
[93]. R2: *Have you heard about minus?*
[94]. Katerina: *Yes*
[95]. R1: *Where from?*
[96]. Maria: *At school perhaps? At school where we do subtraction, we have learnt subtraction*
[97]. R1: *And how would you say this? (point to -2 on x-axis) is* that -1
[98]. Katerina: *-2*
[99]. R1: *Do you agree? Eleni, did you understand what Katerina has just said?*
[100]. R2: *How did you have the idea?*
[101]. Maria: *I helped her too* (who just mentioned subtraction that they had learnt at school).
[102]. Katerina: *I had the idea because it is below 0*
[103]. R1: *And how come is it minus if it is below 0?*
[104]. Katerina: *Otherwise, how could we distinguish it?*
[105]. R2: *How did you have the idea of minus? Does minus remind you of something?*

[106]. R1: *Have you heard it somewhere else? Have you ever listened on TV about the temperature?*
[107]. John: *When there is snow they say -1, -2, -3*
[108]. R1: *Maria have you heard of it ?*
[109]. Katerina: *I think I have seen it in our car*
[110]. R1: *So, what's in the car? What's in the car that you have noticed?*
[111]. John: *It counts Celsius degrees*
[112]. R1: Do you have such in your car that counts degrees?
[113]. Katerina: *Yes* (and adds -1, -2 etc) *but it goes and above 0.*

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(Maria writes the numbers on her own, she writes -2)
[114]. Maria: *here, I know -3*

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[115]. John: *so as not to get mixed up we can also add down here* (meaning the
y-axis)
[116]. R1: *So what would you put here?*
[117]. John: *-1, -2, -3.*
[118]. R1: *Maria and Katerina shall we also put here some numbers? What shall we put here? Hand me the marker John*
[119]. John: *Can I first put mine, -1-2—3 , I can't put -4*
(The young ones also add their numbers and we ask them to place their ships in the form of square clusters)
[120]. R1: *You can place wherever you like but in pairs* (John and Eleni choose locations (-1,-1) (-3,-1) (-3,-3) (-1,-3))
[121]. John: *and we shall continue saying first the horizontals*
[122]. ……………….

d. Analysis

In an out of school milieu a rich mathematical environment— since a lot of mathematical notions have appeared— was offered in a group of early primary school students. The children, through the game of naval-battle had to negotiate mathematical notions that are not included in the corresponding curriculum. Particularly there is a reference in the text book of 1st grade regarding just the use of quadrangular paper for horizontal or vertical movements. Also there are references on the book of 5th and 6th grade about ordered pair for graph's teaching purposes. The teaching of complete system of orthogonal Cartesian coordinates as well as of negative numbers is not included in any class of primary school.

Part of the categories we used for the analysis of the didactical episodes had been decided before the implementation of the research, while some other emerged through the study of the transcribed discussions. Due to limitations of the paper we attempt a first analysis based on these categories with the note that these are not completely different, but we separate them in this way for analysis purposes.

Mathematical notions and didactical game

As we can see, studying the above episodes students dealt with a lot of mathematical notions involved in the playing of naval-battle. We studied the development of particular mathematics notions that they already knew and their capability to construct new notions under particular situations. Students' interest was motivated from the joy they got playing and trying to win in this game.

Among other notions students developed and they seemed to deal with the notions of: ordered pair, Cartesian system of coordinates, the position for zero, negative numbers, and the determination and properties of quadrangle. Another important notice is the development of effective strategies—very important ability for problem solving.

Ordered pair and Cartesian system. In a lot of lines we can see how effectively children used order pair (see 7-12 lines). It is very interesting the moment they move from the axes with letters to one and numbers to the other to a system that has only numbers in both axes (lines 10, 11) and they conceive the convention of ordering the numbers of axes.

The position of zero and negative numbers. As we can see in lines 15, 16 the younger girl spontaneously conceived that the point in the beginning of the axes was zero's position. Also through this dialogue we could see how Katerina leaded to negative numbers.

Katerina: *It is below 0, since here is 0 and we move even more down*

R1: *How will you say this?*

Katerina: *1 below 0? minus 1!*

Quadrangle. In incident 3 students after discussion and interaction clarified the properties of the quadrangle, although initially had a confused perception of it.

Using strategies, keeping the rules and metacognition

In all activities students had to put the ships in a particular formation. Children conceived the idea of the rule they have in any activity and applied it to put their ships. For example in lines 2, 33-41 it is obvious that students developed strategies to find the rest of the ships. In cases they didn't work children modified them, sometimes after discussion (for example 48, 48) developing in this way metagcognitive skills as well.

Collaboration and ZED

The following pictures depict both groups to communicate through gestures, and in this way to collaborate searching for the solution of the problem posed.

The role of their collaboration and the intervention from adults was very important in this procedure. We considered that Vygotsky's ideas regarding the role of the adults in assisting children to perceive new notions—zone of approximate development—as well as the role of social interaction were asserted in the present work. Students dealt with notions that are not expected for their ages. About all of the above mathematical notions are not supposed from the curriculum not only for their class but for all the classes of primary school. The motivation of the game and our role as researchers who designed the activities and facilitated their construction of knowledge had as a result students to acquire this particular knowledge.

During the total procedure students were encouraged to express their ideas and to discuss them. It is of great interest the 3rd incident discussed above regarding the quadrangle.

Use of informal cognition and negative numbers

In incident 6 we also can see an interesting discussion that resulted students to conceive negative numbers' notion. Children here in an obvious way mobilized informal knowledge, knowledge of everyday life to conceive negative numbers:

[123]. Katerina: *I think I have seen it in our car*

[124]. R1: *So, what's in the car? What's in the car that you have noticed?*

[125]. John: *It counts Celsius degrees*

[126]. R1: *Do you have such in your car that counts degrees?*

[127]. Katerina: *Yes* (and she adds -1, -2 etc) *but it goes and above 0.*

Conclusion

The above research was designed in order to answer if and to which extent young children could deal with important mathematical notions, that are considered suitable for older children, in case a didactical game is offered as a mediation tool. The game we used according to the categorization of Roth mentioned above could be characterized as imaginative, imitative, discriminative, disputative and propulsive. On the other side, as we can see from the above quotation and the analysis the use of the game responses to every category that is referred by Piaget. It was a game for exercise that gave joy to the children; it was a symbolic game; a game with rules that encouraged the socialization and cooperation; a constructive game that led children to undertake responsible roles.

Children in order to participate in the game were very willing to accept and keep the rules and conventions every time they were posed—something useful not only for playing purposes but also for mathematics learning. They were excited all time long they dealt with the activities and became happy in case they had a successful attack to the other group ships. The development of effective strategies made them happier than when succeeding by chance.

Through playing with naval battle they were motivated to develop skills in order to describe their observations, to formulate conjectures and to construct new mathematical notions as a result of interaction. All the children were encouraged to be involved in playing this game developing creativity spontaneity, co-operation as well as competitiveness.

Through our observation in a naturalistic context we had the chance to explore the way they developed their thought, the way they collaborated in exchanging mathematical notions in a creative way that resulted in conceiving mathematical notions ascertain the hypothesis that knowledge is a result of social interaction—here the interaction among children and between children and the adults that had the responsibility.

Concluding, we think more research is needed for exploring the use of other didactical games for mathematics teaching/learning not only in an out of school context but at school as well.

Acknowledgement. I would like to thank Vasilis Tsitsos for his help in the implementation of the research as well as Eirini Kouletsi for her help on the text.

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[i] The teaching experiment is a method to study the interrelated relationship between learning and teaching and is presented as an alternative suggestion to educational research, which is critically positioned towards positivist tradition. As a method it is developed through the need to study the very development process of learning and not merely the result (eg. The students' performance).

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The Use of New Technologies and the Project Method in Teaching Statistics: A Case Study in Higher Education

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Abstract

The purpose of this assignment is to show the importance of the use of New Technologies and the Project Method in the teaching of Statistics in the Department of Primary Education. For this purpose a research was conducted in the Department of Primary Education in the University of the Aegean, Greece, for which the questionnaire SATS of Candace Schau was used. The New Technologies that were used involved the construction and the use of a website and a blog, the use of Projects, teaching in a computer laboratory with statistical software and applications of internet (freeware, applets, etc). The results of the research indicated that by using New Technologies and Project Methods in teaching Statistics the attitudes of students towards Statistics improved.

Keywords: Statistics, Attitudes, Teaching, Project Method, New Technologies

1. Introduction

The progress of New Technologies caused many changes in teaching Statistics [20]. The new ways of teaching focus on the understanding of the statistical notions using learning techniques and real life elements [12]. The applications of Information Technology are very important in Statistics. Most of the times what matters is the correct software use, instead of memorizing.

Internet use has enabled the introduction of freeware for statistical analysis in the teaching of Statistics and in using innovations like applets that enable the simulation of statistical concepts. University students are more interested in the educational procedure through the use of computers, multimedia and statistical software [19].

Spreadsheets can be used in the construction of tables and charts, in the instant calculation of statistical functions. They are widely used, they are not very expensive and they are popular with students and teachers. The statistical packets (SPSS, SPAD etc) are very important tools for the implementation of research and its findings and they are part of the curricula of universities. Furthermore, the freeware and applets that are free on the internet like the SL Gallery of the Stochastic Lab[1], the simulation software of VIAS[2] and the java applets of RVLS[3] are all free and user friendly as well as easy to download.

2. The construction of websites in teaching Statistics

The construction of websites is an innovative approach to teaching in order to achieve contact with the data of everyday life [18]. The use of new teaching methods via the internet is more accessible to both students and teachers. The website construction and the entry of data that have been collected by the students enhance the exchange of information and the cooperation among classes and schools [15]. They also enable the teacher to retrieve all the information that was used in the past in order to use it in a new Project. All the data that will be registered may concern a research that will include the definition of a problem, the measuring and collection of data, coding, analysis and finally the presentation of the results of the research. This procedure that includes real life data is extremely significant for the development of statistical thought [13].

[1] The SL Gallery v1.3 of SL (Stochastic Lab) is a tool of estimating probabilities, constructing charts etc. The SL Gallery is a freeware 480 kB that is downloadable by anyone, http://www.stochastic-lab.com/slgallery.html

[2] The Discrete distributions, Measures of location and the Central limit theorem of VIAS (Virtual Institute of Applied Science) are freeware that can be downloaded from http://www.vias.org/simulations/simusoft_ discretedistris.html

[3] Java applications of RVLS, Rice Virtual Lab in Statistics, http://onlinestatbook.com/rvls.html

3. Project type assignments in teaching Statistics

The teacher of Statistics should pay attention to the development of statistical abilities and statistical thought, the use of real data, the fundamental concepts, the development of active learning, the use of New Technologies and, last but not least, the use of measurements for the estimation of learning [8], [24]. The process of learning, which is based on pursuing initiative actions, enhances the understanding of fundamental notions, cooperation and teamwork [4], [16]. Each student learns in his/her own way, but in an active environment of learning, all students can participate based on the way they comprehend learning [9], [23].

The designing and implementation of a Project and the use of New Technologies is the best method to achieve the above. Through these actions, Statistics is considered as an experimental science and less as a traditional subject [11]. In this way, both the interest of the students and their motives for learning are increased, even for students with low interest in mathematics [10], [22].

The first concern of the teacher of Statistics is to intrigue students' interest and to achieve their active participation in the learning process. This can be achieved through the introduction of Project type assignments in the teaching of Statistics [2]. The Projects are very important for the understanding of statistical concepts and the development of statistical thought that will be applied in real life situations [28]. A fundamental part of these Projects is the provision of opportunities and the encouragement of students to construct their knowledge through experimentation, hypothesis formulation, generalization and justification. Through the strategy of active learning, students can learn Statistics through designing research, as well as gathering and analyzing data and presenting their findings etc [7], [27].

4. The procedure of research

During the winter semester of the academic year 2009-2010, a research was conducted in the Department of Primary Education of the University of the Aegean, Greece. The research concerned students' attitude towards Statistics before and after that taking the subject of Statistics, especially with the use of New Technologies and the realization of Projects and the assistance of modern software and the Internet. Within this framework, easy accessibility is the most important criterion for the selection of the software that is going to be used by the students. Such software is Microsoft Excel, the freeware SL Gallery, the simulations of the Virtual Institute of Applied Science, the java applications of Rice Virtual Lab in Statistics and the SPSS v.17. Afterwards, worksheets were written and given to the students during the lessons. The aim of the worksheets focused on teaching Statistics with easy-to-use software. The lessons took place at the University's computer lab. Within the teaching framework, the basic notions of the theory were presented with the use of PowerPoint. After teaching theory, this theory was applied in the computer lab choosing one of the above worksheets each time.

A website has also been constructed[4]. In this website, the notes of the lessons are presented, as well as the worksheets, the applications from the Internet, the PowerPoint presentations, the connections to useful webpages, and so on. In the framework of the interaction among the educators and the learners but also on the basis of a general communication, a blog was constructed, in which the students had the chance to express their thoughts about lesson planning. They also had the chance to make proposals and answer questionnaires online. The website included webpages in which the basic points of the teaching of Statistics in the Greek Primary Education were presented.

During the semester, a Project was given to the students who attended Statistics. The theme of the Project was "the air temperature in Rhodes" that is included in the curriculum of Mathematics of the 5th grade of Primary school [5], [7]. The Project was about recording the air temperature and then the students analyzed the data with statistical methods [14]. The students who conducted the Project recorded the air temperature of a day every three hours and they made a table, from which they were able to extract the average daily temperature and the chart of the development of temperature. After the results of the average daily temperature, the students were asked to find the average daily temperature of the corresponding month and to compare the results. Afterwards, they collected the average temperatures of six cities including Rhodes for a winter and a summer month. The students made charts and graphs [26]. Data and instructions for the students could be found from meteorological stations, from the blog that was constructed in the framework of the lesson, from the website of the Greek National Meteorological Service[5], and so on [18].

For the research, we used the Candace Schau's SATS-36 (Survey of Attitudes Towards Statistics) questionnaire after getting authorization [21].

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[4] http://www.rhodes.aegean.gr/ptde/statistiki/

[5] Meteorological data from the Meteorological Station of Paradisi Rhodes, http://www.hnms.gr/hnms/ greek/forecast/forecast_city _ html?dr_city=Rodos

The SATS-36 refers to two questionnaires: the PRE-SATS which is handed out before the didactic interventions and the POST-SATS which is handed out afterwards. The two questionnaires provide the results for six factors that demonstrate the respondents' attitude towards Statistics: (a) Affect, (b) Cognitive competence, (c) Value, (d) Difficulty, (e) Interest, and (f) Effort. Component (subscale) scores on the SATS-36 are formed by reversing the responses to the negatively worded items (1 becomes 7, 2 becomes 6, etc.), summing the item responses within each component and dividing by the number of items within the component. The possible range of scores for each component is between 1 and 7. Using the 7-point response scale, higher scores correspond to more positive attitudes.

The six factors of student attitudes towards Statistics include 36 items in all and they are the same for the two questionnaires PRE and POST: (a) Affect: 6 items, (b) Cognitive competence: 6 items, (c) Value: 9 items, (d) Difficulty: 7 items, (e) Interest: 4 items and (f) Effort: 4 items. Apart from these items that refer to the aforementioned six factors, there are also questions about students' Mathematical ability, as well as questions of demographic and academic interest. For extraction of results the SPAD v.4.5 and SPSS v.17 software were used.

5. The results of the research

In the research, 126 students (19 men and 107 women) participated. All the 126 students completed a PRE-SATS questionnaire at the beginning of the semester and the POST-SATS at the end of the semester. Of them, 3 (2.38%) were in the 1st semester, 63 (50.00%) in the 3rd semester, 56 (44.44%) in the 7th semester, 2 (1.59%) in the 9th semester, 1(0.79%) in the 11th semester and 1 (0.79%) was in the 13th semester. From the 126 students only 2 (both of whom were at the 7th semester) had already taken Statistics during their studies at the Department of Primary Education in a previous year.

The 113 students were Greek (89.68%) and the rest13 (10.32%) were Cypriots. The age of the majority of the students ranged between 18 and 22 years (105 students, 83.3%), 10 students were between the 23rd and the 27th year of age (7.94%), 8 students were between 28th and 32nd year of age (6.35%) and 3 were older than 30 years of age (2.38%).

A reliability analysis was conducted for the factors of SATS, at the beginning and the end of the semester with the Cronbach's alpha criterion (Table 1).

Table 1: Reliability analysis with Cronbach's alpha

Attitudes	**Sample item**	**Cronbach's a**
Affect-PRE	I like Statistics	0.79
Affect-POST		0.84
Cognitive competence-PRE	I can learn Statistics	0.82
Cognitive competence -POST		0.86
Value-PRE	I use statistics in my everyday life	0.81
Value-POST		0.87
Difficulty-PRE	Statistics formulas are easy to understand	0.57
Difficulty-POST		0.54
Interest-PRE	I am interested in using statistics	0.79
Interest-POST		0.88
Effort-PRE	I plan to attend every statistics class session	0.77
Effort-POST		0.77

All the Cronbach's alpha values are acceptable apart from those of the attitude Difficulty PRE and POST, which are quite low (0.57 and 0.54 respectively). The aim of research was to note whether students' attitude towards Statistics improved after the teaching of the subject with the use of New Technologies and the Project method [1]. Within this framework, there was a Paired samples T-Test of students' attitude, as these attitudes are shown from the results of the research with the questionnaires of the PRE-SAT and POST-SATS. Cohen's d effect size was also estimated, which shows the effect size correlation (Table 2).

Table 2: Attitudes comparison

Attitudes	PRE-SATS		POST-SATS		Sig. (2-tailed)	Cohen's d
	Mean	Standard deviation	Mean	Standard deviation		
Affect	4.62	1.13	5.37	1.18	0.000	0.65
Cognitive competence	5.12	1.14	5.79	1.07	0.000	0.60
Value	5.38	1.03	5.83	1.02	0.000	0.44
Difficulty	3.82	0.78	3.90	0.70	0.234	0.10
Interest	5.71	1.04	5.90	0.97	0.051	0.18
Effort	5.90	0.95	5.91	1.00	0.857	0.01

From the data that are presented in Table 2, it can be noted that there is a statistically significant difference between the PRE-SATS and the POST-SATS as far as the factors of Affect, Cognitive competence and Value are concerned. Some differentiation can also be discerned in the factor of Interest but not so substantial. In the factors of Difficulty and Effort there is no statistically significant difference before and after the educational intervention with New Technologies and Project Method.

An analysis of variance with the use of General Linear Model (GLM) of repeated measures was conducted. The attitudes were placed in pairs, PRE and POST, in "within subjects factor" and Gender and Mathematical Ability were put in "between subjects factor". The aim was to identify the extent to which the factors of Gender and Mathematical ability influenced the educational intervention. The result of the analysis is that the intervention was not affected either by the Gender or by students' Mathematical ability.

5.1. The results of Multiple Correspondence Analysis

Then we wanted to see how the students, who participated in the research, vary based on the total answers. For this purpose we used a method of Data Analysis, the Multiple Correspondence Analysis, based on the correlation of all variables simultaneously [3]. The results of the Multiple Correspondence Analysis give the factorial axes which are the differentiation criteria of the students.

1st criterion of student differentiation (1st factorial axis - 7.71% of the total inertia)

The factor that differentiates mostly the students who participated in the research is the upper and lower levels of their attitudes towards Statistics. On the one hand, there are students with very positive attitudes referring to (a) Affect, (b) Cognitive competence, (c) Value, (d) Interest and (e) Effort, who do not face any difficulty in Statistics, have high mathematical level, expect to obtain an excellent grade[6] in the final exams of the lesson and believe that the Project Method and the New Technologies facilitate the comprehension of the subject of Statistics. These students and their answers are differentiated from their colleagues' answers who have negative attitudes towards Statistics and low mathematical knowledge and who do not expect to pass the final exams.

2nd criterion of student differentiation (2nd factorial axis - 4.35% of the total inertia)

In the second criterion of differentiation, on the one hand there are students who show positive attitudes towards Statistics, according to the factors of (a) Affect, (b) Cognitive competence, (c) Value, (d) Interest and (e) Effort and do not face any difficulty in Statistics, but have low mathematical level, and do not expect to get a good grade in the final exams. Moreover, they do not believe that the Project Method is helpful in the understanding of Statistics. On the other hand, there are students with an average level of attitudes towards Statistics according to the above factors and an average level in Mathematics, who seem to face some difficulty in the subject of Statistics and do not believe in the usefulness of the Project Method in the teaching of Statistics.

3rd criterion of student differentiation (3rd factorial axis – 3.64% of the total inertia)

The third criterion that differentiates the students of the sample is Origin. On the one hand, there are Cypriot students that are up to 22 years old, with more or less positive attitudes towards Statistics and average mathematical ability, who do not really believe in the usefulness of the Project Method and the use of New Technologies in Statistics. On the other hand, there are Greek students with ages that reach the 33 years of age, with low mathematical knowledge, who they believe in the positive results of Projects

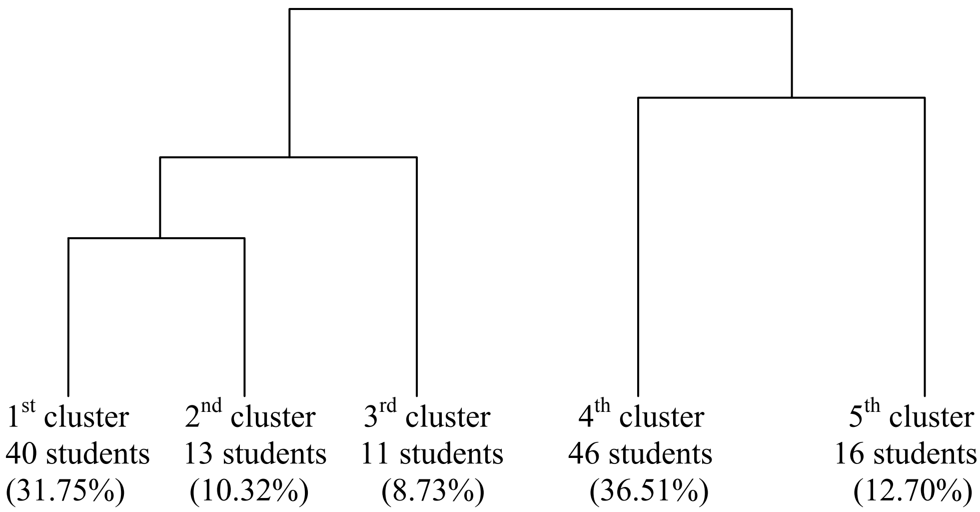
[6] The grades of the students in the Greek Universities vary from 0 to 10, with 10 being the excellent grade.

in the understanding of Statistics and they expect to marginally pass the final exams, although they have difficulty in the realization of the Project.

5.2 The results of Hierarchical Clustering

Finally, we used another method of Data Analysis, the Hierarchical Cluster Analysis, to see how students of the sample can be divided into clusters based on their common answers [3]. The Ward's test was used.

The Hierarchical Clustering[7] led to five clusters. The clusters, their corresponding number of the students and the corresponding percentage are indicated in the Graph 1.



Graph 1: Hierarchical Clustering

The values of the variables that determine the clusters are presented to the following.

1st cluster: 40 students (31.75% of the sample)

The 1st cluster is comprised of Greek students who have average Cognitive competence in Statistics and face some difficulties in Mathematics but they do not have such a difficulty in the Project they were given and they expect to take 7 in the final exams of the subject.

[7] The Hierarchical Clustering was realized with the help of the statistical software SPAD v.4.5.

2nd cluster: 13 students (10.32% of the sample)

The students of the 2nd cluster do not show enough positive attitudes towards Statistics as far as the factors of Cognitive competence, Value, Interest and Effort are concerned and they believe less in the usefulness of Projects and the use of New Technologies in the subject of Statistics.

3rd cluster: 11 students (8.73% of the sample)

In the 3rd cluster there are students who do not have adequate mathematical knowledge, do not have positive attitudes towards Statistics and they expect to marginally pass the final exams of Statistics.

4th cluster: 46 students (36.51% of the sample)

The common element of the students of the 4th cluster is that they have good mathematical knowledge and they expect to get 9 in the finals. They have comprehended the basic principles of Statistics and they have a positive attitude towards the lesson.

5th cluster: 16 students (12.70% of the sample)

The students who comprise the 5th cluster have very positive attitudes towards Statistics, excellent mathematical knowledge and expect to get 10 in the finals.

The last two clusters, although they have different characteristics, they also have a common point: the good knowledge of Mathematics and the positive attitudes towards Statistics. In contrast, in the first three clusters there are students with more limited mathematical knowledge and not positive attitudes towards Statistics.

The attitudes towards Statistics, as they are formed per cluster, are shown in the Table 3. The results of variance analysis ANOVA with the use of Tukey test are also presented in the same Table. The means that don't have any common symbol on their exponent quite differentiate among them.

Table 3: The attitudes towards Statistics per cluster

Attitudes	1st cluster. N=40		2nd cluster. N=13		3rd cluster. N=11		4th cluster. N=46		5th cluster. N=16	
	Mean	Standard deviation	Mean	Standard deviation	Mean	Standard deviation	Mean	Standard deviation	Mean	Standard deviation
Affect-PRE	3.95^{a}	0.85	4.00^{a}	0.91	3.82^{a}	1.17	5.15^{b}	0.82	5.81^{b}	0.91
Affect-POST	4.90^{b}	0.96	4.77^{b}	0.93	3.91^{a}	0.94	5.85^{c}	0.99	6.63^{c}	0.50
Cognitive competence-PRE	4.53^{a}	0.85	4.46^{a}	1.20	4.18^{a}	0.98	5.59^{b}	0.83	6.44^{c}	0.73
Cognitive competence-POST	5.50^{b}	0.75	5.08^{b}	0.76	4.00^{a}	0.89	6.28^{c}	0.81	6.88^{c}	0.34
Value-PRE	5.35^{bc}	0.70	3.77^{a}	0.93	5.09^{b}	1.04	5.70^{bc}	0.84	6.06^{c}	1.00
Value-POST	5.98^{c}	0.8	4.15^{a}	0.69	5.18^{b}	1.25	6.13^{c}	0.75	6.44^{c}	0.63
Difficulty-PRE	3.58^{ab}	0.64	3.92^{ab}	0.86	3.27^{a}	0.65	4.02^{b}	0.80	4.13^{b}	0.81
Difficulty-POST	3.90^{ab}	0.50	4.15^{b}	0.69	3.36^{a}	0.67	3.91^{ab}	0.81	4.06^{b}	0.68
Interest-PRE	5.68^{bc}	0.97	4.54^{a}	0.78	5.36^{ab}	1.36	5.89^{bc}	0.9	6.44^{c}	0.63
Interest-POST	5.88^{b}	0.82	4.31^{a}	0.48	5.55^{b}	1.21	6.15^{bc}	0.67	6.75^{c}	0.58
Effort-PRE	5.68^{a}	0.83	5.54^{a}	0.88	5.36^{a}	1.36	6.02^{ab}	0.83	6.75^{b}	0.77
Effort-POST	5.98^{b}	0.92	5.08^{a}	0.95	5.00^{a}	0.63	6.11^{b}	0.97	6.50^{b}	0.82

The Diagram of the Correspondence Analysis is shown below. In this Diagram the attitudes are coded as follows: Aff (Affect), Cog (Cognitive Competence), Val (Value), Dif (Difficulty), Int (Interest), Eff (Effort). The scale of acceptance of these attitudes is between 1 (strongly disagree) and 7 (strongly agree). The high results correspond to positive attitudes.

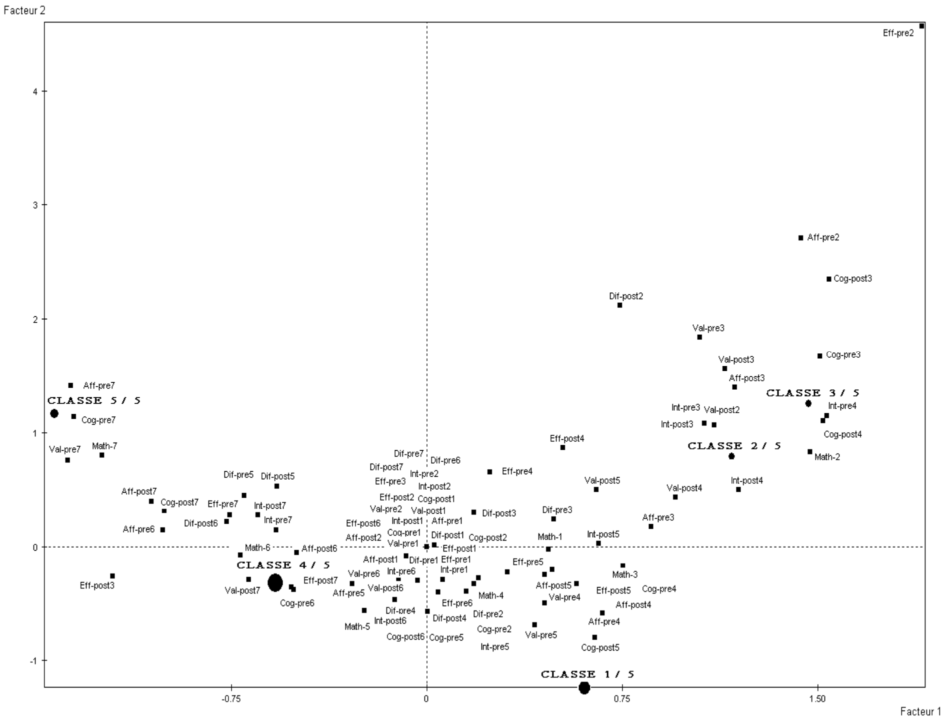


Diagram 1: Correspondence Analysis[8]

6. Conclusions

The purpose of this assignment was to demonstrate the significance of the use of New Technologies and the Project Method in the teaching of Statistics in the Departments of Primary Education. For this purpose, a research was conducted in the Department of Primary Education in the University of the Aegean, Greece, for which the questionnaire SATS of Candace Schau was used. The New Technologies that were used involved the construction and the use of a website and a blog, the use of Projects, teaching in a computer laboratory with statistical software and applications of the Internet (freeware, applets, etc).

In the research, 126 students participated. Most of the students who participated in the research were female (84.92%) that is a common feature of a Department of Primary Education [24]. All the 126 students completed a PRE-SATS questionnaire at the beginning of the semester and the POST-

[8] The Diagram of Correspondence Analysis was realized with the help of the statistical software SPAD v.4.5.

SATS at the end of the semester. The 113 students are Greek (89.68%) and the rest 13 (10.32%) are Cypriots. The age of the majority of the students ranged between 18 and 22 years (105 students, 83.3%), 10 students were between the 23^{rd} and the 27^{th} year of age (7.94%), 8 students were between 28^{th} and 32^{nd} year of age (6.35%) and 3 were older than 30 years of age (2.38%).

A reliability analysis was conducted for the attitudes of SATS, before and after the intervention with the Cronbach's alpha criterion. All the Cronbach's alpha values are acceptable apart from those of the factor Difficulty PRE and POST, which were quite low.

The research shows that there is a statistically significant difference between the PRE-SATS and the POST- SATS as far as the attitude factors of Affect, Cognitive competence and Value are concerned. Some differentiation was also seen in the attitude of Interest but no so substantial. In the attitude factors of Difficulty and Effort, there is no statistically significant difference before and after the educational intervention with New Technologies and Project Method.

An initial implementation of Project Method in the teaching of Statistics was presented at a research of Andreadis and Chandjipantelis [2]. The research was about the intervention in the teaching of Statistics with a group Project and its influence on students' attitudes towards Statistics. The Candace Schau PRE-SATS and POST-SATS questionnaires were also used in this research and the results showed that there was an improvement in the attitudes of students towards Statistics. Seeing things from a similar perspective, we created a new research which differs in many aspects (e.g. size of the sample, use of internet etc).

An analysis of variance with the use of General Linear Model of repeated measures was conducted. The analysis shows that the intervention was not affected either by the Gender or by the Mathematical ability of the students.

The Hierarchical Clustering led to five clusters. The last two clusters, although they have different characteristics, they also have a common point: the good knowledge of Mathematics and the positive attitudes towards Statistics. In contrast, in the first three clusters there are students with less mathematical knowledge and not positive attitudes towards Statistics.

New Technologies in Teaching Statistics can further be applied to Primary and Secondary Education. The supply of interactive whiteboards in public schools of Greece, which starts in the academic year 2010 – 2011, gives a new interest in the research in the use of New Technologies in

Teaching Statistics. The new applications, their acceptance by the educational community, the teachers' training, the students' preparing through the Teaching Departments, the effectiveness of the new methods, the technical and operational problems that may arise, etc. constitute a major challenge for future researchers who will continue the study on the use of New Technology in Teaching Statistics.

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