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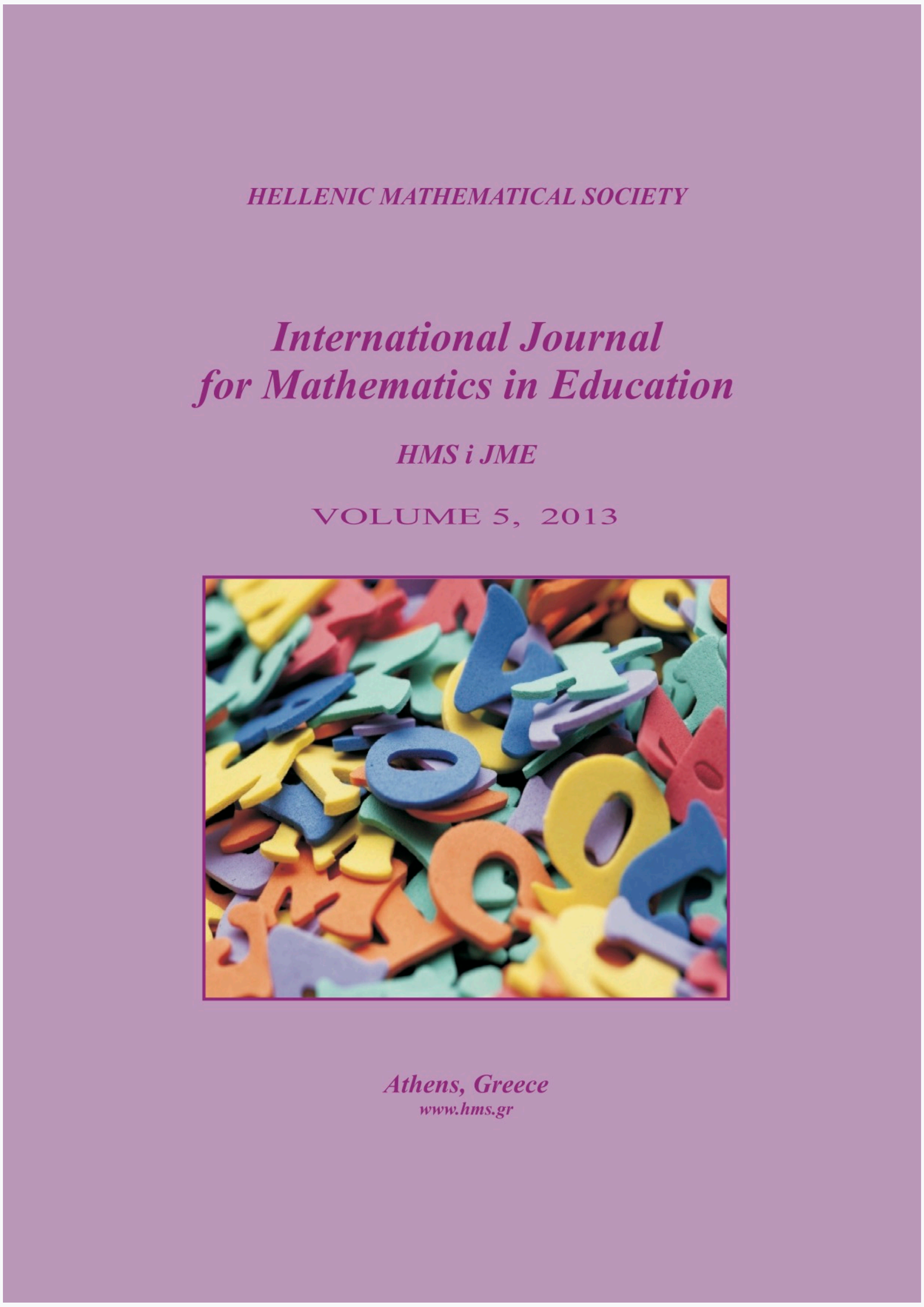

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International Journal for Mathematics in Education (*HMS i JME*)

The Hellenic Mathematical Society (HMS) decided to add this Journal, the seventh one, in the quite long list of its publications, covering all aspects of the mathematical experience. The primary mission of the HMS International Journal for Mathematics Education **(*HMS i JME*)** is to provide a forum for communicating novel ideas and research results in all areas of Mathematics Education with reference to all educational levels.

The proposals must be written almost exclusively in English but may be admitted, if necessary, also in French, in German or perhaps in Spanish.

The proposals could concern: research in didactics of mathematics, reports of new developments in mathematics curricula, integration of new technologies into mathematics education, network environments and focused learning populations, description of innovative experimental teaching approaches illustrating new ideas of general interest, trends in teachers' education, design of mathematical activities and educational materials, research results and new approaches for the learning of mathematics.

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NOTES FOR AUTHORS

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The first page should contain the author's e-mail address and keywords. Papers should be written in English but also, if necessary, in French, in German or in Spanish.

The format of the manuscript: Manuscripts must be written on A4 white paper, double spaced, with wide margins (3 cm), max 20 pages, Times New Roman 12pt. Each paper should be accompanied by an abstract of 100 to 150 words. References cited within text (author's name, year of publication) should be listed in alphabetical order. References should follow the APA style (http//:www.apastyle.org/pubmanual.html).

The submitted papers are reviewed by two members of the International Scientific Committee (blind review). Two positive reviews are necessary for a paper to be accepted.

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Introduction

The existence of many and good scientific international journals on Mathematics Education creates an interesting multivariable space, in which the variety of epistemological views influences the local innovative experiences and vice versa the exchange of the local or the networking experience emerges in new epistemologies in the international level. The complexity of the mathematics education space is also characterized by the growing market of mathematical educational products and services and its antagonistic and complementary relation with the formal education.

In general it is accepted that the improvement of mathematics teaching and learning in the entire world cannot be realized in a homogenous way. We have to work in a different mode the international comparative studies, profiting of the alternative views, the ecosystemic educational research, the digital and networking environment as well as the ethnomathematical approach.

The International Journal for Mathematics in Education (HMS i JME) wishes to contribute in this orientation and we will try to incorporate in its content a number of international comparative studies or results of international conference proceedings as well as studies of the use of materials in the formal or informal learning and teaching of mathematics. We are inviting our colleagues of the international community to send us their contributions.

The fifth volume of the HMS International Journal for Mathematics in Education includes seven research papers.

The first paper written by Marta Ginovart "A simple discrete simulation model to explore in the classroom some rules involved in the decision-making process: the cellular automata" presents teaching material elaborated to work with a very simple discrete model, the cellular automata called "Voting" (free access from the social science section of NetLogo library), and its use in the classroom.

The second paper written by N. Karjanto and R. Osman "On the Transition Period of Implementing New Mathematics Curriculum for Foundation Engineering Students" makes an overview on several mathematics modules in the transition period of introducing a new curriculum for the Foundation program in Engineering at the University of Nottingham Malaysia Campus.

The third paper written by Georgios Kosyvas “Problemes ouverts dans la classe” focuses on the engagement of college students with the solution of three open problems.

The fourth paper “Investigation of possible changes into teaching and learning practices in primary school within the framework of interdisciplinary projects for mathematics teaching” by Lazaridis Ioannis describes the observation of four teachers who implemented interdisciplinary projects in mathematics teaching and how the qualitative analysis of the data lead to common subject categories.

The fifth paper written by Chrysanthi Skoumpourdi “Kindergartners’ performance on patterning” aims to investigate the ability of the kindergarten children (5-6 years of age) to extend and reproduce different types of patterns which are constructed with a variety of materials, before formal teaching.

The next paper “An Interactive Way to Reproduce Structured Matrices” by Kyriaki Tsilika suggests a direction for computer experiments, aimed at students with an interest in Matrix Algebra. The author proposes a programming style in the environment of free computer algebra system Xcas, as an interactive tool to study and/or reproduce the structure of matrices of various forms.

Finally, Michael Voskoglou, in his article “Use of fuzzy logic in students’ assessment”, proposes the use of fuzzy logic as a tool in assessing the students’ knowledge and skills.

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A simple discrete simulation model to explore in the classroom some rules involved in the decision-making process: the cellular automata

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Keywords: computational model, discrete simulation, cellular automata, classroom activity, voting

Abstract

The aim of this contribution is to present teaching material elaborated to work with a very simple discrete model, the cellular automata called "Voting" (free access from the social science section of NetLogo library), and its use in the classroom. A brief description of this agent-based model and its corresponding simulator, jointly with a collection of exercises and queries to investigate how this simulator works and what we can learn from it, are shown. "Voting" is a model in which an agent (individual) changes its state ("vote" or decision) according to the joint states of all of its neighbours. It facilities the exploration of some rules involved in the decision-making process and illustrates how the individual behaviour is responsible for the emergence of global (population) patterns. The results achieved with the implementation in the classroom of this teaching material are discussed.

Introduction

The creation of appropriate learning conditions in mathematics classroom practices, where all the students can develop their abilities to understand and solve increasingly challenging problems is of great interest in the education arena in general. Everybody agrees that students have to become critical thinkers and decision makers both inside and outside the classroom. However, it is nearly impossible to reach a consensus on the most significant students' mathematical knowledge and skills to cope with problems in our society. As to what kinds of learning environments are needed to promote a democratic access to mathematical ideas for all students, or the nature of the critical process into which learners must be initiated, many answers may be collected. In this contribution, in order to deal with these items I propose an attractive response, from my personal point of view and teaching experience, which will be illustrated with a detailed example to be used in classroom practice (in a computer lab).

Nowadays, the exploration and understanding of some social and economic issues through simulation is a reality (Gilbert & Troitzsch, 2005). The use of simulations or computational models to understand and explain social phenomena gives us some opportunities to explore those systems, and it is a rather new idea because simulation only began to be widely used in the 1990s, generating enormous potential to manage ideas and queries in mathematics classes.

Agent-based models (ABMs), with a different philosophy and perspective from classic and continuous models, are computational methods that enable an academic to create, analyse and experiment with models composed of "agents", elements or parts that make up a system and which are treated as autonomous and discrete entities, within an "environment", domain or space with its own characteristics (Gilbert, 2008; Grimm et al., 2006). ABMs, as discrete simulations, introduce the possibility of a new way of thinking about processes, based on ideas regarding the emergence of complex behaviour from relatively simple activities. This kind of computational models with their corresponding simulators can be used successfully in the classroom (Ginovart, Portell, Ferrer-Closas, & Blanco, 2011, 2012; Railsback & Grimm, 2011; Wilensky & Rand, in press).

NetLogo is a multi-agent programmable modelling environment (Wilensky, 1999). It is particularly appropriate to be used in teaching as it provides interesting elements to illustrate ABMs in the classroom as well interacting and working with them (Ginovart et al., 2011, 2012; Railsback & Grimm 2011; Wilensky & Rand, in press). The NetLogo web page http://ccl.northwestern.edu/netlogo/ contains abundant documentation and tutorials, plus a gallery of sample models of diverse areas of application ('Models Library'), ready to be executed, modified or adapted, and which are distributed free of charge for use. Permission to copy or modify the NetLogo software, models and documentation for educational and research purposes has been granted (Copyright 1999-2011 by Uri Wilensky).

Aim

The general aim is to make the characteristics and possibilities of ABMs better known in a teaching and learning context, so that this modelling methodology may be progressively incorporated into academic curricula, complementing other existing modelling strategies and gaining more use in the classroom. To this end, two specific learning goals for the activities of the teaching material elaborated for classroom use are established. The first is to develop a well-founded understanding of what a cellular automata is, identifying and distinguishing its main elements and their relationships, and the processes or rules that can be involved in the behaviour of the set of agents that make up the system. A detailed explanation of the design of a very simple ABM, the cellular automata called "Voting" and its use through a collection of exercises and queries to investigate how it works and what we can learn from it will assist in the achieving of this goal. "Voting" is a model in which an agent changes its state ("vote" or decision) according to the joint states of all of its neighbours. To deal with some ideas involved in the decision-making process, this computational model "Voting" was chosen from the social science section of the NetLogo library and used in the activities designed for the students. The second aim is to familiarize students with the multi-agent programmable modelling environment

NetLogo, a free tool accessible on the Web and especially useful in the classroom for dealing with computational models or simulations.

The participants in this study were a group of 13 third-year students of the Universitat Politècnica de Catalunya (UPC). Students' answers on the cellular automata "Voting" and its use were collected via open-ended queries and face-to-face dialogues, which are analysed and discussed in this paper, along with the outline of the teaching material prepared for the study.

Although, the participants in this study were university students, the teaching material elaborated could also be adapted and used in the final years of secondary or high school.

Material and methods

The previous section that serves to put the computational model "Voting" in context, jointly with the teaching material designed to reach the learning goals and guide the students' modelling and simulation activities, presented in this section, constituted the teaching material that students managed during this study in classroom. Each student had a computer to use in the lab where the activity took place and worked individually and autonomously.

It is possible to install free NetLogo in a computer from http://ccl.northwestern.edu/netlogo/ (it takes only a few minutes), and it runs on almost any current computer. Working with an already developed simple and unsophisticated example like "Voting" model (Wilensky, 1998), chosen from the NetLogo 'Models Library' in the social science section, facilitates the introduction of a computational model in a classroom in an effective way. Also, it allows us to learn how to deal with NetLogo and how to analyse and discuss the results of several simulations. If you have NetLogo installed on your own computer, to open the "Voting" model you can use the following instructions: a) open NetLogo, b) go to File, c) go to the Models Library, d) choose Social Science, e) double click to select and open "Voting". When we open "Voting" model, we find the "Interface" window (Fig. 1). At the top of this main window there are three tabs labelled "Interface" (the current), "Information" and "Procedures". The "Information" window contains some information regarding the model, and in the “Procedures” window can be found the corresponding computer code developed for the implementation of this model (Fig.2).

The “Voting” model was described in Rudy Rucker's "Artificial Life Lab", published in 1993 by Waite Group Press. It is a model in which an agent or individual of a system changes its state according to the joint states of all of its neighbours. It is a simple cellular automata that consists of a number of identical spatial cells (the agents) arranged in a regular grid, a squared bi-dimensional array.

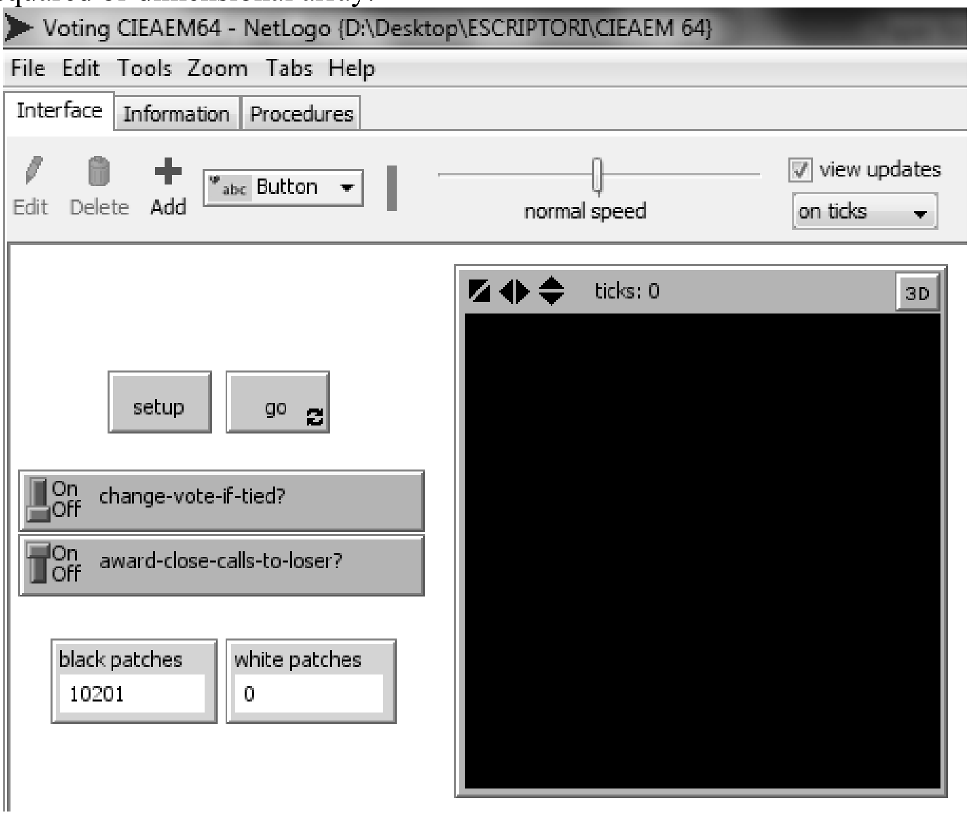

Fig. 1: The “Voting” model from the NetLogo Models Library: “Interface” window.

In a social representation of the “Voting” model, the agents (spatial cells) stand for individuals. Each spatial cell or agent can be in one of two states, “off” or 0 (white patch), or “on” or 1 (black patch). The model simulates voting distribution by having each patch take a “vote” from its eight surrounding neighbours (Fig. 3) then, perhaps its own “vote” changes according to the outcome from the rules assumed. We can adapt the situation of this model, the “vote”, to examples in which the states represent opinions (such as supporting a political party or not), individual characteristics (having them or not) or attitudes (for instance, cooperating or not with others). For example, people might adopt a fashion only if the majority of their friends (neighbours) have already adopted it. At each time step during the evolution of the system, the state of each spatial cell or agent may change.


```
patches-own
[
  vote   ;; my vote (0 or 1)
  total  ;; sum of votes around me
]

to setup
  clear-all
  ask patches
    [ set vote random 2
      recolor-patch ]
end

to go
  ask patches
    [ set total (sum [vote] of neighbors) ]
  ;; use two ask patches blocks so all patches compute "total"
  ;; before any patches change their votes
  ask patches
    [ if total > 5 [ set vote 1 ]
      if total < 3 [ set vote 0 ]
      if total = 4
        [ if change-vote-if-tied?
          [ set vote (1 - vote) ] ]
      if total = 5
        [ ifelse award-close-calls-to-loser?
          [ set vote 0 ]
          [ set vote 1 ] ]
      if total = 3
        [ ifelse award-close-calls-to-loser?
          [ set vote 1 ]
          [ set vote 0 ] ]
      recolor-patch ]
  tick
end

to recolor-patch  ;; patch procedure
  ifelse vote = 0
    [ set pcolor white ]
    [ set pcolor black ]
end


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; The full copyright notice is in the Information tab.
```

Fig. 2: The computer code of "Voting" model as shown in the "Procedures" window of NetLogo.

The state of a spatial cell after any time step is determined by a set of rules which specify how that state depends on the previous state of the spatial cell and the states of the spatial cells' immediate neighbours (Fig. 3). The same rules are applied to update the state of every spatial cell in the grid. Fig. 4 shows an initial distribution of votes on the grid, distribution of white and black spatial cells after the "setup" button in the "Interface" window has been pressed, and the distribution at the end of a run, the distribution achieved after some time steps using the default values and rules for the "Voting" model of NetLogo. In this case, once the cells have achieved this speckled pattern (Fig. 4, bottom), there is no longer any opportunity for change. It is a very simple discrete simulation model that enables classroom investigation of diverse rules involved in the decision-making process.

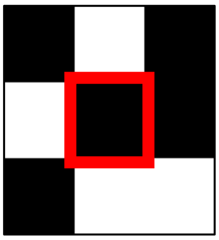


Fig. 3: Each spatial cell or agent can be "off" or 0 (white patch) or can be "on" or 1 (black patch), and the agent changes its state according to the joint states of all of its eight (immediate) neighbours according certain rules.

Some indications to understand how this computational model works are suggested through a set of exercises. These exercises, with the corresponding queries to be answered and discussed, can assist the examination and analysis of the "Voting" model and its simulation results. These tasks are as follows.

1) Read and check carefully the documentation in the "Information" window and try to understand the computer code of the "Procedures" window to identify the default rules (both switches "change-vote-if-tied" and "award-close-calls-to-loser" are off). Carry out, for instance, 5

successive simulations. What do you see? Why don't the evolutions and final distributions in the space of the two sub-populations, white cells and black cells, look exactly the same? What tendencies or patterns of behaviour are observed in the system?

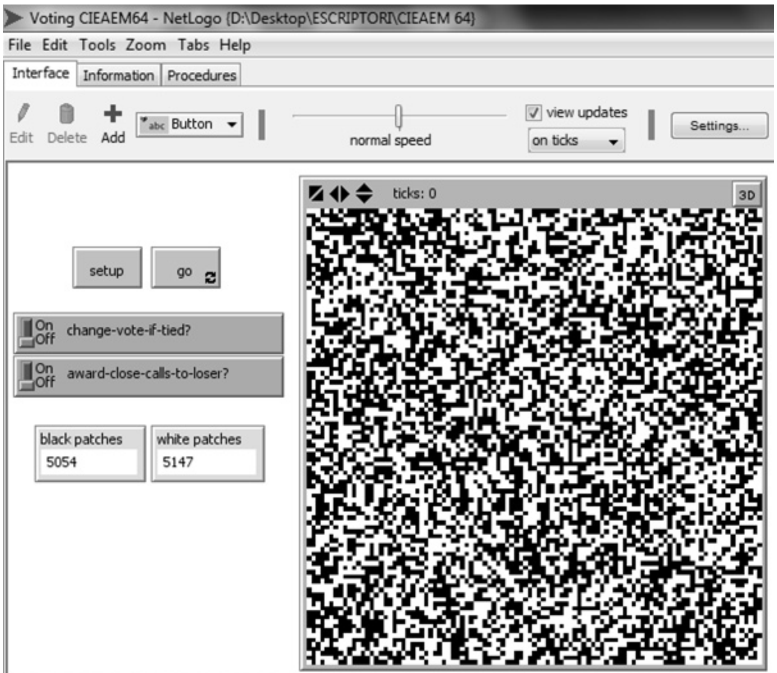
2) You can open up the Model Settings, by clicking on the "Settings..." button at the top of the "Interface" window. You will notice that "max-pxcor", "max-pycor" are fixed to 75. Change these two values to 1 and also modify the "Patch size" from 3 to 140. Setup the new conditions and run a simulation. After that perform the default rules over this initial configuration of the "Voting" model by hand. You can apply them to the central spatial cell. Compare the simulation results with those that you achieve by hand. Try different initial conditions to repeat these calculations.
3) Carry out different simulations by modifying the values of "max-pxcor", "max-pycor" and "Patch size". For instance, you can use the following set of values: 1, 1 and 140; 2, 2 and 70, 4, 4 and 35; 8, 8 and 18; 16, 16 and 9; 32, 32 and 5; 64, 64 and 3; and finally, 128, 128 and 2. Examine the simulations that you have obtained. Watch how any arrangement settles to a static state. Approximately, how many time steps are required to achieve these final static states in the different cases evaluated? How can you explain this behaviour?
4) Read and check carefully the documentation in the "Information" window and try to understand the computer code of the "Procedures" window to identify the rule behind the option "change-vote-if-tied". Watch what happens when only the "change-vote-if-tied" switch is on. How are the simulation results different from those achieved in Exercise 1?
5) Do the same with the option "award-close-calls-to-loser" and watch what happens when only the "award-close-calls-to-loser" switch is on. How are the results different from those achieved in Exercises 1 and 4?
6) What happens when both switches "change-vote-if-tied" and "award-close-calls-to-loser" are on?

7) Can you imagine any other rules to change the status of the spatial cells of the system?

Results and discussion

The results of the implementation of this teaching material in the classroom and the answers given by the students will be shown and discussed as follows.

At each simulation performed with "Voting", the initial set of agents or spatial cells is established randomly in the domain (Fig. 4), which permits replications of the behaviour of the system as in an experimental trial, in contrast to the deterministic resolution of other types of modelling that use continuous functions. The possibility to carry out successive simulations with parameters fixed in the program, but with different initial spatial distributions of the agents, was discussed and very well appreciated by the students. If both switches "change-vote-if-tied" and "award-close-calls-to-loser" are off, the operating procedure is as follows: the new cell state is the state of the "majority" of the cell's immediate neighbours or the cell's previous state if the neighbours are tied. The rule says that a cell is white (or 0) if there are five or more white cells surrounding it, black (or 1) if there are five or more black cells around it, and remains in its current state if there are four whites and four blacks. See Fig. 2 to identify this rule in the computer code with the following sentences: "*if total>5 then vote=1*" and "*if total<3 then vote=0*" where the total is the sum of the numeric values of the spatial cells.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white patches
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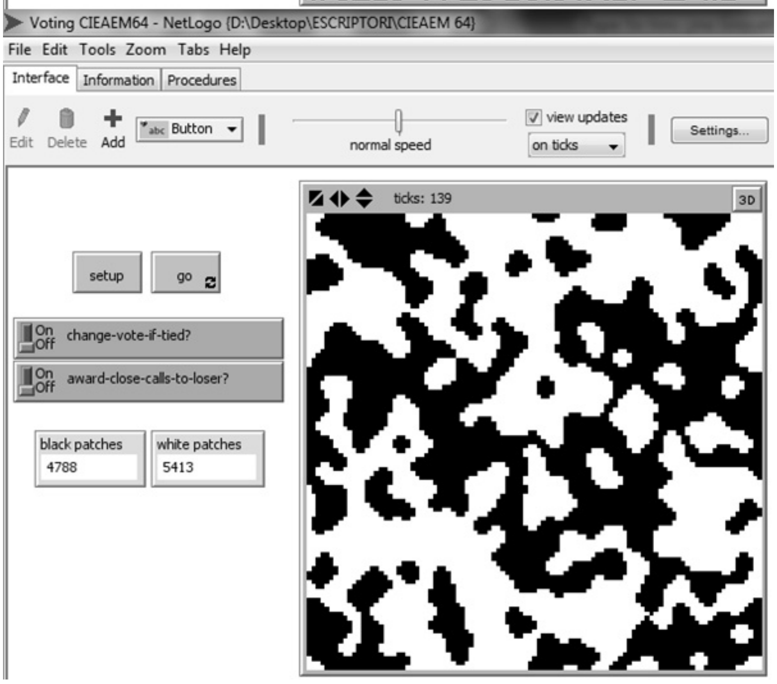
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Interface
Information
Procedures
Edit
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Add
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normal speed
view updates
on ticks
Settings...
ticks: 139
3D
setup
go
On Off
change-vote-if-tied?
On Off
award-close-calls-to-loser?
black patches
4788
white patches
5413

Fig. 4: An initial distribution of white and black spatial cells (top) and the distribution at the end of a run (bottom) using the default rules of the NetLogo "Voting" model.

Starting from a random distribution of white and black spatial cells, a random distribution of votes on the grid, and applying the rules in each spatial cell of the grid and in each time step, successive configurations of the system can be observed for different time steps (Fig. 5). The result of running this default rules implemented in the NetLogo program is a patchwork of small white and black blocks (Fig. 4). Cells surrounded by cells of the other colour change to the colour of the majority so that isolated cells merge to form blocks of one colour. Cells whose neighbours are four white cells and four black cells stay unchanged and form stable boundaries to these blocks. Once the cells have achieved a speckled stable pattern, there is no longer any opportunity for change as Fig. 5 shows.

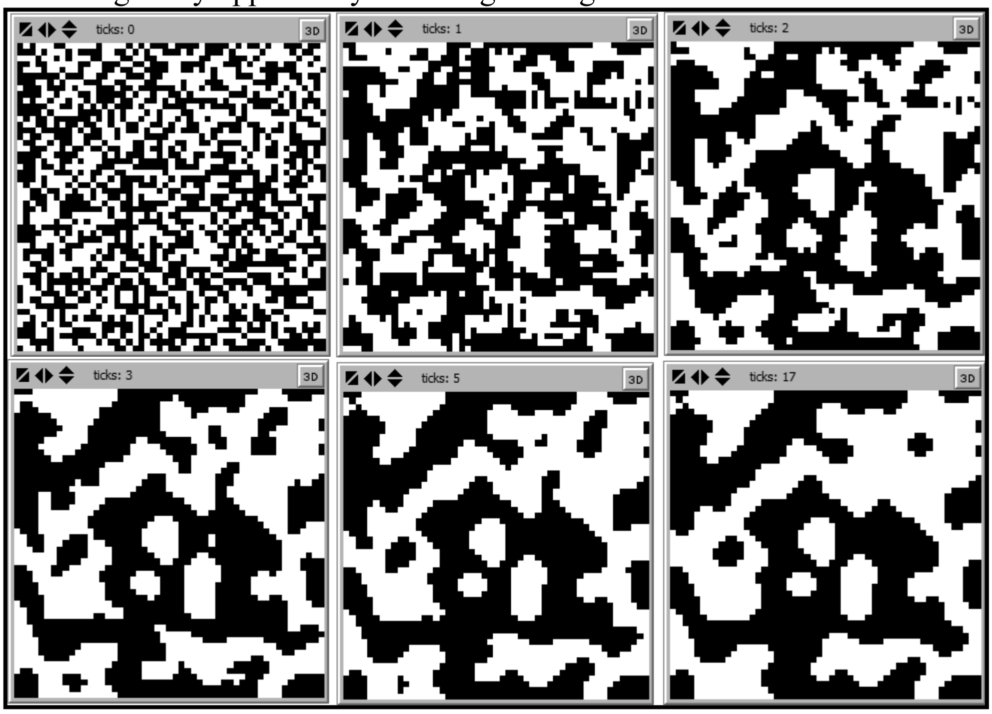


Fig. 5: Different time steps of an evolution of the system (ticks 1, 2, 3, 5, and 17) from an initial distribution of white and black cells (tick 0) until a stable

final pattern is achieved with the default rules of the NetLogo "Voting" model.

Fig. 6 shows some simulated evolutions that achieve different final distributions from diverse initial systems. They do not look exactly the same due to the fact that the initial configurations for those systems are randomly constructed, but the tendencies or patterns of behaviour observed in the systems are similar. Although the final patterns achieved are not identical, they are the same type (Fig. 6).

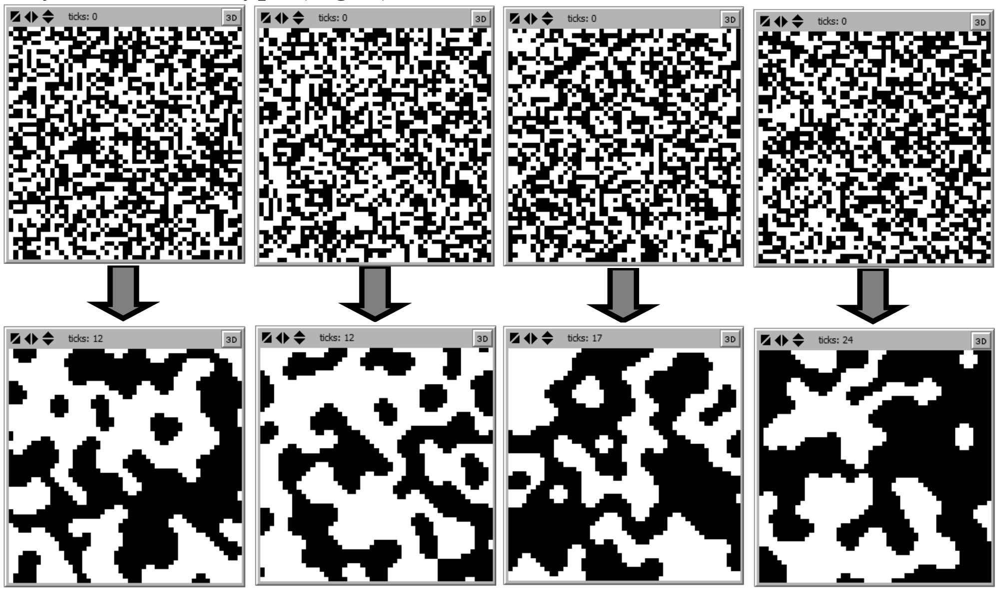


Fig. 6: From different initial configurations, final distributions of the system achieved with the default rules of the NetLogo "Voting" model.

Some facts regarding the change in the dimensions of the domain ("max-pxcor", "max-pycor") can be observed and discussed. Firstly, if the domain is very small, as shown in Fig. 7, the default rules can be performed by hand on any initial configuration. The possibility to apply the default rules of "Voting", first to the central cell, and after that to the rest of the cells of the domain, exemplifies very well the actions carried out by the computer in a quicker way. On the other hand, the number of agents involved in the system has a decisive role in order to permit the emergence of patterns of behaviour. With a small number of agents for the system, which it would be possible to manage by hand, the patterns do not appear in the same way (Fig. 7), therefore the use of the computer is indispensable when dealing with large domains and their corresponding global behaviours. In addition, this task of applying the rule by hand should make the student think about how to choose the cells in order to apply the rules, and look for the answer in this simulator, which is randomly within the NetLogo environment (in order to avoid privileged cells, first or last acting cells, and the consequent non-desirable effects). Also, a key question related to this task (performing by hand the rules) is the effect and way to deal with the boundaries of the system's domain. What can be noted is that the patches around the edge of the domain have fewer than eight neighbours, the corners have three and the rest have five. Nevertheless, the topology of the NetLogo "world" has four potential values: torus, box, vertical cylinder, or horizontal cylinder. The topology is controlled by enabling or disabling wrapping in the x or y directions, and the default "world" is a torus, which is that used in all the simulations performed in this study (with the change in sizes of the bi-dimensional domain, this option of boundaries appears and it is very clear). A torus wraps in both directions, meaning that the top and bottom edges of the "world" are connected, and the left and right edges are also connected, it is like having eight replications of the main system.

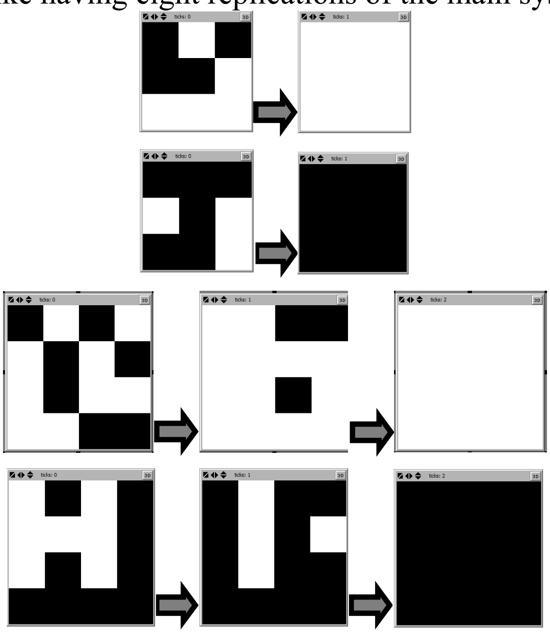
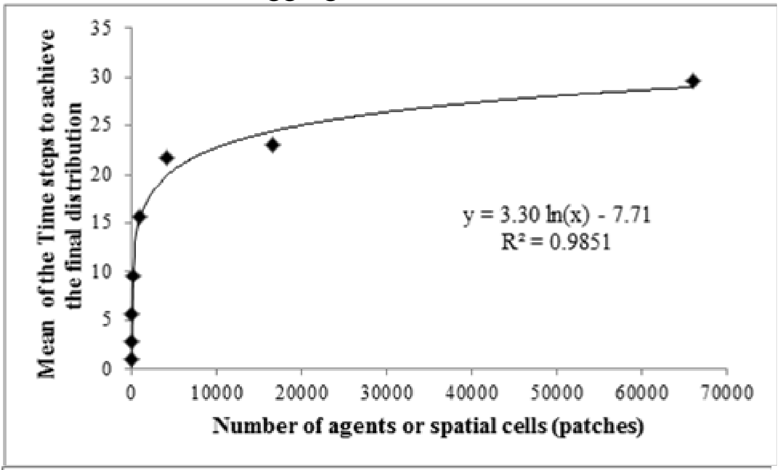


Fig. 7: Initial and final distributions of some systems with small number of spatial cells in the domain using the default rules of the NetLogo "Voting" model.

In order to examine the necessary time that different conditions in the size of the system need in order to achieve the stable states, simulations modifying the values of "max-pxcor", "max-pycor" of the default domain were carried out. The domain used in the simulations was always a square, and the following set of values were the corresponding values for these parameters: 1, 2, 4, 8, 16, 32, 64, and 128; leading to bi-dimensional domain or squares with sides of 3, 5, 9, 17, 33, 65, 129, and 257 respectively. The corresponding numbers of the spatial cells (patches) in each of those cases were: 9, 25, 81, 289, 1089, 4225, 16641, and 66049. Fig. 8 shows two different ways to represent the simulation results collected from students in connection with the relationship between the number of spatial cells (agents) involved in the system and the necessary time to achieve the stable final state. Some of them used the mean values of the set of simulations performed in each case and others the original individual values achieved in each simulation (Fig. 8). For certain sizes of the system, the increase of the time necessary for a stable configuration does not seem to be noteworthy. Using a logarithm scale gives us the connection between these two variables, a linear relation between "Time" and Ln("Number of spatial cells") is observed in Fig. 8. The increase of the time until the final distribution is achieved is very important for small domains, but for larger domains this increase is not significant. Large systems self-organize quicker than smaller ones.

It was very easy to identify the rule behind the option "change-vote-if-tied?" The central spatial cell or the agent changes its colour or state to match the majority vote, but if there are four blacks and four whites, then it does not change when this option is off. If the "change-vote-if-tied?" switch is on, in the case of a tie (4 and 4), the central patch will always change its colour or state according to the procedure "*if total=4 then vote=(1-vote)*" (Fig. 2). When the vote is 0, then the next tick will be 1-0=1, but if the vote is 1, at the next tick the vote will be 1-1=0. Fig. 9 shows three successive screenshots of the final distributions achieved when this option on. The pattern is different from that achieved when this option is off, because the system does not get a static or stable final distribution, it moves from one configuration to another, back and forth systematically, with non-stable boundaries in the blocks or aggregates.



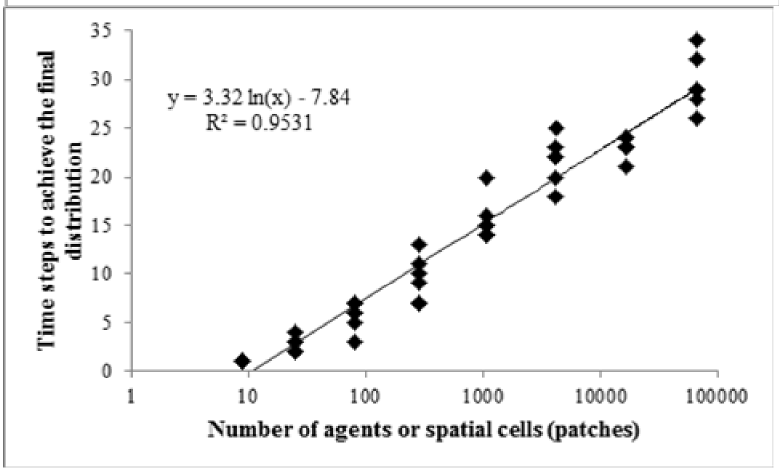


Fig. 8: Relationship between the number of spatial cells (agents) involved in the system and the time necessary to achieve the stable state, using the default rules of the NetLogo "Voting" model: with mean values of the set of simulations performed in each case (upper), or with original individual values achieved in each simulation and in logarithmic scale in the horizontal axis (bottom).

The rule behind the option “award-close-calls-to-loser?” works in a different way. When the “award-close-calls-to-loser?” switch is on, if the result is 5 or 3, the central patch votes with the losing side instead of the winning side. That is, if the total is 5 then the vote will be 0 (instead of 1 that would correspond to the “majority” model), and if the total is 3 the vote will be 1 (instead of 0 that would correspond to the “majority” model). The results of this variant in the evolution of a system are shown in Fig. 10, where the patterns achieved are very different from those achieved with the “majority” model (Fig. 5), with larger blocks or aggregates and much more compact, and with small instabilities in some of the frontier positions. Also, it takes much more time to get this type of semi-static or semi-stable final distributions. After many, many time steps, the domain will usually be completely occupied by black spatial cells (vote 1) or white spatial cells (vote 0) with very small islands of different colour. The domains used in the simulations need to be large enough to be able to show some of these enormous blocks or aggregates, otherwise they do not appear at all or disappear very quickly (Fig. 10).

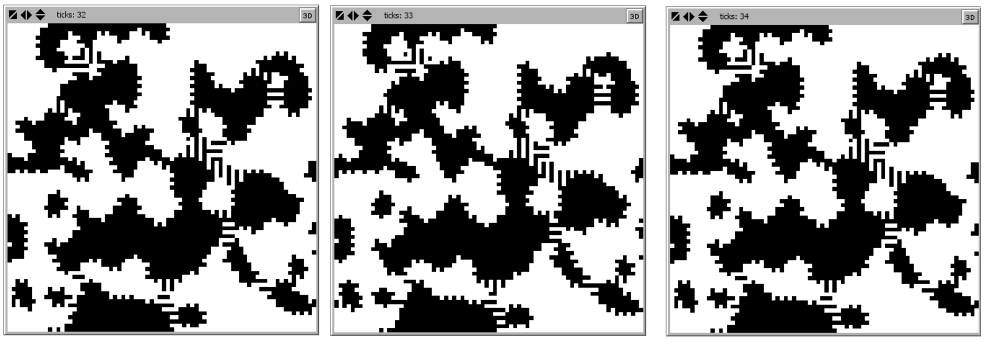


Fig. 9: At the end of the evolution, three successive screenshots of the distributions achieved when the “change-vote-if-tied?” switch is on in the NetLogo “Voting” model.

When both switches “change-vote-if-tied?” and “award-close-calls-to-loser?” are on, no specific patterns for the evolutions and final distributions are observed (graph not shown). The system maintains a random distribution of votes, states or colours, and any structure or arrangement can be identified. No pattern emerges from the system with those individual rules for the spatial cells or agents.

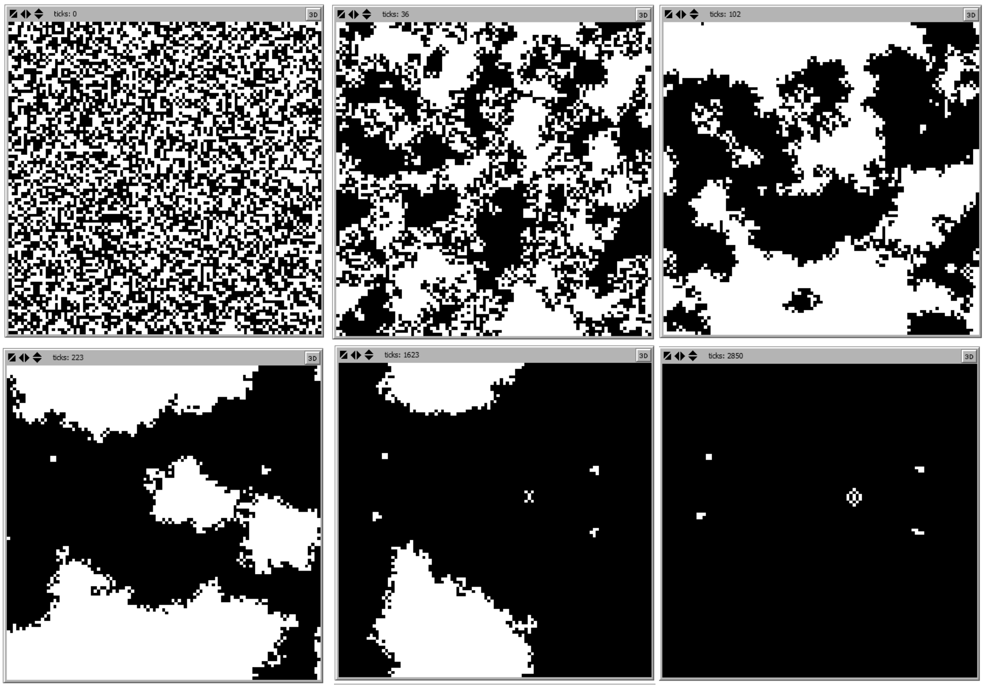


Fig. 10: Different time steps of an evolution of the system (ticks 36, 102, 223, 1623 and 2850), from an initial distribution (tick 0) of white and black spatial cells when the "award-close-calls-to-loser?" switch is on in the NetLogo "Voting" model.

With the use of this simulator (the implementation of a computer code of 30 very simple lines approximately) it was shown that the set of rules for each element (agent, spatial cell, position or site) of a system generates specific arrangements of the states of these elements, discovering or disclosing different patterns in the global system viewed as a whole. If a small change in this set of rules affecting the individual behaviour of the elements of the system is performed, the resulting organization of the elements in the global system can vary significantly. This idea of "emergence" was transmitted to the students by generating successive simulations performed according to the tasks 1, 4, 5 and 6 proposed (Figs. 4, 5, 6, 9 and 10), and was very well understood. There is no control over the global system configuration (as a population variable), only throughout the individual elements that constitute the system. The different patterns shown in those figures are the consequence of small changes in the assignation of the new states of the elements.

Different options were collected from students regarding the possibility of imagining other rules to change the status of the agents or spatial cells. Because this task would directly evolve changes in the computer code with the rewriting of procedures in the computing language, different levels of difficulty in the modifications were implemented. Some students were only capable of changing colours for the spatial cells or introducing specific conditions when the total votes took certain values. Others considered major changes such as those defining three different values or positions for each spatial cell (0, 1 and 2) and then modified threshold values in order to apply other types of rules. Others tried to see what would happen if random changes of state for some spatial cells or agents were included. The idea of incorporating randomness in these rules was very interesting for the observation and comparison of the evolutions of the systems without it. Nevertheless, the most significant aspect in this task, from my point of view, was to offer students an attractive tool and an environment (or context) where they were able to modify and adapt individual rules for the agents according to their own interests, and observe what would happen to the global behaviour of the system.

This simulator, as an interesting tool for using in classroom activities has its own limitations if it is considered in the complex social simulation arena, but a more complex tool would not be suitable for use with students during their first interactions with simulations and computational models, in a learning context, in college or in the first university courses. “Voting” is very easy to explain from a computational point of view and it is sufficient to generate different attractive results. At the same time, the model behind this simulator allows the representation (in a very simple way) of some social actions generated by proximity or connection with neighbourhoods, the reason why, with the help of a computer and following the teaching material designed for this activity, some of the simple ideas involved in a decision-making process can be managed by students. In addition, just for the sake of learning simulation techniques, the design of this activity has also proved to be suitable (as has been observed with the implementation carried out in the pilot group of UPC students).

Conclusions

The development of the activities designed with the use of the simple agent-based model "Voting", a cellular automata chosen from the social science section of the NetLogo library, allows students to investigate how a simple computational model works and diverse aspects related with its implementation and execution in a suitable frame. The analyses of the outcomes or simulation results were used for the comprehension of its core elements. The students were able to deal with the emergence of macro-level patterns in a system (population of agents) from the operation of simple rules applied to the micro-level parts of it (spatial cells or agents), which is nearly impossible without a computer. The use of computational models makes the introduction of some ideas related to complex systems easier, which jointly with 'virtual' experiments (different runs or replications of the evolutions of the system) help to deal with them. If a good mathematical activity is an activity to invite students to take their own decisions, involve them in exploration, development and contrast conjecture, also encouraging reflection and interpretation, favouring originality and inventiveness, while stimulating questions like "what will happen or not if…", I am convinced that the tasks carried out with this computational model will meet these expectations.

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On the Transition Period of Implementing New Mathematics Curriculum for Foundation Engineering Students

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Abstract

An overview on several mathematics modules in the transition period of introducing a new curriculum for the Foundation programme in Engineering at the University of Nottingham Malaysia Campus is discussed in this paper. In order to progress to Undergraduate programmes in Engineering, previously the students must complete three mathematics modules of 40 credit points in total, for which one of them was a year-long module with 20 credit points. Currently under the new curriculum, the students are required to complete five mathematics modules with 10 credit points each. The new curriculum gives positive impacts for both the lecturers and the students in terms of material organization, fully utilizing textbooks and a new arrangement for tutorial sessions. The new curriculum also provides the students with stronger mathematical background in critical thinking and

problem solving skills to equip them to embark the Undergraduate programmes in Engineering.

Keywords: *mathematics modules, new curriculum, Foundation programme, Undergraduate programme, credit points*

1 Introduction

1.1 Background and motivation

The University of Nottingham in Malaysia is one of the branch campusses of the University of Nottingham in the UK. Another overseas branch campus of the university is located in Ningbo, a seaport city of northeastern Zhejiang Province, eastern part of People's Republic of China. Our campus houses four Faculties and the Department of Applied Mathematics belongs to the Faculty of Engineering. Although our department does not really offer any study programme majoring in Mathematics, it plays an important role as a service department not only for the Undergraduate programme but also for the Foundation programme in Engineering. Currently, there are four engineering departments within the Faculty of Engineering that are being served by our department for their mathematics modules: Electrical and Electronic Engineering, Chemical and Environmental Engineering, Mechanical, Materials and Manufacturing Engineering and Civil Engineering.

In contrast to the North American system where the students will normally spend four years in their undergraduate study, the British system of higher education is adopted in our curricula, where the undergraduate level is divided into three years (framework). This means that the students who enter to Year 1 have an entry requirement of A (advanced) level or a similar qualification. However, a significant number of our potential students possess only an O (ordinary) level qualification, particularly the local Malaysian students who have completed their SPM (*Sijil Pelajaran Malaysia* or the Malaysian Certificate of Education). This SPM is equivalent to the British GCSE (General Certificate of Secondary Education), and provides the opportunity for the students to continue their studies to pre-university (Foundation) level. For those who are interested to obtain a better grasp on the education system in Malaysia, including the status of mathematics teaching and learning in the country, the reader is strongly encouraged to consult an article by Sam *et al*. (2009).

In order to accommodate the students with an O level qualification, the Faculty made a special arrangement, known as the Foundation programme in Engineering. Similar Foundation programmes in Computer Science, Bioscience and Business are also offered by the other Faculties in the branch campus. The Foundation programme in Engineering lasts for three semesters which normally begins either in April or in July and is completed in May of the following year. The aim of this programme is to support the students with merely an O level qualification to enter the Undergraduate programme in Engineering. In this case, at the end of the third semester of their Foundation study, the students would have a similar level to those with an A level qualification. After completing the Foundation programme and satisfying the progression rules, the students may proceed to choose one study programme offered by the four engineering departments. The Foundation programme in Engineering covers a set of modules that contains topics on Mathematics, Physical Sciences (including Physics and Chemistry), Information Technology, English and some other modules for the local needs.

In this paper, the curricula for mathematics modules at the Foundation level will be discussed. The purpose of this article is not only to provide an overview of these mathematics curricula but also to explain some pedagogical and content thoughts behind the need to adjust and to modify the curricula. The paper also presents some problems associated with the teaching of a number of mathematics modules to students enrolled in the Foundation programme. Explicit connections between our new curriculum and current development in mathematics curriculum are discussed in this paper. In a more general context, we show that by implementing the new curriculum, some major problems faced in teaching mathematics at the early undergraduate (Foundation) level for Engineering programme could be solved. The solutions proposed are in line with current thoughts on teaching and learning mathematics in general. Although empirical results might not be readily available, some anecdotal evidences and initial outcomes from the implementation of the new curriculum are also included.

Since the focus of this paper is specifically on mathematics curriculum for mathematics modules at the Foundation level, a spacious room is absolutely still available for discussion of mathematics curriculum at the undergraduate level for Engineering programme in general or even for other study programmes that need mathematics.

1.2 Literature review

A general overview on problem solving in the mathematics curriculum is reported by Schoenfeld (1983). The four phases of ‘Multidimensional

Problem-Solving Framework' (MPSF), i.e. orientation, planning, executing and checking, that characterizes various problem solving attributes are described by Carlson and Bloom (2005). Other authors proposed five-phased model of engagement, transformation-formulation, implementation, evaluation and internalization to describe (meta)cognitive approaches for problem solving (Yimer and Ellerton, 2010). A theory of 'goal-oriented decision-making' in mathematics problem solving is proposed by Schoenfeld (2010). Recently, an approach of 'grounded theory' for problem solving skill in engineering problems has been published by Harlim and Belski (2013).

A research on students' understanding of functions and its importance for the undergraduate mathematics curriculum is discussed by Thompson (1994). The design of the mathematics curriculum for engineers in an Australian university has been addressed by Varsavsky (1995). Her findings indicate that to achieve cohesion and to make the course more meaningful to students, the design of the mathematics curriculum must be done in close collaboration between the mathematics department and the engineering faculty. A reform in undergraduate mathematics curriculum with more emphasis on social and pedagogical skills is presented by Pesonen and Malvela (2000). The readers who are interested in the history of mathematics curriculum, the culture of mathematics, gender, and social justice issues in mathematics may consult an article written by Teese (2000).

An examination of the various forces which act on mathematics curriculum and on curriculum trends in both at local and US national level is presented by Hillel (2002). The article covers some issues including undergraduate programmes, specific courses, mathematical content, degree of rigour, modes of delivery and interaction and assessment schemes. An article discussing the importance of mathematics in the university education for engineers is reported by Kent and Noss (2003). The authors emphasize mathematical skills, studio-based and problem based learning techniques and use of information technology (IT) in teaching mathematics to engineering students. An identification and enhancement of mathematical understanding among engineering undergraduate students at MIT by improving mathematics curriculum is presented by Willcox and Bounova (2004). A report describing research to create explicit links between engineering courses and upstream mathematics courses at MIT is discussed by Allaire and Willcoc (2004).

A suggestion for a theoretical model based on the anthropological notion of a modern-day rite of passage for a transition from secondary school to university mathematics in the context of North-American educational context is featured by Clark and Lovric (2008). An enriched version of the model, with the added notions of cognitive conflict and culture shock, has been addressed by the same authors (Clark and Lovric, 2009). Their model suggests that the transition from high school to university mathematics requires a proper environment and there is a need for filling in the temporal gap between the two stages with a set of meaningful activities.

The transition of mathematics learning from secondary school to university from the students' perspective has been addressed amongst others by Barnard (2003) and Hernandez-Martinez *et al.* (2011). A large body of literature exists discussing surrounding issues of the transition of teaching and learning in mathematics from secondary to tertiary levels, amongst others are (Brandel *et al.*, 2008; Brown *et al.*, 2008; Jourdan *et al.*, 2007; Wood, 2001). Some findings from a project analyzing the transition from secondary to tertiary education in mathematics from the teachers' and lecturers' perspectives is reported by Hong *et* al., (2009). The results provide evidence that each group lacks a clear understanding of the issues involved in the transition from the other's perspective and there is an urgent need to improve the communication between the two parties. An excellent article on the changes in thinking involved in the transition from school mathematics to formal proof in pure mathematics at university has been written by Tall08. It is interesting for the students enrolled in the Foundation level to experience this transition level from high school mathematics to university mathematics.

This paper fills the literature gap of the transition level from secondary mathematics education to undergraduate mathematics education and is organized as follows. After this introduction, Section 2 addresses some problems and issues in teaching mathematics modules at the Foundation levels. Additionally, Section 3 discusses current thinking on mathematics teaching and learning, in particular the curriculum which emphasize the students acquiring problem solving skills. Furthermore, Section 4 gives an overview on the old and the new curricula of mathematics modules at the Foundation level, including some changes that have been implemented. Next, Section 5 provides some initial reactions to the implementation of this new curriculum. Specific descriptions addressing the concerns are also discussed in this section. Finally, Section 6 gives the conclusion of our discussion.

2 Issues surrounding teaching Foundation mathematics

It is observed that many students who are currently enrolled in the Undergraduate programmes in Engineering possess weak background in basic mathematics. This observation is based on our experience in teaching several mathematics modules to the Undergraduate students. Many of the students have forgotten some basic mathematical concepts that were introduced to them earlier when they were enrolled in the Foundation programme. For instance, some are confused to conclude that when $f(x-2)=1/x$, then $f(x)=1/(x+2)$. Many find also difficult to conclude that if **Σφάλμα!**, then **Σφάλμα!**. It can even be surprising that a Year 3 student was not able to perform simple integration and derivation, such as evaluating **Σφάλμα!**. Actually this is simply the First Fundamental Theorem of Calculus, which is taught at the Foundation level; see Smith and Minton (2008) or any other books on Calculus.

In addition, many students lack of ability in critical thinking and problem solving skills. Even though the enrollment to the programme has been through a selection process, we may still enrolled the students who are rather weak in mathematics but quite strong in other fields and thus the current situation is simply inevitable. The crème de la crème students in the country prefer to choose well-known local universities instead of our private institution, mainly for financial reasons. A similar situation occurs for our international students. Even though we manage to capture excellent students, the indisputable fact is that on average, many of our students are academically at the intermediate level, particularly in mathematical ability. It is our responsibility to train them to think critically and to possess problem solving skills.

Furthermore, it is also observed that a year-long module of 20 credit points is always tougher for the students than two separate modules of 10 credit points each. In the context of our university curriculum, a 10 credit points' module would provide the students a two-hour lecture and a one-hour tutorial sessions each week. A final exam covering the semester's material is conducted at the end of each semester. For a year-long module with 20 credit points, a similar class session with the one from the 10 credit points module is also adopted. However, there is no final exam at the end of the first semester; the final exam is conducted at the end of the second semester and covers the material from both semesters. This particular type of arrangement imposes a heavy burden to the students since they have to cover two-semester materials for the final exam.

With this in mind, we have proposed to split the 20 credit points modules into two modules with 10 credit points each. As a consequence, there will be two final exams at the end of the first and the second semester, respectively. Under this new arrangement, the burden for the students will be alleviated since they simply need to prepare the material for each semester only. Moreover, in response to improve students' problem solving skills and their critical thinking ability, we have also developed our curriculum with respect to these needs. This strategy is in line with the current thoughts on teaching and learning mathematics in general and engineering mathematics in particular, as it will be discussed in the following section.

3 Current thoughts on mathematics teaching and learning

The main aim of implementing this new curriculum is to improve the confidence, mathematical knowledge and fluency of the students in problem solving. The expected learning outcome consists of four important components: knowledge and understanding, intellectual skills, professional skills and transferable skills. Basically, the first component will depend on the contents of the module. However, the latter three components can be described in a more general framework as follows.

- Concerning the intellectual skills, the students who complete the module should be able to reason logically and work analytically, perform high levels of accuracy, manipulate mathematical formulas, algebraic equations and standard functions and apply fundamental mathematical concepts to routine problems in engineering or science.
- Concerning the professional skills, the students who complete the module should be able to construct and present mathematical arguments with accuracy and clarity as well as apply basic solution techniques learned to mathematical problems arising in the study of engineering or science.
- Finally, the students who complete the module are also expected to obtain transferable skills, i.e. to communicate mathematical arguments using standard terminology, express the ideas of solution of mathematical problems appropriately and effectively and use an integrated software package to enhance learning and practice their problem solving skills.

It is important to note that our new curriculum heavily emphasizes on teaching and learning through problem solving. Problem sheets are distributed each week and a number of selected questions are discussed in the following week during the tutorial session. The type of questions in the problem sheets ranges from simple to more challenging ones.

The implemented teaching and tutorial sessions still rely heavily on the traditional method, where the instructor tends to dominate the entire teaching session. Although the students are also strongly encouraged to participate actively during the tutorial session, it remains a challenge to have an interactive teaching session, since many students are timid by nature.

The third component of our theoretical framework is the process of gathering and analyzing data. For this, we still yet have to wait for the outcomes of implementing the new curriculum. However, the previous data of the students' results, as well as the class observation of the students' abilities, suggest that there is an urgent need to improve the curriculum and teaching emphasis.

It is interesting to note that our new curriculum is developed in line with current trends in teaching mathematics in general, as we have discovered in the literature. A description of a particular framework for research and curriculum development in undergraduate mathematics education with some examples of its application is given by Asiala *et* al., (1996). The authors describe certain mental structures for learning mathematics, including actions, processes, objects, and schemas, also known as the acronym APOS, and the relationships among these constructions. The components of the ACE teaching cycle (activities, class discussion, and exercises), cooperative learning and the use of a mathematical programming language are also explained thoroughly.

The theory of APOS based on Piaget's principle that an individual learn mathematics by applying particular mental mechanism to build specific mental structures and uses these to solve problems related to particular situations (Piaget, 1970). The main mechanisms are interiorization and encapsulation and the related structures are APOS themselves. There are three stages for the implementation of APOS theory as a framework for teaching and learning mathematics, i.e. theoretical analysis, instructional sequences and data collection and analysis (Dubinsky and McDonald, 2001). Based on this APOS theory, the pedagogical approach of ACE teaching cycle that encourages active student learning is developed. Although the activities meant by the previous authors are computer related activities (Asiala, *et* al., 1996), in our new curriculum, the activities also include the traditional classroom activities, where the instructor poses some mathematical problems to the students before explaining and covering a particular topic (Maharaj, 2010).

The following provides an example mentioned earlier in the context of APOS theory. **Action**: A student who wants to find an expression of the function $f(x)$ given the form $f(x-a) = 1/x$, $a \neq 0$, $x \in R$ or finding $g(t)$ given a rational function $g(t-b)=h_1(t)/h_2(t-b)$, where h_1 and h_2 are other functions in t and $b \neq 0$, $t \in R$, can find the expression for the functions f and g directly by observing the expression at the right-hand sides. **Process**: an individual with a process of understanding the substitution will construct a mental process by replacing old variables x and t with new variables $\xi = x - a$ and $\tau = t - b$ and rewrite the new variables ξ and τ with the old ones x and t once the process is complete. **Object**: the student could think those functions as geometrical object and changing the variables mean shifting the graph horizontally to the left (a, $b > 0$) or to the right (a, $b < 0$). **Schema**: the individual organizes all the other three components of actions, processes and objects into a coherent framework, thus a complete understanding of problem solving is attained.

The three components of the theoretical framework for mathematics education mentioned in (Asiala *et* al., 1996) are adopted implicitly in our new curriculum. The first component, i.e. theoretical analysis, where the students acquire knowledge and understanding, is covered in the first learning outcome of the new curriculum. The second component of the theoretical framework is implementation of instruction. This postulates certain specific mental construction that the instruction should foster. In connection to our new curriculum, the instruction of the new curriculum is tailored to the desired learning outcome, which includes the intellectual, professional and transferable skills. The implemented instruction method will help the students to use certain constructions in different situation and to develop problem solving skills. Finally, the third component of the theoretical framework is the collection and analysis of data. Although at this stage the outcome of the new curriculum is not fully documented yet, some anecdotal evidences and initial reactions, however, are discussed in Section 5.

Another work emphasizes problem solving as a basis for reform in mathematics curriculum and instruction (Hiebert *et al*., 1996). The authors discuss the history of problem solving in the curriculum that has been infused with a distinction between acquiring knowledge and applying it. They propose an alternative principle by building on the idea of reflective inquiry, arguing that the approach would facilitate students' understanding.

A number of mathematics curricula for engineers have also been designed to tailor the need for critical thinking and problem solving skills. See for instance amongst others (Gainsburg, 2006; Hurford, 2009; Lesh and Doerr, 203; Lichtenberger, 2002) and the references therein.

The problem solving approach proposed by Hiebert96 allows students to wonder why things are so, to search for solutions and to resolve incongruities. Our new curriculum is also designed in a similar line of ideas. The instruction is tailored so that the students are engaged in problem solving activities and thus being trained to reason logically and to think critically (intellectual skills). Historically, the mathematics curriculum in general has been shaped by concerns about preparation for the workplace and for life outside of school (Stanic and Kilpatrick, 1988). Problem solving has been used as a vehicle to reach this goal. The professional skills of our new curriculum include some aspects of students' construction when they encounter problems in science and engineering, later during their study of even after they leave university. Finally, the transferable skills emphasized by Hiebert *et al*. (1996), who propose an implementation of problem-based learning or case-based instruction, are also adopted in the curriculum.

More recent views on problem solving has been mentioned in the introduction of this paper. New insights on problem-solving process by offering multidimensional framework to investigate, analyze and explain mathematical behaviour are described by Carlson and Bloom (2005). The authors explains four phases of 'Multidimensional Problem-Solving Framework' (MPSF) of orientation, planning, executing and checking, where various problem-solving attributes, including their roles and significance during each phases, are characterized by MPSF.

An excellent scholarly work on the understanding of how and why pedagogical decision-making happens in the course of teaching and learning can be found in (Schoenfeld, 2010). The author proposes a theory of 'goal-oriented decision-making' and provides particular examples of problem solving in mathematics where teachers' instructional decision making is a function of instruction goals, resources and orientations. A five-phased model to describe the range of cognitive and metacognitive approaches used for mathematical problem solving is proposed by Yimer and Ellerton (2010). The five-phases are engagement, transformation-formulation, implementation, evaluation and internalization. Findings on how engineers develop their problem solving skills with implications on general educational strategy, the development and the implementation of computer technology for engineering problem solving are published recently (Harlim and Belski, 2013). The authors utilized an exploratory approach of 'grounded theory' to understand the complexities of engineering problem solving.

The following section explains the differences between the old and the new curricula and a new tutorial arrangement conducted after implementing the new curriculum.

4 The old and new mathematics curricula

4.1 The old curriculum

The old curriculum referred in this paper is the one that has been implemented until the academic year 2008/2009. Under this former curriculum, three mathematics modules with total of credit points of 40 are given to the Foundation students. These are HG1BMT Basic Mathematical Techniques (10 credit points) offered in Semester 0, HG1FND Foundation Mathematics (20 credit points) offered in Semesters 1 and 2 and HG1M02 Applied Algebra (10 credit points) offered in Semester 2. Thus, the students who have completed their Foundation programme have taken three mathematics modules with total of 40 credit points.

The following are the summaries of the content of each of the mathematics modules under the former curriculum.

- HG1BMT Basic Mathematical Techniques
  This module provides a basic course in algebra and introduces some basic knowledge on functions and analytic geometry. This module cover basic algebra, inequalities, polynomials, functions and graphs, coordinate geometry, conic sections, sequences and series, binomial expansion and partial fraction decomposition.
- HG1FND Foundation Mathematics
  This module provides the basic topics of differential and integral calculus. It covers trigonometric functions, complex numbers, differentiation, applications of derivatives, integrals, numerical integration, curve sketching and the binomial theorem for any rational index.
- HG1M02 Applied Algebra
  In this module, some of the fundamental concepts of vector and linear algebra that arise naturally in many engineering circumstances are introduced. Problems that demonstrate the applicability of the theory are covered and this enables students to develop a facility at applying the techniques for themselves. This is considered to be a crucial aspect of the training of a modern engineer. This module covers

determinants, vector algebra and its applications to three-dimensional problems in geometry, vector differential operators, matrices and systems of equations.

4.2 The new curriculum

The new mathematics curriculum is the one that is currently being implemented from the academic year 2009/2010 onward, starting since April 2009. Under this new curriculum, there is no significant change to HG1M02 Applied Algebra. However, the 20 credit points' module HG1FND Foundation Mathematics is split into two 10 credit points modules, to become F40CA1 Calculus 1 and F40CA2 Calculus 2, with some slight variations. These modules are offered in Semester 1 and Semester 2, respectively and basically contain differential and integral calculus. A 10 credit points' module which is similar to HG1BMT Basic Mathematical Techniques is brought in under a new name: F40FNA Foundation Algebra, which is offered in Semester 0. Furthermore, a new 10 credit points module is introduced under the new curriculum and is offered in Semester 1, namely F40FMT Mathematical Techniques.

The following are the summaries of the contents of each mathematics module under the new curriculum.

- F40FNA Foundation Algebra
  This module is offered in Semester 0 and provides a basic course in algebra and trigonometry. It also introduces the students to skills in core mathematical techniques. This module will cover indices and rules of algebra, quadratic equations, polynomials, partial fraction decomposition, trigonometry, logarithms, inequalities, sequences, series and binomial expansion.
- F40FMT Mathematical Techniques
  This module is offered in Semester 1. It introduces the complex number system and it provides a basic course in elementary statistics. Algebraic manipulation and operations on complex numbers are introduced with the aid of an Argand diagram in the complex plane. Initial key elements of definition, manipulation and graphical representation of data are introduced prior to establishing statistical techniques used in the analysis of problems in engineering and physical sciences. Application to solving real life problems is developed. The module will cover complex numbers, basic set theory, graphical representation of data, numerical descriptive, probability and counting techniques, discrete probability distribution (binomial

distribution) and continuous probability distribution (normal distribution).

- F40CA1 Calculus 1
  This module is also offered in Semester 1 and provides a basic course in differential and integral calculus. Initial key elements of definition, manipulation and graphical representation of functions are introduced prior to establishing techniques of calculus used in the analysis of problems in engineering and physical sciences. Applications in solving real life problems are developed. The module covers functions and graphs, limits and continuity, techniques of differentiation, applications of differentiation and curve sketching.
- F40CA2 Calculus 2
  This module is offered in Semester 2 and provides a basic course in integral calculus. Initially, formulas for integration as an antiderivative of functions are introduced prior to establishing techniques of integration for more complicated functions. For applications, integrals are used to evaluate the area under a graph, the volume of revolution, mean value, and root mean square value of a function. For a non integrable function, numerical integration is introduced. The technique of the Newton-Raphson method is also demonstrated. Finally, the conic sections with their equations and the parametric equations of curves are explained. This module covers indefinite and definite integrals by formulas, techniques of integration, integration by parts, applications of the integral, differential equations with separable variables, numerical integration, conic sections and parametric equation of curves.
- HG1M02 Applied Algebra
  This module is also offered in Semester 2 and provides the basic elements of vector algebra and linear algebra and their applications to simple engineering situations. It introduces the modeling of basic engineering situations in terms of multi-dimensional models. Initially the key elements of definitions and manipulations of basic mathematical skills and mathematical techniques in matrices and vectors are introduced prior to modeling and analyzing problems related to engineering situations. The module covers vectors, matrices, system of equations, Gauss and Gauss-Jordan elimination and Cramer’s rule.

We observe that trigonometry and logarithms which formerly were presented in Semester 1, they are presented in Semester 0 under the new curriculum. Curve sketching moves from Semester 2 into Semester 1. Complex number stays in Semester 1, but it is taught under the new module Mathematical Techniques. Furthermore, this new module simply contains new materials that are not really covered in the old curriculum. These are basic set theory and introduction to probability and statistics. Introduction to differential equations with separable variables is a new topic and is covered in Semester 2 under Calculus 2. Basically, there is no significant change between the former and the new curricula for HG1M02 Applied Algebra.

4.3 A new tutorial arrangement

A new arrangement on tutorial session has been introduced starting in the current academic year for the Foundation students enrolled in April 2009 intake. This arrangement is particularly beneficial for those who are less strong in the mathematical competency compared to the average of the class within the same batch. A scholastic test is given at the beginning of the semester to distinguish the weaker group of students from the stronger one. Thus, instead of following the 2 + 1 sessions, the weaker students will follow 3 + 1 sessions. It means that they will receive a three-hour lecture and a one-hour tutorial session. Since a measure of flexibility is allowed, the lecturer might implement 2 + 2 sessions, where the students will have more opportunity to do more problem solving and exercises during the class, since a two-hour session is devoted for the tutorial. After implementing this arrangement, it is expected that at the end of the semester, the gap of the mathematical competency between the two groups can be decreased. We are eagerly awaiting and curiously anticipating to the success of this novel idea.

5 Research methodology and its initial findings

This section explains the research methodology implemented in study. It also reports some anecdotal evidences and initial findings concerning the implementation of new curriculum from both the instructors' and students' viewpoints.

5.1 Research methodology

A methodology of simple qualitative research interview is implemented in this study, without explicit questionnaire and realizes upon discussions and arguments. Since there is a wide variation among interviewing approaches, the approach of 'unstructured interview' is adopted. Although no interview can truly be considered unstructured, what is meant by unstructured interview in this context is more or less equivalent to a guided conversation, where the interview is conducted in conjunction with the collection of observation data. This is different with semi-structured interview where the interview is conducted as the sole data source for a qualitative research project and usually clinical type in nature. Information on different formats of qualitative interviews and qualitative methods used in mathematics education research has been explored by Romberg (1992), Schoenfeld (1994) and DiCicco-Bloom and Crabtree (2006).

We interviewed six faculty members of the Department of Applied Mathematics at UNMC. All of these colleagues have experience in teaching a number of mathematics modules at the Foundation levels using the old curriculum. Out of six faculty members, four of them have experience teaching mathematics modules using the new curriculum. Two of the faculty members are not interviewed since they have left the faculty to continue their PhDs but they have contributed to the development of the new curriculum.

We also interviewed the students who have completed their Foundation programme and the students who are still enrolled in the Foundation programme. The former cohort of students experiences the old mathematics curriculum but does not experience the new curriculum. Conversely, the latter cohort of students experiences the new curriculum but does not have any idea the situation for the old curriculum. For each cohort, we explain the differences in the curriculum and some changes that we have implemented. Since the students are generally a little bit intimidated when their instructors interviewed them, they were interviewed in a rather informal settings, i.e. during office hour, during lunch break, during class intermission and during the study week period before the final exam.

The following are several examples of the questions during the guided conversation of the unstructured interviews. What are some differences between the old and the new curricula? What aspects make the new curriculum more suitable for faculty and for students? Why is splitting a 20 credit points' module to two-10 credit points' modules beneficial? What are the benefits of a new arrangement for tutorial sessions? Why adopting a particular textbook for certain course is very practical? What do you think regarding the overall impression of the new curriculum? Some anecdotal evidences and initial reactions regarding the implementation of the new curriculum is reported in the following subsection.

5.2 Initial reactions

All members of academic staff of the department have responded positively to the content, topical arrangement and textbook choice implemented in the new curriculum. Formerly, the interdependence between some topics covered in the old curriculum is rather clumsy. For instance, topics on trigonometry and complex number are presented in the earlier part of HG1FND Foundation Mathematics and binomial expansion for any rational index is presented at the end of the module. In fact, the content core of Foundation Mathematics module is Calculus. Thus, although these three topics might be essential for students' understanding in mathematics, they may distract the main purpose for the module, i.e. to cover Calculus. Furthermore, having these three topics included has made it very challenging for us to select a suitable textbook. Although many Calculus textbooks cover these topics in the appendix or as a quick review to remind the readers, they are not as thorough as we would like to be. For instance, a thorough discussion on trigonometry and trigonometric functions are generally found in the textbooks that also discuss Algebra, for instance *Algebra and Trigonometry* by Sullivan (2011). The reason is that trigonometry is considered as a prerequisite for any general course in Calculus.

On the other hand, the topics covered from each module in the new curriculum are more specific and have strong interrelation. As instructors, we are very glad that some topics have been moved to another module. The coverage for Calculus has also been extended. For instance, trigonometry and binomial expansion have been transferred to F40FNA Foundation Algebra. Complex number is given in the new module F40FMT Mathematical Techniques. The division of the materials between F40CA1 Calculus 1 and F40CA2 Calculus 2 is quite distinct. Basically, Calculus 1 covers differentiation, its applications and all related things to it and Calculus 2 covers integration and its accompanying techniques and applications. From the instructors' perspective, we affirm that the new curriculum has a better organization with respect to the topics distribution.

As instructors, we also respond positively to the admonition of implementing textbooks in our teaching. In the old curriculum, there is no particular textbook that neither we and the students could use due to rather peculiar topical arrangement. As a consequence, we spend more time in compiling lecture notes for the students, with inevitable minor typographical errors here and there. This definitely took our time for teaching preparation and in turns may affected the quality of our teaching. However, in the new curriculum, some particular textbooks have been chosen as main sources of reference for the corresponding modules. For example, we adopted an A-level mathematics textbook from UK for our Foundation Algebra module, written by Bostock and Chandler (2002).

Adopting and implementing textbooks for mathematics modules will not only help us in our teaching preparation but also it saves us uncountable precious time in preparing lecture notes as well as preparing sets of problem sheet. Consequently, a high quality teaching is delivered every session. In addition, both teaching and tutorial sessions are much easier to handle since we simply refer to the textbook instead of heavily depended on our self-built lecture notes. In addition, when assigning certain exercise questions to the students, we simply refer to the textbook instead of rewriting in sheet of papers and then distribute them to the students. Since textbooks have gone to editorial process, the number of errors is usually very minimal. This is another advantage of adopting and implementing textbooks as part of the new curriculum. Overall, the new curriculum has saved us time both from teaching preparation and from administrative aspects.

The students have expressed their positive responses toward the implementation of the new curriculum. The new students who are enrolled in the Foundation programme are high school graduates and might not be aware regarding the change of the curriculum at the university. However, the students who have completed their Foundation programme and enrolled in the Undergraduate programme have discovered that the new curriculum is better than the old one. They respond positively in term of re-arrangement of the topics, an added extra module and an extra hour for class interaction. They wish that that could enjoy the privilege and they affirmatively confirm that the new curriculum is more favorable for their younger peers. These comments particularly come from the older students who are rather weak in their mathematical and analytical skills, but have a little opportunity to improve them and yet they are willing to learn. Provided that these students put sufficient effort to study, the new curriculum contributes in helping them to improve their study skills.

Furthermore, even though the new students might not be aware regarding the difference between the old and the new curriculum, many still respond enthusiastically to the new curriculum. One particular positive response is to the arrangement of an extra hour class interaction. This extra hour might be use either for teaching or for tutorial session or both, depending on the need arises in the classroom. The instructor in this case has a measure of flexibility to implement which session is best suited for the need his/her class. The students who are rather weak in mathematics would appreciate since the teaching is paced appropriately. The students are also beneficial when a step by step explanation is given, rather than jumping some steps. Importantly, the instructor could give guidance in problem solving and the students have more opportunity to ask and discuss the problems. Interestingly, some students who are qualified for a normal session have expressed their desire to join their peers on the slower session with an extra contact class hour. Although initially we are a little bit hesitant to arrange two groups with different number of hours for class interaction, the result is quite the opposite. We also observed that the students are working more diligently on the problem solving session and class discussion. So, the overall impression from the students' perspective regarding the new curriculum is also very positive and encouraging.

6 Conclusion

A brief overview on mathematics curriculum for students enrolled in Foundation programme in Engineering at the University of Nottingham Malaysia Campus has been presented in this paper. The relationship between the new curriculum to current trends in teaching mathematics in general has also been discussed. It is observed that under the new curriculum, more materials are given to the students compared to the previous one. The purpose of this new curriculum is to increase the students' mathematical ability in order to be able to thrive successfully later in their study, particularly during the undergraduate period. Some initial reactions from both the lecturers and the students show that both parties respond positively toward the implementation of the new curriculum.

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PROBLEMES OUVERTS DANS LA CLASSE

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Résumé

Ce travail est centré sur la résolution de trois problèmes ouverts dans des classes d'un collège grec. Les pratiques pédagogiques qui sont décrites ici sont liées au programme officiel. Les expérimentations sont effectuées par un enseignant-chercheur qui essaie d'impliquer ses élèves dans de véritables démarches de recherche en mathématiques. Les résultats de la recherche montrent de quelle manière des situations ouvertes peuvent conduire les élèves à réfléchir et à développer des stratégies mathématiques très variées. Chaque problème ouvert suscite l'intérêt des élèves et sert de motivation pour l'investigation des erreurs et l'approfondissement des concepts mathématiques. Les élèves développent des échanges argumentés, un esprit réflexif, ils formulent des interrogations, élaborent des conjectures et recherchent en commun les meilleures solutions. Ce sont des activités riches, qui favorisent la communication et encouragent le raisonnement mathématique.

Introduction

En mathématiques, le terme *problème ouvert* se réfère habituellement aux problèmes qui sont restés non résolus pendant une longue période, comme par exemple le dernier Théorème de Fermat qui a été résolu en 1993 ou la Conjecture de Goldbach qui reste encore sans solution. En didactique des mathématiques, le terme "problème ouvert" renvoie à un problème de recherche qui n'engage pas les élèves à suivre une méthode spécifique de solution. Ce n'est pas un problème de routine quotidienne de la classe. C'est plutôt un problème inhabituel pour lequel l'élève ne dispose d'aucune procédure de résolution éprouvée.

L'introduction du terme "problème ouvert" est d'origine japonaise, il est apparu durant les années 70 et il avait pour but de développer la pensée mathématique au niveau élevé en réformant l'enseignement des mathématiques à l'aide d'approches ouvertes en pratique pédagogique (Becker & Shimada, 1997; Pehkonen, 1991). Chez les didacticiens des mathématiques, il n'y a pas une définition commune du problème ouvert (Kosyvas, 2010a). D'après "l'approche ouverte" japonaise, « *les "problèmes ouverts" ou "incomplètes" sont définis les problèmes qui ont des réponses multiples* » (Becker & Shimada, 1997, p. 1). Durant les années 1993-96, il y avait un groupe de discussion de PME (Psychology of Mathematics Education) sur le sujet « *Using Open-ended Problems in Mathematics* ». D'après les résultats de ce groupe, les problèmes ouverts répondent aux catégories suivantes: « *investigations, Problem Posing, situations réelles vivantes, projets, problèmes sans questions, problèmes avec variété de réponses, problèmes de champs (ou problèmes de séquence)* » (Nohda, 1995; Silver, 1995; Stacey, 1995; Pehkonen, 1997). En psychologie cognitive, la classification des problèmes revêt une importance capitale pour les activités de la classe, parce qu'elle présente deux grands types de problèmes: problèmes bien définis et problèmes mal définis (well-structured and ill-structured). Dans les problèmes ouverts ou dans les problèmes mal structurés, les questions ou les données ne sont pas claires ou sont insuffisantes (Davidson & Sternberg, 2003; Xun & Land, 2004). Il est évident que tous les problèmes ouverts ne sont pas pertinents d'un point de vue pédagogique. Nous pouvons distinguer entre autres quatre catégories de problèmes ouverts : les problèmes ouverts avec variété de stratégies de résolution, ceux dont les résultats sont multiples, ceux dont l'interprétation de l'énoncé est ouverte et enfin les problèmes ouverts-énigmes (Kosyvas, 1995; Kosyvas, 2010b).

Dans les problèmes ouverts, la question est formulée avec clarté seulement du point de vue grammatical-rédactionnel. Contrairement au niveau sémantique, il peut exister une ambiguïté dans la question. Ceci ne signifie pas que le problème soit vague en tant que problème, mais plutôt que sa formulation implique aussi la réflexion des élèves. Nous allons préciser la nature du problème ouvert à l'aide d'un exemple: *Un cube avec une arête de 10 cm est peint en rouge. Il est ensuite découpé en 1000 petites cubes d'une arête de 1 cm. Pouvez-vous prévoir combien de petits cubes sont colorés en rouges sur 6 faces, combien sur 5, 4, 3, 2, 1 ou aucune ?* D'après Kalavasis F. (1990), il s'agit d'une situation de réflexion. La classe scolaire fonctionne en tant que petite communauté scientifique. Les élèves discutent, formulent des hypothèses et prouvent. Les élèves doivent concevoir le cube dans leur esprit et ils peuvent alors trouver:

1. 4, 5 et 6 faces rouges : il y a 0 petits cubes.
2. 3 faces rouges : ils peuvent compter 8 petits cubes.
3. 2 faces rouges : ils peuvent calculer 8×4+8×4+8×4=96 petits cubes.
4. 1 face rouge : ils peuvent calculer : (8×8)×6= 384 petits cubes.
5. Aucune face rouge : ils peuvent calculer : (8×8)×8= 512 petits cubes, ou 1000-(8+96+384)=512.

La notion de problème ouvert peut être expliquée de la manière suivante : un problème est fermé si la situation initiale et la situation finale sont bien définies. Un problème est considéré comme bien défini dans la mesure où les données initiales, les contraintes et le but sont énoncés de façon explicite et opérationnelle. Dans l'énoncé du problème, la personne a en sa possession, sans devoir les définir elle-même, tous les éléments et les critères concrets et précis pour évaluer la démarche et non le but (Tardif, 1997). Si la situation initiale ou la situation finale est ouverte, nous avons un problème ouvert. Le degré de précision de l'état initial et de l'état final, obtenu à la suite de la résolution des problèmes, comporte le critère du caractère ouvert (Pehkonen, 1995).

Une étude pertinente du "problème ouvert" a été réalisée par une équipe de l'IREM de Lyon, elle a étudié des problèmes posés à des élèves du collège et du lycée, qui visaient au développement d'attitudes de recherche et de capacités de méthodologie scientifique (Arsac & Mante, 2007). Une recherche scientifique développe des capacités de méthodologie multiples, comme la formulation des hypothèses de travail, la préparation du projet expérimental ou de recherche, le choix de l'échantillon, la mise en place des outils d'évaluation, l'analyse et l'interprétation des résultats. Certaines de ces capacités apparaissent à chaque problème ouvert. Selon l'équipe de l'IREM de Lyon, le problème ouvert, proposé et examiné par les élèves, présente les caractéristiques suivantes :

- *L'énoncé est court.*
- *L'énoncé n'induit ni la méthode, ni la solution (pas de questions intermédiaires ni de questions "montrer que"). En aucun cas, cette solution ne doit se réduire à l'utilisation ou à l'application immédiate des derniers résultats présentés en cours.*
- *Le problème ouvert se trouve dans un domaine conceptuel avec lequel les élèves sont assez familiarisés. Ainsi, peuvent-ils prendre facilement « possession » de la situation et s'engager dans des essais, des conjectures, des projets de résolution, des contre-exemples. (Arsac & Mante, 2007, p. 20)*

L'équipe de l'IREM de Lyon a voulu rendre au problème une place importante dans l'activité des élèves. Ella a voulu les placer dans une situation d'apprentissage qui les amène à : *essayer-conjectures-tester-prouver*. De manière plus analytique, l'adoption des caractéristiques ci-dessus, que le groupe de recherche de l'IREM de Lyon considère comme nécessaires à chaque problème ouvert, est justifiée comme suit : l'énoncé du problème est habituellement court et formulé en langage courant ou mathématique ; l'énoncé simple et court favorise la lecture rapide, la compréhension et crée des conditions de facilité en ce qui concerne ce qui se maintiendra en mémoire et la gestion des données. En outre, il peut donner l'impression que le problème est facile et incite ainsi à s'y intéresser ; en aucun cas, cette solution ne devra se limiter à l'utilisation simple ou à l'application directe de conclusions ou de règles qui se sont présentées durant les derniers cours il constituerait alors un problème d'application directe et non un problème ouvert. Cependant, il est fondamental que l'énoncé du problème ouvert ne résulte pas directement de la méthode et de la solution ; le problème ouvert doit être fondé sur des notions avec lesquelles les élèves sont assez familiarisés. Il est indispensable afin que les élèves, dans le cadre des restrictions habituelles de l'horaire scolaire, puissent formuler des résultats ou produire des idées.

L'attribution du problème ouvert et l'insertion de phases a-didactiques (Brousseau 1986) provoque une rupture avec le "contrat didactique" et révèle certains aspects invisibles qui aident l'enseignant à opérer un réexamen critique de sa pratique habituelle (Balacheff, 1988). L'élève conçoit très peu que, pour résoudre de tels problèmes, il n'applique pas directement des connaissances enseignées recherchant dans l'énoncé des "mots-clefs" pour trouver la bonne opération qui fournit le résultat et effectuer correctement cette opération conduisant à une seule solution. Au contraire, il faudra qu'il cherche seul et qu'il prenne des initiatives. Dans ces conditions, les élèves doivent pouvoir saisir facilement la situation et prendre part à des essais, formuler des conjectures, établir des démarches de vérification, des projets de résolution et des contre-exemples, qui visent à la découverte et à la création de la solution ou des solutions du problème ouvert.

En Grèce, les auteurs des manuels scolaires des trois classes du collège qui sont utilisés depuis 2007, malgré leurs différences, soulignent que l'amélioration de l'enseignement des mathématiques est liée à la qualité des opportunités d'apprentissage offertes aux élèves. Le contenu mathématique est organisé avec, pour axe central, les «activités». Ils visent à l'incitation active des élèves dans le processus d'apprentissage et particulièrement le déploiement du raisonnement et de la communication à l'aide du langage mathématique. Cependant, les professeurs donnent moins d'importance à « l'activité des élèves » et plus de priorité à leur propre activité. En outre, dans le cadre du nouveau programme officiel, il est recommandé d'utiliser autant des problèmes de la vie quotidienne qui ont un sens pour les élèves que d'autres problèmes, originaux ou inhabituels, qui mobilisent leurs capacités logiques. Dans cet esprit on privilégie l'apprentissage participatif des problèmes ouverts et on demande de l'initiative et une action innovatrice de la part des enseignants. Il est mentionné de manière significative: *L'activité vise à l'encouragement de la coopération et du travail en groupe…*(Argyrakis et al., 2007, p. 9). *Dans cette voie, les activités d'apprentissage qui comprennent des travaux de recherche et de travail en petits groupes d'élèves constituent un outil important* (Vandoulakis et al., 2007, p. 32). De plus, les problèmes ouverts sont mentionnés dans le même livre. *En général nous appelons problème ouvert ce qui peut être interprété de nombreuses manières et par conséquent admettre des solutions différentes…* (Ibid., p. 33). En général, dans les pratiques quotidiennes, les approches d'enseignement alternatives ne sont pas largement étendues et les problèmes ouverts représentent une activité marginale dans les pratiques de classe les plus courantes et les professeurs habituellement les évitent. Par la suite, dans le cadre d'une certaine liberté relative, nous décrirons et analyserons certaines pratiques pédagogiques avec des problèmes ouverts dans le collège grec.

Dans ce travail, nous allons étudier les raisonnements mathématiques des élèves du collège expérimentant et élaborant des solutions aux problèmes ouverts liés aux expériences de leur vie quotidienne. Seront présentées ici les données de la solution collaborative de trois problèmes. C'est sur ces expérimentations didactiques que va se focaliser l'intérêt de la recherche des interactions mathématiques de la classe et particulièrement sur la formulation des conjectures, des échanges argumentés et des stratégies élaborées par les élèves. On émet l'hypothèse que les problèmes ouverts ont un intérêt pédagogique pour les mathématiques du collège (d'apprentissage, méthodologique, d'autorégulation métacognitive, effectif) et qu'ils constituent des occasions créatives qui suscitent la curiosité et le goût pour la découverte en mathématiques, stimulent l'intérêt et motivent l'engagement, la participation et la persévérance des élèves.

La méthodologie de l'expérimentation.

La recherche s'est déroulée tout au long de l'année scolaire 2008-2009 dans les trois classes du 1er Collège Expérimental d'Athènes. Les élèves avaient de 11 à 15 ans et provenaient surtout des couches sociales moyennes. L'enseignant était le professeur de mathématiques de l'école.

Nous avons observé les interactions entre les élèves pendant la durée de leur coopération (par groupes de quatre ou de deux) et pendant la présentation commune à toute la classe. Au 1er problème la classe a été séparée en 6 groupes et au 2^{d} en 7 groupes. Pour cette expérimentation, nous avons utilisé généralement deux séances d'une heure (pour le troisième problème une heure). La première heure consistait en la recherche individuelle et collective et la rédaction des diapositives. La deuxième heure était consacrée au débat ouvert en classe. Lors de la première séance, nous avons pris des notes et nous avons considéré les écrits individuels et collectifs. Les séances ont été enregistrées en vidéo et certains épisodes des données produites par des méthodes audiovisuelles sont présentés ici. L'analyse des données est qualitative et concerne l'observation participative de la classe. Les axes d'observation sont les suivants : la rectitude et la pertinence des arguments, la généralisation de stratégie (avec les explications et les justifications), les moyens de représentation et l'utilisation de symbolisme. Nous examinons principalement le développement du raisonnement mathématique et de la justification au cours des activités (Arsac et al., 1992; Kosyvas & Baralis, 2010). La résolution des problèmes ouverts est liée à un travail collaboratif de la classe (Arsac & Mante, 2007; Sauter et al., 2008; Kosyvas, 2005).

On a choisi des problèmes ouverts qui pourraient susciter un véritable travail expérimental et favoriser le questionnement des élèves, l'élaboration de conjectures et les échanges argumentés. Dans chaque classe a été organisée une expérimentation. Les problèmes qui ont été cherchés sont deux problèmes d'arithmétique (quatre opérations, proportionnalité) et un problème de géométrie. Tous ces problèmes concernent des notions mathématiques compatibles avec le programme scolaire du collège grec. Avant l'expérimentation d'autres problèmes ouverts ont été proposés dans les classes. C'est pourquoi les élèves ont été familiarisés avec les problèmes ouverts et le travail en groupe. On va se limiter à une description synoptique des découvertes les plus importantes qui en ont résulté tant au cours de la phase de collaboration en groupes que pendant les débats dans les classes. Pendant la présentation commune à toute la classe les élèves soutiennent leurs solutions ou sont convaincus par leurs camarades de les dépasser. Les arguments divers et les cheminements multiples des élèves sont liés à des types différents de raisonnement mathématique. On expose ci-dessous les comptes rendus des expérimentations.

Les résultats de l'expérimentation du premier problème (*La tirelire*)

Ce problème favorise l'expérimentation féconde tant au troisième cycle de l'école primaire qu'au collège. Le problème a été posé et expérimenté dans une classe de 1re année du collège grec (Kosyvas, 2011), qui correspond à la cinquième du collège français. Ce problème a été intégré dans l'unité des « problèmes des quatre opérations ». Les élèves avaient terminé le premier chapitre de leur livre qui concernait l'enseignement combinatoire de nombres entiers et décimaux. L'énoncé du problème est le suivant :

Enoncé 1 **:** *Les élèves d'une classe ont ramassé dans leur tirelire une somme de 120 euros en 15 billets de 5 et 10 euros. Combien de billets de chaque valeur sont dans leur tirelire ?*

L'objectif du problème n'était pas seulement l'émergence de la méthode par essais et erreurs, laquelle constitue essentiellement une priorité de l'école primaire, mais surtout la découverte de stratégies arithmétiques subtiles de la part des élèves. Ce problème concerne la vie quotidienne et on s'attend à ce que les élèves de la 1re année du collège grec le résolvent avec des connaissances d'arithmétique pratique et non d'algèbre, puisque ils n'avaient pas encore acquis des connaissances de ce type. La structure mathématique du problème renvoie à un système linéaire de deux équations avec deux inconnus.

L'appropriation du problème par les élèves des 6 groupes était efficace. Après quelques interventions du professeur pour les inciter à travailler en groupes, les élèves se sont bien impliqués dans l'investigation et la résolution du problème. Les procédures de résolution étaient très variées. Les stratégies spontanées que les élèves ont suivies pour la solution du problème peuvent être divisées en trois niveaux :

Niveau intuitif : raisonnement holistique ou explorations insuffisamment justifiées (la méthode essai-erreur, réponses sans cohésion)

Certains élèves ont deviné la solution sans réflexion ou ils ont donné une réponse correcte sans qu'ils puissent convaincre de la justesse de leur choix. Le dialogue suivant est caractéristique :

M.: *Il y aura dans la tirelire 6 billets de 5 € et 9 billets de 10 €.*

Ens. : *Pourquoi y aura-t-il* 6 *billets de 5 € et 9 billets de 10 €?*

M.: *Nous avons trouvé le résultat correct. Le voilà.*

Ens. : *Pourriez-vous écrire sur la diapositive votre justification ?*

M.: *Nous avons dessiné notre solution. La tirelire va contenir 6 billets de 5 € et 9 billets de 10 €. C'est sûr.*

Ens. : *Comment vous l'expliqueriez ?*

M.: (Silence).

Ens. : *Comment vous avez trouvé la solution ?*

D.: *... Au hasard !*

Ici, la solution du problème commence par une bonne prévision naturelle, mais la découverte réussie n'est pas liée à d'autres essais expérimentaux et des vérifications. Lorsque les élèves ne disposent pas d'une méthode ou d'un algorithme de solution ils font des observations empiriques, des anticipations arbitraires ou des conjectures « hâtives» et ils contrôlent leur validité avec un nombre limité de cas numériques. Au premier niveau sont réunies surtout des réponses réussies « d'essai et d'erreur » sans une justification suffisante ou d'autres affirmations sans cohésion logique qui mènent à une solution erronée du problème et à des démarches ratées. Souvent les élèves ne contrôlaient pas si la solution qu'ils trouvaient satisfaisait aux contraintes exigées. La solution du problème se réduit à la découverte d'une paire de nombres qui vérifient l'équation $5x+10y=120$ sans vérifier l'autre, c'est-à-dire le $x+y=15$ ou même inversement.

Niveau analytique et descriptive : Explorations numériques organisées, hypothèses et justifications

Au deuxième niveau, les stratégies des élèves sont plus systématiques. Habituellement les élèves font des tableaux sur lesquels ils placent leurs observations dans des structures organisées. Ils mettent les données par écrit (enregistrements complets ou d'ampleur limitée) et ils choisissent la solution numérique qui satisfait les deux conditions du problème.

De plus, au cours de la communication mathématique, les élèves formulent et contrôlent des conjectures et font paraître des explications. Cependant leurs solutions ne sont pas les plus brèves possibles. Ils font apparaître des stratégies « de pluralité invariable de billets de la tirelire » et « de somme totale invariable de la tirelire ». Ces découvertes sont en accord avec d'autres recherches de problèmes semblables de « fausse position » (Porcheron & Guillaume, 1984; Silver et al., 1990; Sauter, 1998; Pluvinage, 2008). Les stratégies précédentes ont conduit à deux types de tableaux :

- Au premier type, qui a été observé surtout chez les groupes A et B, les élèves maintiennent invariable la somme des billets de la tirelire (x+y=15) et forment successivement des sommes de 15, lesquelles ils vérifient en essayant de créer progressivement le nombre 120 (5x+10y=120).
- Au deuxième type de tableau (groupe D, voir le tableau ci-dessous) ils travaillent en constituant des compositions du chiffre 120, et ensuite ils procèdent à des contrôles pour constater s'ils forment la somme 15. Dans ce cas est effectuée une numération de remplacements : Nous avons des remplacements répétés, où les billets de 5 € sont réduits de deux et donnent leur place à un billet de 10 €, donc il augmente d'un la pluralité des billets de 10 €, tandis que la tirelire a toujours la même somme totale d'argent.

Le dénombrement organisé de toutes les données spécifiques du tableau prend du temps, cependant il a une valeur pédagogique importante. Bien que les opérations numériques qui sont insérées dans le tableau soient fondamentales et récurrentes et que l'équilibre penche davantage vers les explorations numériques et non pas vers la force convaincante de l'argumentation mathématique élaborée, il s'agit d'une méthode scientifique qui exige vérification exacte et organisée, où les élèves prennent confiance. La présentation du tableau engendre la réflexion et favorise l'observation de structures et de relations.

Dans ce cas, l'utilisation de la méthode essai-erreur n'est pas arbitraire, occasionnelle ou imprévue. Nous reformulons brièvement une voie régulière et adaptative des enfants que nous avons constaté aux groupes B et F:

Si les 15 billets de la tirelire étaient des billets de 10 €, alors la valeur totale serait 150 €, tandis que si tous étaient des billets de 5 €, alors la tirelire aurait 75 euros. Cependant puisque le contenu de la tirelire est 120 €, on doit essayer des combinaisons intermédiaires appropriées:

Avec 11 billets de 10 € et 4 billets de 5 € nous trouvons 130 €.

Avec 7 billets de 10 € et 8 billets de 5 € nous trouvons 110 €.

De nouveaux essais s'approchent davantage du nombre 120 :

Avec 10 billets de 10 € et 5 billets de 5 € nous trouvons 125 €.

Avec 8 billets de 10 € et 7 billets de 5 € nous trouvons 115 €.

Enfin en essayant 9 billets de 10 € et 6 billets de 5 € nous trouvons 120.

Dans ce processus de sens double, les élèves examinent des exemples particulières, ils développent des essais raffinés, des relations spécifiques et ils arrivent au résultat de la solution. Pour cette méthode les élèves s'occupent des essais étudiés. Chaque nouvel essai est justifié, il se forme sur la base du précédent et l'améliore. Progressivement l'erreur est corrigée, alors que les essais successifs s'approchent de plus en plus du résultat souhaitable. C'est pourquoi il vaut mieux parler « d'essais successifs multiples », « de corrections successives » ou « d'approches successives » (Polya, 1962). L'usage de ce raisonnement adaptatif a une valeur didactique distincte améliorant la compréhension des élèves et les aidants à clarifier ce qui est vrai ou faux.

Niveau de raisonnement sur les relations arithmétiques : Abrégement et généralisation de la solution

Au troisième niveau, les élèves, tant en groupes qu'en classe entière, justifient leurs conjectures et perfectionnent leurs stratégies en inventant des solutions élégantes et économiques qui sont caractérisées par la reformulation et l'organisation originale de l'argumentation logique. Trois solutions ont été observées qui s'intègrent à cette catégorie (Kosyvas, 2011). On va mentionner une de celles-ci.

Groupe D La solution du groupe a été donnée par un tableau différent de celui des groupes A et B. Ici, on maintient stable la somme totale en € (5x+10y=120) et on recherche la combinaison appropriée des billets de 5 et

10 € qui forment une somme de 15 (x+y=15). Les explications du représentant du groupe ont pris la forme du dialogue suivant :

P.: *Dans notre groupe nous avons pensé faire l'hypothèse que les 120 € qui existent dans la tirelire sont tous des billets de 5 €* (Il a montré le tableau suivant sur la diapositive).

S.: *Si ceci était vrai, nous aurions 120:5 =24 billets de 5 €. Cependant dans notre tirelire il existe au total 15 billets et non 24.*

D.: *Vous dites qu'il existe 15 billets qui font 120 €. Je ne comprends pas pourquoi nous voulons qu'il en existe 24. Puisque ce ne sont pas tous des billets de 5 €.*

P.: *Je ne soutiens pas qu'ils sont réellement 24. Eh bien ! Nous prétendons qu'ils sont 24 ! Nous savons que ceci n'est pas vrai. C'est pourquoi nous avons pensé à diminuer les billets deux par deux. Si, des 24 billets de 5 €, nous enlevons deux billets de 5 € et nous les remplaçons par un billet de 10 €, nous aurons à nouveau 120 € (120-2×5+1×10=120). Ainsi maintenant nous aurons 23 billets (24-2+1=23). En continuant de la même façon nous avons construit le tableau suivant.*

TABLEAU DES REMPLACEMENTS DU GROUPE D				
numérotation des remplacements	**nombre de billets de 5€**	**nombre de billets de 10 €**	**total stable de la tirelire en €**	**somme des billets**
1.	24	0	24×5+0×10=120	24+0=24
2.	22	1	22×5+1×10=120	22+1=23
3.	20	2	20×5+2×10=120	20+2=22
4.	18	3	18×5+3×10=120	18+3=21
5.	16	4	16×5+4×10=120	16+4=20
6.	14	5	14×5+5×10=120	14+5=19
7.	12	6	12×5+6×10=120	12+6=18
8.	10	7	10×5+7×10=120	10+7=17
9.	8	8	8×5+8×10=120	8+8=16
10.	6	9	**6×5+9×10=120**	**6+9=15**

Il faut souligner, au cours de la solution précédente, l'importance d'exploration de la conjecture. L'émission de l'hypothèse erronée « *Eh bien ! Nous prétendons qu'ils sont 24 !»,* (*des billets de 5 €*) cette « fausse position » était un processus mathématique important lié au raisonnement d'exploration elle implique en plus des étapes de justification et de validation. Dans le tableau précédent, nous observons que les élèves essaient des remplacements successifs, où ils enlèvent deux billets de 5 € et donnent leur place à un billet de 10 €, donc ils augmentent la pluralité des billets de 10 €, tandis que la tirelire a toujours le même total d'argent. La solution sera trouvée lorsque la somme des billets sera devenue 15, 6 billets de 5 € et 9 billets de 10 €. Lors des échanges argumentés, qui ont suivi, un élève a réussi à trouver une autre solution originale.

A.: *J'ai regardé le tableau avec attention. J'ai observé que nous pourrions trouver dès le début combien de remplacements sont nécessaires, sans construire un tableau et ainsi gagner du temps.*

Ens. *C'est-à-dire ?*

A.: *Alors! Chaque remplacement réduit les billets de un et nous voulons que les 24 billets deviennent 15. D'accord ?*

Ens. *Continue!*

A.: *Pour trouver combien de remplacements sont nécessaires, nous ferons une soustraction. De 24 nous retirons 15, c'est-à-dire 23, 22, 21, 20, 19, 18, 17, 16, 15. Donc nous trouvons 9 remplacements* (il compte avec les doigts).

Ens. *Continue !*

E.: *Pourquoi ? Je n'ai pas compris !*

Ens. *Pour comprendre il faut que vous fassiez attention !*

A.: *Donc! Ainsi les remplacements qui sont nécessaires sont 24-15=9. Je crois que c'est mieux de calculer avec le cerveau plutôt que de les compter un par un.*

M.- K.: *Ni compter avec les doigts. Moi je ne suis pas d'accord ! Il nous faut le tableau entier. Comme celui-ci que nous avons construit.*

Ens. *Chacun a sa propre manière. Respectons-le. Je vous prie ne pas interrompre. Continue A !*

A.: *Le tableau est bon mais il exige beaucoup de temps pour le construire. Donc nous observons que grâce aux 9 remplacements existeront dans la tirelire 120 € à 15 billets. Nous constatons qu'avec les 9 remplacements que nous ferons on en enlèvera au total 9×2=18 billets de 5 €. Alors nous aurons 24-18=6 billets de 5 € et enfin nous aurons 9 billets de 10 €, puisque 15-6=9.*

P. *Le tableau est utile. Si nous avons fait ce tableau…*

A.: *Le tableau nous a aidés à trouver une solution plus courte …*

…..

A.: *Notre solution est la meilleure. Si la tirelire avait par exemple 100 billets, de 5 € et de 10 €, ce qui ferait une valeur de 950 €, alors construiriez-vous un tableau ?*

M.- K.: *Alors non…* (silence).

A.: *C'est impossible ! Avec notre idée on peut résoudre le problème pour 100 ou 1000 billets dans la tirelire. Bien que les nombres du problème puissent être complètement différents, l'argument arithmétique reste toujours le même.*

M.- K.: *Je ne suis pas sûre que je puisse réfléchir à votre propre solution.*

L'argumentation précitée a été répétée par un autre membre du groupe et est devenue acceptable, suscitant pour certains de leurs camarades un regard réflexif sur leur pratique. Il est évident que la création du tableau a donné à cette élève l'idée de la description de relations numériques générales qui ont conduit à la solution simple et synoptique précédente, laquelle s'applique aussi à de grands nombres. L'observation subtile de l'organisation systématique du tableau et la corrélation mentale de ses éléments ont contribué à l'omission des calculs inutiles, à la découverte de régularités cachées importantes et à l'abstraction logique. La condensation de la solution montre une compréhension plus profonde.

À la fin, une discussion a portée sur le choix de la meilleure solution. Certains considéraient qu'il est meilleur d'utiliser des méthodes analytiques et de faire des tableaux où on peut essayer de nombreuses solutions (peut-être toutes) et choisir la correcte et d'autres croyaient qu'il valait mieux utiliser les méthodes d'élaboration d'hypothèses et la vérification via des opérations numériques ou autres arguments qui se sont enchaînés de façon logique. Il a été remarqué qu'avec des raisonnements originaux les solutions sont brèves et élégantes. Il est remarquable que la manipulation dynamique du problème ait engendré de nouvelles solutions dérivées que les élèves ont recherchées avec de nouvelles stratégies. La stratégie précitée est améliorée, elle ne s'appuie ni sur la chance ni sur le recensement complet de nombreux cas, quand ce n'est pas nécessaire. Des solutions intégrées à ce niveau ont été observées dans les trois groupes (à cause du manque d'espace, elles ne sont pas mentionnées). Ces solutions mettent en valeur les éléments empiriques du problème et sont confrontées avec succès à d'autres problèmes concernant de grands nombres. Elles ont un caractère de généralisation, puisque la réponse numérique unique du problème peut être considérée comme un cas particulier. Par conséquent, le rapport étroit entre le raisonnement numérique et algébrique devient évident. La pensée algébrique n'est pas liée automatiquement à l'utilisation de lettres. Ces solutions sont composées et comprennent des méthodes sophistiquées, fondées sur l'introduction de conjectures au problème, en développant un raisonnement généralisé, numérique, ou presque algébrique (bien que les nombres du problème puissent être complètement différents, l'argument arithmétique reste toujours le même). Cette caractéristique les place en relation de prééminence par rapport aux mises en tableaux inefficaces.

De plus, puisque les raisonnements mathématiques des élèves sont directs et éloquents, ils ont un avantage par rapport aux formalismes où domine le maniement algorithmique mécanique, qui souvent aboutit à la perte du sens d'idées importantes. L'utilisation des lettres en tant que symboles numériques abstraits représentant les variables inconnues du système linéaire constitue une évolution des stratégies informelles. Cette méthode exige la traduction de la langue quotidienne en langue algébrique. Elle peut être appliqué avec succès tant pour les petits nombres que pour les grands nombres. Elle n'exige pas l'invention d'une stratégie originale et ainsi elle est efficace pour une grande catégorie de problèmes analogues. Cela constitue le stade suivant auquel parviennent les élèves lorsqu'ils se trouvent en 3^e^ année du collège Grec (correspondant à la troisième du collège français).

- La généralisation algébrique comprend seulement certains aspects du partiel, tandis qu'au cas spécial sont inhérents des avantages mathématiques rajoutés (structures, propriétés, régularités) et esthétiques (beauté, élégance) qui habituellement restent dans l'obscurité, cependant à notre expérience d'enseignement se sont été révélés aux solutions originales des élèves.

- En outre, les élèves utilisent diverses procédures mathématiques et algorithmes, sans réaliser le besoin de s'engager sur leur sens sous-jacent et décortiquer ces concepts. Avec la force puissante des symboles algébriques coexiste l'absence de sens référentiel. Leur déconnexion des expériences directes du monde réel aux yeux des enfants les rend incompréhensibles et arbitraires.

Enfin, nous pouvons généraliser davantage le problème en remplaçant les nombres du deuxième membre avec des variables paramétrées. Cette généralisation est enseignée au lycée.

En résumé, le problème ouvert que nous avons expérimenté s'est avéré propice à l'action coopérative des élèves, en révélant des différences qui ont facilité l'interrogation et la recherche. Les résultats de cette étude montrent que 5 des 6 groupes ont résolu le problème de la tirelire avec des stratégies spontanées multiples, qui n'avaient pas été enseignées aux élèves. Bien sûr la plupart des 26 élèves ont déployé des aptitudes impressionnantes d'émission de conjectures, en exposant avec leurs propres moyens des stratégies logiquement argumentées. Les élèves réunissent, enregistrent, écrivent des données sur les tableaux, observent et organisent des cas particuliers, examinent des conclusions, généralisent, expliquent et communiquent leurs découvertes. Parmi ces solutions, il y en a trois arithmétiques, originales et uniques, qui ont été inventées et présentées par les élèves. *L'élégance et la beauté de ces solutions arithmétiques sont perdues dans les équations algébriques abstraites*. De plus, il en résulte que les raisons subtiles et les généralisations des élèves sont étroitement liées entre elles en élargissant leur pensée mathématique.

Les résultats de l'expérimentation du second problème
(*Les caravaniers*)

L'énoncé a été posé et expérimenté (Kosyvas, 2010b) dans une classe de 2^de^ année du collège grec (correspondant à la quatrième en France). Deux observateurs étaient présents au cours de cette expérimentation. Le problème décrit de manière amusante une situation qui renvoie au partage, mais non pas au partage équitable. Grâce à son analogie, nous avons intégré ce problème dans l'unité de la proportionnalité. On avait enseigné aux élèves des équations simples et la proportionnalité l'année précédente, cependant ce problème possède une structure différente et ne constitue pas une application directe de la proportionnalité. Les élèves devaient distinguer avec une attention subtile les grandeurs qui sont proportionnelles. Le problème n'introduit pas de notions mathématiques nouvelles, mais suscite une réflexion critique sur des méthodes et des notions déjà connues.

Enoncée 2 *: Deux caravaniers A et B qui voyageaient dans le désert avaient avec eux l'un 2 pains et l'autre 3. Sur leur route, ils ont rencontré un voyageur riche C, qui avait faim. Après avoir mangé tous ensemble le voyageur leur a laissé 15 écus d'or. Comment faudrait-il procéder au partage de l'argent ?*

C'est un problème traditionnel que nous avons reformulé. Incontestablement, l'énoncé le rend très attractif et ludique. C'est pourquoi nous avons considéré qu'il mobiliserait l'intérêt et la curiosité des élèves. La solution exige des opérations arithmétiques avec des nombres entiers ou fractions et la notion du partage en parties proportionnel. La raison de l'utilisation de la proportionnalité renvoie aux conventions sociales. Dans ce problème on suppose que le nombre des écus d'or que C a payés est proportionnel à la quantité des pains qu'il a reçue. Le choix de stratégie dépend de facteurs liés au type du problème et aux relations qui dépendent des données numériques. En accord avec ce qui avait été enseigné, on s'attendait à ce que les élèves résolvent le problème en utilisant la méthode suivante :

Chacun a mangé $\frac{5}{3}$ *de pains. On partage le nombre 15 en parties proportionnels aux nombres* $\frac{1}{3}$ *et* $\frac{4}{3}$*. On construit le tableau suivant:*

	A et B	*A*	*B*
Quantité de pains que C a reçue (fractions de pains)	$\frac{5}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
Nombre d'unités monétaires que C a payé (écus d'or)	*15*	*x*	*y*

On multiplie diagonalement : $\frac{5}{3}x = \frac{1}{3} \cdot 15$ *ou* $5x = 15$ *ou* $x = 15 : 5$ *ou* $x = 3$

Donc A va recevoir : x=3 écus.   Et   B : 15-3=12 écus.

Cependant la méthode précédente n'est pas apparue. Il se peut que son absence soit due aux difficultés de représentation correcte de ce problème. Ou bien ce mode standardisé de raisonnement n'avait pas été bien maîtrisé par les élèves dans le cas des fractions.

Pendant la phase de recherche (individuelle et en groupe) le problème semblait difficile, il y a eu beaucoup d'incompréhension ne disposant pas d'un algorithme prêt ils étaient bloqués. Au début, les essais et les tâtonnements des élèves se faisaient au hasard et sans aucune organisation systématique. On a fait une mise au point générale en encourageant les élèves à réfléchir au problème. Progressivement ils ont mis en place leurs différentes idées, qui ne se sont pas toujours organisées en solutions complètes. On présente brièvement un bilan de l'élaboration des productions de certains de 7 groupes.

Première rédaction (une diapositive): D'après cette production, l'argent devrait être partagé équitablement entre A et B. Ils ont écrit : « *Un partage juste ! Donc 7,5 l'un et 7,5 l'autre* ». Il est évident que ces élèves avaient une représentation inadéquate et erronée du problème. Pendant le débat beaucoup d'élèves ont formulé l'argument « *B a donné plus de pain que A. Pourquoi devraient-ils recevoir le même somme* ? ». La discussion était très riche et a conduit à la notion de proportionnalité.

Seconde rédaction (quatre diapositives): Une source d'erreurs était l'erreur de compréhension suivante : l'argent a été partagé entre A et B proportionnellement aux pains que chacun avait. Avant la présentation, on a parlé des difficultés de compréhension initiale comme d'une étape nécessaire et naturelle ainsi que de la reconnaissance du droit des élèves à l'erreur. On présente deux solutions. Les autres sont semblables :

GROUPE 5

On complète le tableau de proportionnalité:

	A et B	A	B
Nombre de pains	5	2	3
Nombre d'écus	15	x	y

On a : $\frac{x}{2}=\frac{y}{3}=\frac{x+y}{5}=\frac{15}{5}=3$.

Par suite: $\frac{x}{2}=\frac{3}{1}$ *or* $x \times 1 = 2\times 3$ *or* $x=6$.

Donc A va recevoir 6 écus d'or.
Et B : 15-6=9 écus.

GROUPE 3

On utilise la méthode de passage à l'unité.

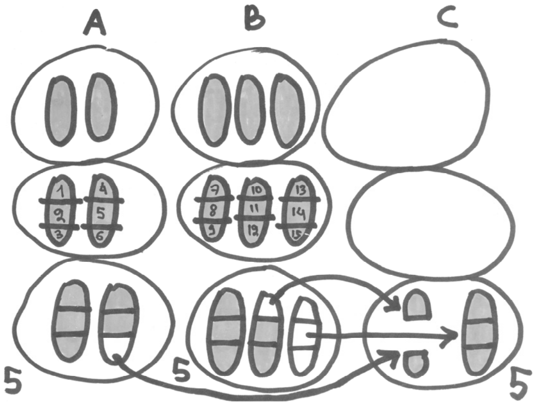
Les 5 pains coûtent 15 écus d'or.
1 pain coûte 15 :5=3 écus.
Les 2 pains de A font $2\times 3=6$ *écus.*
Les 3 pains de B font $3\times 3=9$ *écus.*

Vérification : 6+9=15 écus.

Pendant le débat de mise en commun, les élèves ont discuté de l'application correcte de la proportionnalité et ont mis en évidence la validation de la stratégie. Une élève a objecté. « *Ils ont utilisé la proportionnalité pour les pains qu'ils avaient et non pour les pains qu'ils ont donnés. C'est faux* ». On peut constater que les erreurs commises ont rendu le débat très intéressant. Quelques élèves ont montré un esprit plus critique sur leurs productions écrites.

Troisième rédaction (une diapositive): Premièrement, il faut que les pains soient partagés équitablement entre A, B et C et que les écus soient partagés proportionnellement entre A et B (conformément à la quantité de pain qu'ils ont donnée à C).

GROUPE 2



Les A, B, C partageront équitablement les 5 pains. Ainsi nous séparons chacun des 5 pains en 3 morceaux égaux, un pour chacun. Sur ce dessin nous avons compté 15 morceaux de pain, c'est-à-dire que chacun devait manger 5 morceaux. A avait 6 morceaux. Il a gardé les 5 et il en a donné 1 à C. B avait 9 morceaux. Il a gardé les 5 et il en a donné 4 à C. C a reçu au total 5 morceaux, 1 de A et 4 de B (1+4=5). C pour 5 morceaux de pain a payé 15 écus d'or, c'est-à-dire 3 écus pour chaque morceau. Donc A a vendu 1 morceau et a reçu 3 écus et B 4 morceaux et a reçu 4×3=12 écus.

Ce groupe n'a pas résolu le problème avec la méthode formelle de proportionnalité enseignée, comme le groupe 5 l'avait fait sans succès; mais les élèves ont étayé leur rédaction en utilisant un langage naturel, libéré de tout formalisme. Les images qu'ils ont dessinées ont facilité leur raisonnement arithmétique. D'après la présentation de leur solution, ils ont utilisé des stratégies informelles de partage et ils ont préféré des modèles additifs par rapport à des modèles proportionnels. L'idée de l'utilisation de la proportionnalité comme modèle social nécessite la réalisation de l'équité ou l'égalisation des parts ce qui constitue le premier niveau de l'abstraction de la part des élèves, « *C a reçu au total 5 morceaux* ». En réalisant une distribution « régulière », s'est créé intuitivement le concept de ratio de part constante d'écus par morceau, « *3 écus pour chaque morceau »*. Par conséquent « *A a vendu 1 morceau et a reçu 3 écus et B 4 morceaux et a reçu 4×3=12 écus* » qui est influencé par la compréhension de la multiplication et de la division et nécessite un deuxième niveau d'abstraction.

La découverte de la solution a été obtenue par la collaboration de deux élèves. Les deux autres membres du groupe se sont enthousiasmés et ont changé leur attitude en cours de mathématiques, devenant actifs et demandeurs. La solution trouvée par ce groupe était compréhensible et convaincante. Au cours de la discussion en commun, certains élèves ont contesté la méthode du dénombrement sur le dessin en utilisant des stratégies de l'arithmétique mentale: « *Au lieu de compter on peut calculer : Puisque il y a 3 personnes A, B et C, on partage les pains de chacun en 3 morceaux égaux. A avait 2×3=6 et B 3×3=9. Au total il y a 6+9=15 morceaux du pain pour les trois. Chacun a mangé 15:3=5 morceaux. A a donné 6-5=1 et B 9-5=4 etc.* ».

L'élaboration de stratégies informelles par les élèves peut préparer des stratégies formelles. Resnick et Singer (1993) soutiennent que des concepts fondamentaux des mathématiques, comme sont les rapports et la proportionnalité, peuvent être développés avec succès seulement s'ils s'appuient sur ou intègrent les connaissances informelles et non standardisées des élèves. Les concepts de rapport et de proportionnalité ne sont pas isolés, ils s'intègrent aux champs multiplicatifs conceptuels et se forment en interaction avec la multiplication, la division, la fraction, les nombres rationnels et les fonctions linéaires (Vergnaud, 1988). Avant même que les enfants reçoivent l'enseignement de ces notions, ils disposent déjà d'un répertoire riche de stratégies sur leur résolution.

Quatrième rédaction (une diapositive):

GROUPE 1

Personnes	*Il a eu*	*Il a mangé*
A	$2=\frac{6}{3}$ *pains*	$\frac{5}{3}$ *pains*
B	$3=\frac{9}{3}$ *pains*	$\frac{5}{3}$ *pains*
C	$0=\frac{0}{3}$ *pains*	$\frac{5}{3}$ *pains*

A a donné $\frac{1}{3}$ *de pains.*

Ce groupe n'a pas complété sa solution. Les élèves ont fait la division 5:3 mais c'était impossible de trouver un nombre décimal fini. La périodicité **5:3=1,66…** était un obstacle.

Elève *: -Est-ce qu'on peut simplifier et le faire 1,7 ?*

Enseignant *: -Tu pourrais réfléchir un peu plus.*

Distinguer un nombre d'une de ses valeurs approchées n'est pas transparent pour certains élèves. Cette différence a des conséquences visibles sur les résultats du problème. Pendant le débat, un élève a donné la solution suivante (passage à l'unité):

Les $\frac{5}{3}$ *des pains (C) font 15 écus d'or.*

Le $\frac{1}{3}$ *du pain (que A a donné à C) fait 15 :5=3 écus.*

Les $\frac{4}{3}$ *de pains (que B a donnés à C) font* $4\times 3=12$ *écus.*

Cette stratégie a attiré l'attention des élèves qui s'étaient trompés et a suscité une attitude réflexive de leur part.

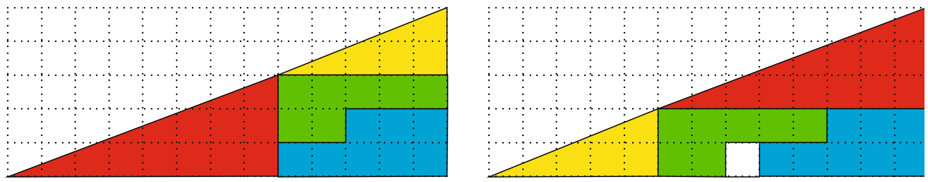
Le passage à l'unité est considéré par Nesher et Sukenik (1991) comme une méthode performante parce qu'il inclut la notion de la partition qui aide à la compréhension conceptuelle du rapport. Conformément à celle-ci, les élèves trouvent d'abord la valeur d'un objet et ensuite la valeur de nombreux objets similaires. Elle est efficace lorsque le prix d'un objet est un nombre entier, tandis que le passage à la fraction de l'unité fonctionne négativement, et conduit à une impasse lorsque les données ne mènent pas à un prix entier de l'unité.

En résumant, les résultats de cette activité montrent que deux solutions correctes du problème ont été exposées. Bien que le problème se situe dans le champ des connaissances des élèves, nombre d'entre eux ont exposé des représentations inadéquates. Quatre groupes ont utilisé une proportionnalité erronée « *pour les pains qu'ils avaient et non pour les pains que chacun a donnés*», et un groupe a procédé au « *partage équitable des écus* ». Incontestablement, la discussion riche dans la classe fut une procédure active et créative, ce qui souligne que les mathématiques ne sont pas la mémorisation de formules mystérieuses et de règles injustifiables, mais *ce que font et pensent les élèves lorsqu'ils communiquent entre eux*. C'est un défi intéressant avec des motivations abondantes pour les élèves.

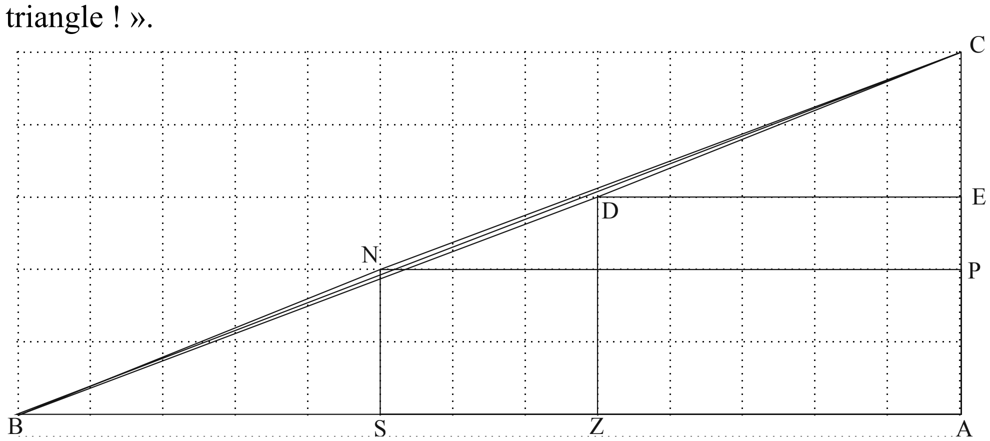
Les résultats de l'expérimentation du troisième problème (Situation de surprise : *32=33!*)

Le problème suivant est un paradoxe géométrique fameux, présenté par Paul Curry, magicien amateur, à New York en 1953, il a été nommé paradoxe de Curry (Gardner, 1995; Cohen, 2005). Il s'agit de découper une figure et de réarranger les morceaux de façon à en « faire disparaître » une petite partie. Il existe plusieurs variantes.

Enoncé 3: *Les figures suivantes montrent deux puzzles qui ont été construits avec les mêmes morceaux. Si nous réarrangeons les morceaux du premier puzzle, alors il se forme un deuxième auquel cependant un carreau blanc est en trop. Comment expliquer ce paradoxe ?*



Les deux figures ci-dessus semblent être constituées toutes les deux par un grand triangle rectangle, deux triangles rectangles plus petits et dans la première figure par un rectangle de dimensions 5×3 et dans le deuxième par un rectangle de dimensions 8×2 (divisés en deux hexagones non convexes), alors que dans le deuxième cas apparait en plus un petit carré blanc. Les deux puzzles couvrent apparemment « des triangles égaux » dont l'aire diffère ! L'explication pour ce paradoxe est que « le grand triangle n'est pas un triangle ! ».



À l'agrandissement de la figure nous pouvons observer que l'hypoténuse réelle du triangle ABC relie les points B et C, tandis que les « hypoténuses » BNC ou BDC du « triangle» ne sont pas des segments de droite, mais sont constituées par deux parties « brisées ». Par conséquence, l'« hypoténuse » de la figure gauche est infléchie légèrement vers l'intérieur, tandis que l'« hypoténuse » de la figure droite est infléchie vers l'extérieur. Tout ceci montre que nous ne pouvons pas avoir une confiance illimitée dans notre intuition, en supposant que les points D et N se trouvent sur l'« hypoténuse ».

La surprise, l'admiration et la curiosité sont considérées comme des subterfuges pédagogiques ludiques et attrayants de l'enseignant pour exciter l'intérêt des élèves, la progression de l'apprentissage et la croissance de la pensée critique. La surprise constitue essentiellement un agréable étonnement ou suscite la curiosité d'un événement inhabituel ou inattendu et elle inclut les caractéristiques de la joie, de l'admiration, du divertissement et de la satisfaction. La surprise peut produire de la tension et de l'inquiétude aussi bien que de l'excitation et de l'énergie. Lorsque les élèves sont surpris, ils vivent une attraction intense et générale et ils sont impliqués existentiellement. Il reste à examiner si le rôle de la surprise dans la classe de mathématiques est didactiquement fécond. Dans ce puzzle, on s'attendait à ce qu'il provoque un conflit cognitif chez les élèves entre la perception optique et leurs connaissances des aires.

L'énoncé précédent a été posé dans une classe de 3^{e} année du collège grec (Troisième française). Ce problème est intégré à l'unité de la trigonométrie du triangle rectangle. Les élèves avaient appris le théorème de Pythagore et le théorème de Thalès. La durée de l'expérimentation était d'une heure. Le professeur distribue l'énoncé et présente la séance, il annonce qu'il y aura un temps de recherche puis un temps de bilan, que les élèves doivent travailler par deux, que ce travail ne sera pas noté. On observe les interactions mathématiques entre les élèves pendant la durée de leur collaboration par deux et lors de la mise en commun des résultats dans la classe.

La majorité des élèves a trouvé bizarre que les mêmes morceaux laissent un carreau blanc au deuxième puzzle. Les réactions émotionnelles des élèves variaient : d'un côté il y a ceux qui ont exprimé leur désir de résoudre le problème et de l'autre ceux qui ont manifesté de la perplexité, de la peur, du mécontentement, de l'inquiétude, de la confusion et de la déception. Des 28 élèves de la classe, bien que la plupart aient voulu connaitre la solution du problème, ils ressentaient de l'incertitude pour ce qu'ils voyaient et ne savaient pas par où commencer. Les élèves ont bien senti qu'ils ne se trouvaient pas dans un domaine de certitudes habituelles. Ils considéraient que bien qu'en essayant sérieusement, ils ne trouveraient pas la solution. Tandis que les élèves travaillaient par binômes, certains premiers échanges explicatifs, pour éclairer le paradoxe ont été nécessaires.

Ens. *Comment pouvons-nous calculer l'aire du puzzle gauche?*

Élè.: *C'est un triangle rectangulaire. Pour calculer l'aire, on choisit un des côtés perpendiculaires comme base et l'autre comme hauteur. Son aire est :* $\frac{1}{2} \times$ *base* $\times$ *hauteur.*

Ens. *C'est-à-dire combien est-il, l'aire du premier puzzle ?*

Élè.: *Son aire est égale à* $\frac{5\times 13}{2} = \frac{65}{2} = 32{,}5\ cm^2$.

Ens. *On découpe le puzzle en morceaux et on assemble les morceaux d'une autre façon. Alors ces morceaux peuvent-ils reconstituer une figure différente de la même aire?*

Élè.: *Oui c'est sûr! Avec la décomposition, l'aire ne change pas. Si on ajoute les aires de tous les morceaux, l'aire totale sera la même et égale à 32,5 cm*^{2}*. D'ailleurs, deux triangles ayant la même base et la même hauteur, ont la même aire.*

Ens. *Je vous propose à tous de calculer les aires des deux puzzles en utilisant leurs morceaux. Qu'observez-vous ?*

.....

Un élève écrit au tableau:

Premierr puzzle: $3\times\frac{8}{2}+3\times 5+2\times\frac{5}{2}=32\ cm^2$. *Second puzzle:*

$$2\times\frac{5}{2}+2\times 8+3\times\frac{8}{2}=33\ cm^2.$$

La notion de la conservation des aires après découpage, déplacement et reconstitution était bien naturelle, une connaissance vraiment intuitive et les élèves l'utilisaient dans la résolution de problèmes analogues. Cependant ce problème conduisait à une conclusion bizarre, opposée au bon sens. Les élèves en agissant par formules, ont trouvé les aires des figures constituées de « triangles » et ils les ont ajouté : 32 cm^2 le premier et 33 cm^2 le second. La réorganisation des morceaux du premier puzzle a conduit au second. Tout paraissait correct. Alors ? Où se trouvait le truc magique ? Il était évident que quelque part existait une erreur. Le problème a suscité chez les élèves un conflit cognitif profond et une déstabilisation et il en a poussé certains à chercher une justification.

Ens. *Qu'avez-vous constaté pour l'aire de la première figure ?*

Élè.-1: *D'un part nous avons trouvé que l'aire du grand triangle est 32,5 cm^2 et l'autre, si nous le divisons, est 32 cm^2.*

Ens. *Et pour la seconde figure ?*

Élè.-2: *Le grand triangle a encore une aire 32,5 cm^2, tandis que si nous le séparons c'est 33 cm^2. Il y a un carreau blanc en trop. Je me suis embrouillé et je ne sais plus ce que je pense. Quel est le résultat correct ? Est-il 32 ou 33 ? Peut-être que l'aire correcte est 32,5 cm^2 qui est au milieu. Qu'en dites-vous Monsieur ?*

Ens. *J'attends que vous l'examiniez. Comment expliquez-vous ces résultats différents ?*

Élè.-3: *Celui-ci est impossible! Vous nous avez donné un problème qui est incorrect !*

Élè.-1: *Moi, je pense qu'il n'y a qu'un seul résultat juste ! Pour expliquer quel est le résultat correct on a besoin d'une preuve.*

Ens. *Essayez donc de trouver une preuve !*

Bien que les interventions précédentes aient apporté une contribution bénéfique à l'éclaircissement du problème et la conscience de la nécessité d'une preuve, peu d'élèves ont fait des progrès dans leur recherche. Malgré l'encouragement continu de l'enseignant, certains manifestaient une confusion et une déception et ils n'étaient pas disposés à réfléchir sur ces différences. Il y avait aussi des élèves qui s'étaient engagés dans un véritable travail de recherche et l'enseignant a apprécié leurs efforts et leur persévérance. Une élève a pris la parole et a dit:

"*Puisque la conclusion est erronée, quelque part il y a une erreur. Il est possible que l'idée initiale que le grand triangle a une aire de 32,5 cm^2 soit incorrecte ou il peut exister une erreur dans les raisonnements intermédiaires… On pourrait com-*

mencer inversement. Les deux rectangles de dimensions 5×3 et 8×2 ont des aires 15 cm² et 16 cm². Donc ils sont inégaux. En ajoutant les deux triangles rectangles plus petits, qui sont par deux isométriques, on forme les deux puzzles triangulaires avec des aires égal à 32 cm² et 33 cm², c'est-à-dire on n'obtient pas une aire de 32,5 cm² ... Ceci n'est pas possible".

Cette élève a posé le sujet sur une base correcte. L'erreur peut être inhérente soit à l'hypothèse soit à la procédure de la preuve, avant le résultat obtenu. Mais, ce raisonnement inverse a été compris par peu d'élèves. Le nouveau contrôle des calculs des aires n'a pas apporté de résultat. Deux élèves ont tracé un dessin agrandi, ils se sont lancés dans les mesures des deux segments « brisés » de l'hypoténuse avec une règle graduée et des vérifications sur la figure, mais ils avaient un doute sur l'exactitude de leurs mesures. C'est pourquoi les élèves se sont tournés principalement vers le raisonnement géométrique. Certaines dyades d'élèves ont reformulé et raffiné les hypothèses convergentes suivantes :

- *Nous n'avons pas vraiment des triangles, mais des quadrilatères puisque l'un « entre dedans » et l'autre « sort en dehors ».*
- *Lorsque nous reconstituons les morceaux, comment connaissons-nous qu'on va créer la deuxième configuration ? J'estime que ça n'est pas sûr ! Peut-être s'agit-il d'une illusion optique de triangle.*
- *Il ne se forme pas un triangle rectangle réel, puisque les trois points sur l'hypoténuse supposée ne s'alignent peut-être pas.*

Les formulations précédentes sont des résultats provisoires et plausibles. Polya (1962) a beaucoup insisté sur l'importance du processus heuristique et du raisonnement de plausibilité pour la découverte de la solution. D'un côté de nombreuses dyades d'élèves ont manifesté des doutes et des hésitations et n'ont pas laissé de trace de leur travail, ni rédigé leur diapositive. Il est probable que ces élèves n'ont pas abouti à des solutions complètes ou ont cru que leurs idées ne seraient pas acceptables. De l'autre côté certains élèves ont développé de véritables attitudes scientifiques : expérimenter, conjecturer, prouver. Pour la démonstration de l'hypothèse émise, ils ont utilisé le théorème de Pythagore, la similitude des triangles, la pente et la trigonométrie. Nous présentons certains extraits des élaborations de preuves (pas de démonstrations formelles) de trois dyades qui ont été exposées pendant la mise en commun des résultats dans la classe.

Première rédaction (Giannis et Theodoros)

Nous avons pensé que lorsque certaines dimensions de la figure sont connues alors les autres ne sont pas arbitraires, mais dépendent de celles-ci. Si nous appliquons le théorème de Pythagore aux triangles DEC et BZD de la première figure nous aurons:

$$DC = \sqrt{DE^2 + EC^2} = \sqrt{5^2 + 2^2} = \sqrt{29} \simeq 5{,}4 \quad et \quad BD = \sqrt{BZ^2 + ZD^2} = \sqrt{8^2 + 3^2} = \sqrt{73} \simeq 8{,}5.$$

Du triangle ABC on va obtenir:

$$BC = \sqrt{BA^2 + AC^2} = \sqrt{13^2 + 5^2} = \sqrt{169 + 25} = \sqrt{194} \simeq 13{,}9.$$

On a: BC=13,9, BD+DC=8,5+5,4=13,9. Par suite: BC=BD+DC et les point B, D, C sont alignés. Même raisonnement pour les points L, N, M.

Ainsi, les élèves ont placé incorrectement le point D sur [BC] et le N sur [LM], ils ont donc considéré ABDC et KLNM comme des “triangles”. Conformément à la discussion dans la classe entière, l’égalité 13,9=8,5+5,4 était à la base de l’illusion optique et a conduit à une interprétation erronée du dessin. Il était impossible de comprendre les conséquences des approximations rationnelles des nombres irrationnels précédentes (Voskoglou & Kosyvas, 2011). Deux élèves ont proposé les justifications suivantes :

- *En utilisant la calculatrice on a trouvé des calculs plus exacts:*

  $$DC = \sqrt{29} \simeq 5{,}385, \quad BD = \sqrt{73} \simeq 8{,}544 \quad et \quad BC = \sqrt{194} \simeq 13{,}898.$$

  Puisque BC=13,898 et BD+DC=8,544+5,385=13,929 est valable ***BC<BD+DC*** *et non BC=BD+DC. Il en résulte que les points B, D, C ne sont pas alignés. Même conclusion pour les points L, N, M.*

- *Si la figure était exacte, le nombre* $\sqrt{29} + \sqrt{73}$ *serait égal à* $\sqrt{194}$ *. Cependant si:*

  $$\left(\sqrt{29} + \sqrt{73}\right)^2 = \left(\sqrt{194}\right)^2 \Rightarrow 29 + 73 + 2\sqrt{29 \times 73} = 194 \Rightarrow$$

  $$2\sqrt{29 \times 73} = 194 - 29 - 73 \Rightarrow 4 \times 29 \times 73 = 92^2 \Rightarrow \mathbf{8468 = 8464}! \ \textit{Incorrect.}$$

Les élèves de la classe ont infirmé l'idée que « *les point B, D, C sont alignés* » en faisant appel à un ensemble de savoirs connus et ils ont compris et validé leurs preuves avec une argumentation logique dépassant l'incertitude.

> *Seconde rédaction* (Irène et Marie)
>
> *Si on compare la pente des triangles DBE et CDE, c'est-à-dire les $\frac{3}{8}$ et $\frac{2}{5}$ avec la pente du triangle rectangle «réel» ABC les côtés perpendiculaires 13 et 5, on constate que : $\frac{2}{5} \neq \frac{5}{13}$ et $\frac{5}{13} \neq \frac{3}{8}$. Puisque on a des pentes différentes, ainsi les aires différeront. Le même raisonnement peut être appliqué à la deuxième figure.*

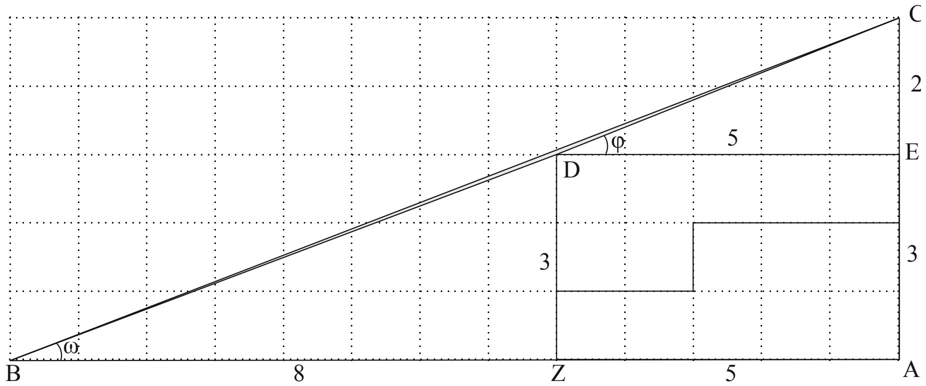
Avec les pentes (coefficients directeurs) la discussion s'est déplacée vers la vérification de la similitude des triangles. Le travail précédent a conduit solidement à la conclusion que nous ne pouvons pas combiner les deux plus petits triangles pour obtenir un grand triangle. Pourtant, puisque

$$\frac{2}{5} > \frac{5}{13} > \frac{3}{8}, \quad 2 \times 13 - 5 \times 5 = 26 - 25 = 1 \quad \text{et} \quad 5 \times 8 - 3 \times 13 = 40 - 39 = 1$$

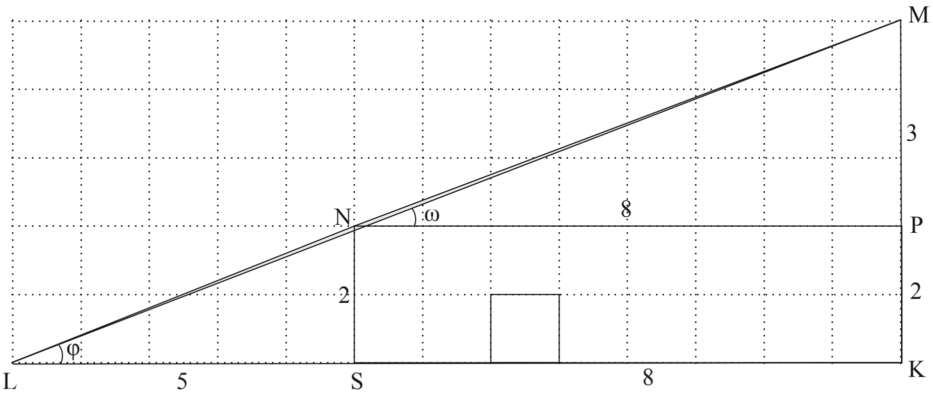
nous observons que les trois fractions sont proches, ainsi les deux « triangles » paraissent semblables.

Troisième rédaction (Elli et Sophie)

Les triangles rectangles DBZ et MNP sont isométriques puisque ils ont leurs côtés perpendiculaires égaux une par une. Cela s'applique aussi pour les triangles CDE et NLS.



Puisque les deux triangles sont isométriques, on a : $D\hat{B}Z = M\hat{N}P = \omega$ *et* $C\hat{D}E = N\hat{L}S = \varphi$. *Puisque DE//BA et NP//LK, les angles φ et ω ont des positions d'angles correspondantes. Aux deux figures pour les angles aigues φ et ω on déduit:* $tan\omega = \frac{3}{8}$ *et* $tan\varphi = \frac{2}{5}$. *Mais* $\frac{3}{8} \neq \frac{2}{5}$. *Par conséquent* $tan\omega \neq \tan\varphi \Rightarrow \omega \neq \varphi$. *Donc les points B, D, C ne sont pas alignés. De même pour les points L, N, M.*



Dans la première figure le triangle ABC a un demi carreau d'aire de plus que le puzzle (32,5-32=0,5), tandis qu'au deuxième triangle, le KLM a un demi carreau de moins que puzzle avec le carré blanc (33-32,5=0,5). Ces deux demi carreaux expliquent la présence étrange du carreau blanc.

On a vu des inférences d'élèves clairement argumentées des hypothèses jusqu'à la conclusion. Ces processus de justification sont associés aux manipulations géométriques et au travail algébrique. La démonstration précédente a été accompagnée par la construction des figures correspondantes au tableau comme support visuel à la résolution et mathématisation et elle est devenue acceptable par les élèves de la classe. Un élève a observé que les côtés perpendiculaires aux trois triangles sont (5, 2), (8, 3) et (13, 5), ne sont pas arbitraires et l'enseignant a ajouté qu'ils renvoient aux régularités de la suite de Fibonacci. La recherche n'est pas terminée, le problème reste toujours ouvert et il y a d'autres prolongements intéressants.

En résumant, nous avons constaté qu'initialement les élèves ont conclu que les sommets des morceaux qui visuellement sont placés sur l'« hypoténuse du triangle » constituent des points alignés. Le paradoxe est dû à l'illusion d'optique. Les quatre morceaux du puzzle ont aussi dans les deux cas la même aire, mais ils ne forment pas des triangles rectangles comme il apparaît trompeusement. Avec l'aide de papier quadrillé ou d'instruments de géométrie, les mesures directes sont inexactes parce que les imperfections du dessin sont imperceptibles et à la limite de l'erreur expérimentale. Cependant, bien que le tracé ne soit jamais exact, l'expérience accumulée des élèves par rapport aux dessins, constitue une base intuitive précieuse qui les a incités à imaginer les dessins comme modèles de figures précises ou idéales et les a aidés à déduire des propriétés géométriques et à composer des preuves convaincantes. Progressivement, certains élèves se sont rendus compte de la fraude perceptive et ont été conduits à une interprétation appropriée en décodant des informations condensées de la figure. Ils se sont appuyés sur les propriétés de la figure prises comme hypothèses qui pourraient être validées par une démonstration. Ainsi, en mobilisant un raisonnement géométrique, ils ont mis en valeur diverses idées géométriques comme l'isométrie de triangles rectangles, le parallélisme, la perpendicularité, les estimations de côtés et d'angles, des pentes, des proportionnalités etc., et en faisant une étude théorique de la figure avec des processus logiques comme l'utilisation du théorème de Pythagore et la trigonométrie, ils ont abouti à des conclusions sûres.

La figure géométrique est un guide utile de raisonnement, mais elle peut conduire à des erreurs. Dans la pratique de la géométrie et particulièrement dans son apprentissage, les figures géométriques jouent un rôle heuristique et constituent un support intuitif en facilitant la transition du concret à l'abstrait. Duval (2005) distingue au moins deux modes d'appréhension de la figure : *la perceptive et la conceptuelle*. Nous avons une perception optique lorsque nous voyons la figure en tant que forme iconique. Il s'agit d'une reconnaissance spontanée, à partir d'objets géométriques élémentaires tels que la droite, le triangle rectangle, les différents quadrilatères, etc. Nous avons une appréhension opératoire quand on a une mobilisation par un raisonnement géométrique, une démarche faite de réflexions et d'initiatives. Cette appréhension conceptuelle est plus «dynamique» que la perceptive. On pense les objets géométriques à travers leurs propriétés, on pose des conjectures et on construit des structures et des relations. Cette appréhension peut s'approfondir en complétant la figure par de nouvelles traces (points, lignes, figures) et en procédant à des «reconfigurations» (Duval, 1994). De plus, Fischbein (1993) met en lumière l'importance des « *concepts figuraux* » qui constituent « *un processus idéal de fusion et d'achèvement entre les aspects logiques et iconiques*» (Fischbein, 1993, p. 150). Le concept figural du triangle rectangle constitue simultanément l'ensemble de toutes les images mentales (propriétés de la figure, position, magnitude) et les propriétés conceptuelles (relations abstraites et générales comme le théorème de Pythagore, les rapports trigonométriques etc.). Le concept figural peut expliquer non seulement le raisonnement géométrique qui amène aux solutions correctes, mais aussi les erreurs qui pourraient résulter de la fusion incomplète entre les deux aspects.

Dans notre problème, nous avons constaté des difficultés évidentes de la plupart des élèves dans la gestion des « concepts figuraux», ce qui est dû à notre avis à une *inflexibilité perceptive* liée à la prédominance de la visualisation de la figure géométrique au détriment d'une étude théorique fondée sur des définitions géométriques et des preuves. Les élèves sont attachés aux représentations iconiques visibles et ils étaient dans l'impossibilité de se déplacer vers la connaissance abstraite.

Enfin, le paradoxe perceptif avec la faille du raisonnement a aidé certains élèves à réaliser la nécessité de la preuve. La création de la situation d'incertitude les a conduits à surmonter la conception « réaliste » initiale et à tourner leur attention vers l'étude théorique des objets géométriques (Zimmermann & Cunningham, 1991; Hershkowitz et al., 1996). Incontestablement, la spécificité de la preuve géométrique est initiée par le statut de la figure et ses difficultés.

Conclusions

En résumant les résultats ci-dessus, on aboutit à la conclusion que la pratique pédagogique avec les trois problèmes ouverts dans les classes du collège grec s'est avérée un apprentissage fécond. Au problème de la tirelire les enfants ont développé le raisonnement arithmétique (assai-erreur, exploration des relations arithmétiques et enchaînement des idées de façon logique, généralisation de la solution), au problème des caravaniers ont déployé des stratégies informelles de la proportionnalité et au problème géométrique (situation de surprise) ils ont émis et validé des hypothèses et ils ont fait des preuves.

D'une certaine manière, le problème ouvert a piqué la curiosité des élèves et a motivé leur recherche, en leur donnant le goût de faire des mathématiques. Les élèves ont réussi à se rapprocher des notions mathématiques familières d'une manière nouvelle et inhabituelle qui leur a causé des surprises, des désaccords et des déstabilisations cognitives. Les expériences d'enseignement précitées dévoilent l'importance pédagogique du problème ouvert et révèlent ses divers aspects, méthodologiques, métacognitifs, émotionnels et fondamentalement cognitifs. Dans les situations ouvertes de problématisation qui ont été exposées, les élèves sont obligés de penser en profondeur, de prendre part activement aux échanges mathématiques de la classe, en défendant leur raisonnement et en faisant apparaître des arguments convaincants.

Les problèmes ouverts étaient des opportunités propices qui ont suscité des conflits cognitifs en favorisant des restructurations qui ont mené à une plus grande cohésion intérieure des connaissances des élèves. C'est à des conclusions analogues qu'ont abouti aussi d'autres recherches (Becker & Shimada, 1997; Arsac & Mante, 2007; Silver et al., 1990; Brown &

Waltet, 1983; English, 1997; Sauter et al., 2008). Cependant, il y avait aussi des cas d'élèves où la technique didactique du conflit cognitif a causé une confusion passagère et une déception (particulièrement pour le problème géométrique où la surprise a motivé la curiosité et l'effort actif de peu d'élèves). Avec l'encouragement on obtenait un renforcement de la confiance en soi et un retour de l'espoir.

Les observations nous montrent que la résolution coopérative de problèmes ouverts stimule d'excellentes motivations, en favorisant chez les élèves une réflexion et un réexamen critique des connaissances mathématiques qui sont exigées et encourage un apprentissage fondé sur le sens. Lorsque les élèves travaillent en groupes à résoudre les problèmes engageants et non routinier, ils deviennent habiles à justifier leurs idées en formulant des arguments mathématiques. Ils apprennent aussi à prendre des risques, à persévérer et à avoir confiance en leur capacité à résoudre des problèmes. Il est évident que l'engagement et la persévérance des élèves à la solution des problèmes ouverts sont inhérents au mode de gestion de la classe par l'enseignant (Potari & Jaworski, 2002). Pendant cette riche confrontation scientifique, les élèves vivent un climat fait de certitude et de doute. Le doute éveille la curiosité et leur intérêt et la certitude leur donne la force de penser et de soutenir leurs arguments. Les élèves qui prennent la parole s'expriment dans un langage quotidien en commençant par leurs propres questionnements.

La "méthode du problème ouvert" en mettant l'accent sur la découverte active des mathématiques, améliore le cours quotidien. L'expérience nous assure que la méthode précitée, avec des adaptations appropriées, peut être appliquée à toutes les classes. Elle enrichit autant les élèves que l'enseignant : elle implique personnellement les élèves « dans les défis mathématiques » et les aide à explorer leurs idées, à scruter en profondeur et mieux comprendre les notions qu'ils étudient. Elle permet aussi, à l'enseignant de mathématiques de découvrir ses élèves d'une nouvelle manière, de suive leurs stratégies informelles ou formelles et peut-être, d'apprendre eux-mêmes de leurs erreurs. Incontestablement, le problème ouvert est un contexte principal d'apprentissage des mathématiques, qui nous permet de concevoir de manière critique notre pratique habituelle. De plus, les activités ouvertes peuvent susciter le raisonnement, la compréhension conceptuelle, l'argumentation, la justification et la communication comme moyen excellente de développement et d'expression de la pensée mathématique. Dans le monde d'aujourd'hui, le problème ouvert doit être intégrés à tous les niveaux de l'enseignement des mathématiques, jouant un rôle vital pour l'épanouissement de la personnalité des élèves.

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INVESTIGATION OF POSSIBLE CHANGES INTO TEACHING AND LEARNING PRACTICES IN PRIMARY SCHOOL WITHIN THE FRAMEWORK OF INTERDISCIPLINARY PROJECTS FOR MATHEMATICS TEACHING

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Abstract

Within the framework of a doctoral research, the observation of four teachers who implemented interdisciplinary projects[1] in mathematics teaching and the qualitative analysis of the data lead to common subject categories. The purpose is an investigation, into the new learning context, of the possible differentiation from the traditional[2] teaching approach to mathematics. The methodological approach is based on case studies of school classrooms and in participatory observation with borrowed elements from ethnography and action research. Presented here in short, are interdisciplinary activities which were applied in classes using projects about nutritional matters and traffic safety education. In the results we find indications of temporary change and enrichment of traditional teaching of mathematics.

Keywords: interdisciplinarity, project method, interdisciplinary projects, mathematics teaching, case studies, teachers' beliefs – attitudes, teaching practices.

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1 Interdisciplinary teaching through projects. One project can be interdisciplinary, but it can also not be. If its central theme refers to various disciplines, then it is considered to be interdisciplinary.

2 By the term "traditional teaching" we mean the anachronistic, teacher-centered teaching style which is related to the "authoritative type" of teacher, where teacher is considered authority and source of the knowledge, he has the absolute guidance of learning, he monopolizes the possession of the speech in the classroom and the process of learning is developed by frontal teaching with lecture or demonstration.

1. INTRODUCTION

Over the last decades, there has been an emphasis on the relationship between mathematics and daily life. The interdisciplinary approach and the use of projects in teaching can contribute to enhancing this relationship. At Greek Association for Research in Mathematics Education (GARME, Greek: ΕΝΕΔΙΜ) conferences (Lazaridis 2005, 2007, 2009) I presented three case studies of primary school classes which I observed for my doctoral thesis. In this article, after completing the research, I attempt to combine and present all four case studies - school classes, which implemented interdisciplinary projects based on mathematics in "Nutrition" and "Traffic Safety Education" in the two-hour sessions of Flexible Zone[3] (teaching period in the timetable for interdisciplinary activities), and to present the conclusions reached.

The purpose of the study was to investigate the possible changes, even temporary ones, in traditional practices of teachers and students in primary school within the framework of interdisciplinary projects for mathematics teaching.

Studies show that teachers' attitudes change with difficulty, especially through new teaching approaches in mathematics (Rodriguez & Kitchen 2005, Chapman 2008). A proper context of suitable teaching activities is needed. The main research question in our study is if and how the context of interdisciplinary projects offers the opportunity for change in traditional attitudes[4] and practices in the mathematics lesson. In order to identify changes comparing the data, we observed the classes also, before and after, during

3 Flexible Zone/Flexible Time Zone = A two-hour teaching period during the week which has been incorporated into the Elementary school weekly timetable for the implementation of interdisciplinary activities e.g. Environmental Education and Health Education.

4 By the term "traditional attitudes", the term "traditional" we have explained already in the previous foot-note, and by the term "attitudes" we mean generally the practices of teaching and learning, the style of communication in the classroom and the way of management of the mathematical knowledge, in other words, the sociomathematical norms.

the usual mathematics lesson and not only during the interdisciplinary approach.

2. THEORY

The present study is based on the project method and interdisciplinarity. The origin of the project method is found in Pragmatism and in the movement for 'progressive education' whose main spokesperson is Dewey (1916). The project method was spread by Dewey and Kilpatrick (1925) who defined it as a hearty purposeful act. The project method has many strengths which one discovers as a project develops (Frey 1994). It may include interdisciplinarity, 'open' learning situations, group work, experiential learning in authentic settings of everyday life and the opening of school into society. Novick (1996) points out that the role of the teacher in project work is a departure from the role of the teacher who merely examines students, towards a teacher who is a collaborator in the context of group co-investigation. The central educational concept associated with Kilpatrick was, "We learn what we live." He often elaborated on this idea in such ways as: "We learn what we live and then live what we learned" (Beineke 1998). The renewed interest in the project method is based on newer studies on learning which favour a holistic approach in dealing with the curriculum as well as on globalization and the changes that fast-paced technological development has brought about. The goal of a project is to learn about a topic rather than to seek correct answers to questions posed by a teacher (Katz & Chard 1999). Advocates of the project approach do not suggest that project work would constitute the whole curriculum; they consider it to be complementary. Main component of a project is the interdisciplinary, experiential-communicative method (Chrissafides 1996) through thematic connection.

Jacobs (1989) defines interdisciplinarity as "a knowledge view and curriculum approach that consciously applies methodology and language from more than one discipline to examine a central theme." Fogarty (1991) describes ten levels of curriculum integration. Definitions agree that the interdisciplinary approach prepares children for lifelong learning and do not consider education merely the teaching of discrete subject matter, but rather a factor in the development of skills needed in the 21st century, such as skills to make connections and perceive combinations, solve problems with alternative possibilities and incorporate ideas from different fields.

In 2003, the interdisciplinary approach began to be applied in Greece with the Cross Thematic Curriculum Framework (Greek: ΔΕΠΠΣ), (Ministry of National Education, 2003) and with the implementation of Flexible Zone (FZ). Since 2011, within the framework of the Timetable of the General High School (Lyceum), (Timetable, 2011), students will be required to produce two projects each year and a two-hour teaching period per week has been incorporated into the timetable for implementation of projects.

As for the necessity of applying interdisciplinary approach to mathematics, Dörfler and McLone (1986, p 75-76) mention, “we referred to the extension of the range of applications of mathematics in recent times... This process was heavily accelerated by the advent of electronic computers in recent decades...School mathematics can be seen to be in the centre of a highly complex net which connects it to most other subjects. These connections could and should be twofold. The other subjects are both sources for mathematical ideas and fields of application... Meaningful teaching and learning of mathematics cannot occur in an isolated place if one takes seriously the understanding of mathematics as a human means to understanding reality and to solving problems in it". Whitehead[5] (1929) mentions that the only solution to improving the educational process would be to do away with the harmful separation of subject matter. This separation or isolation of subject matter damages the vitality of the contemporary syllabus. The National Council of Teachers of Mathematics in the USA (NCTM) at the Principles and Standards for School Mathematics (2000, p 46 & p 49) in the standard “Connections” mentions: “Instructional programs from pre-

5 One of Whitehead's central themes: Education cannot be dissected from practice. Whitehead's synthesis of knowledge and application contrasts sharply with educational theories that recommend mental training exclusively. His general philosophical position, which he called "the philosophy of organism," insists upon the ultimate reality of things in relation, changing in time, and arranged in terms of systems of varying complexity, especially living things, including living minds. Mind is an organic element of an indissoluble mind/body unit, in continuous relationship with the living environment, both social and natural. Whitehead's philosophy of organism, sometimes is also called "process philosophy".

kindergarten through grade 12 should enable all students to…recognize and apply mathematics in contexts outside of mathematics…These connections can be to other subject areas and disciplines as well as to students' daily lives…The opportunity to experience mathematics in context is important." Referred to in the bibliography are, cross curricular or interdisciplinary activities connecting mathematics with other subjects such as: physical education (Banister & Harlow 1997), physics (Berlin and White 1992) and art (Willet 1992).

Concerning the literature review, Lake (1994) classifies the studies related to the interdisciplinarity in three categories. The second one includes all the studies from primary up to high school which describe the positive effect of interdisciplinarity in the perceptions and the attitudes of teachers and students (Jacobs 1989, Edgerton 1990, MacIver 1990, Greene 1991). Concerning the use of the project approach in mathematical education, Rodriguez and Kitchen (2005) mention the resistance of teachers in the effort of changing pedagogy and they propose contemporary methods, like the project approach, to teach mathematics. In other studies are mentioned the advantages of the application of the project method in the mathematical education of minorities that present linguistic difficulties (Reyhner and Davison 1992, Austin and Fraser-Abder 1995). Finally, in a study by Boaler (1998) there were positive results concerning children's understanding of the subject matter, when comparing traditional teaching and the project approach in mathematics teaching in two British high schools.

The originality of our study lies in that the combination of examining teachers, pupils and learning context for possible changes and for enrichment of teaching and learning strategies in mathematics, through the combination of interdisciplinary projects, has not been examined to such a degree before.

3. METHODOLOGY

The methodological approach of my study is purely qualitative (Merriam 1988, Patton 2001) and is based partly on participatory observation, with elements taken from ethnographic methodology (Heath 1982, Anderson

1989) and action research (Elliott 1991, Altrichter, Posch & Somekh 2001) and partly on case study research (Merriam 1988, Knirk 1991, Gilgun 1994, Yin 1994, Thaller 1994) where case studies are the four school classes that implemented the projects which we observed. Empirical data from the whole content of the teaching was collected by continuous classroom observation (Gall et al. 1996) and by distributing questionnaires (Kvale 1996) to teachers and students. The empirical data were then recorded and described. The ensuing analysis, comparison and categorization of observed data leads to the emergence of thematic categories and the extraction of conclusions (Willis 1977, Miles & Huberman 1994, Boulton & Hammersley 1996). As a researcher, I took on the role of the second teacher in each class. My interaction was necessary as a participator while also an observer in order to focus on the interdisciplinary learning procedure. Written notes, recording and memory were used to record the data. Initially, I took notes while reading the data. Then while rereading, I discerned patterns (Altrichter, Posch & Somekh 2001) and similarities and distinguished certain categories. I gave titles to the topics related to the data. Data having to do with the same or similar category was noted with the same colour pen. Afterwards, I gathered sections of data from the observation notes and questionnaires which were related to a specific category. In the end, I compared all the data within the same category. This is the process I used to produce texts analyzing each case study / class. Finally, I combined these texts (Merriam 1988) and presented them in a unified form, by topic. In order to ensure the validity and reliability of the research, the use of triangulation was constant (Lincoln & Guba 1985, Patton 1990). I combined data for each situation from three points of view: a) the teacher's, b) the individual student's and c) mine as researcher. Through the methodological triangulation I was able to cross check data from questionnaires, discussions, texts and observations, documenting each category. In analyzing the data, I preferred initially to use the structural (Pigiaki 1994) thematic categories "Teacher, Researcher, Student, Activities". Later through consecutive levels of comparison/grouping similar structural categories and lower order functional associative categories in each case study, I reached common higher order functional associative categories (Pigiaki 1994). Through the ‘method of constant comparison’ (Glaser & Strauss 1968) and different levels of abstraction, the thematic categories developed as follows:

Higher Order Functional Associative Categories

⇧

Comparison of similar functional categories

⇧

Lower Order Functional Associative Categories

⇧

Comparison of similar structural categories

⇧

Structural Categories

3.1. The classroom setting in the four case studies

The characteristics of the teachers were different in the case studies. The teacher in the 1st case study, TK, with 19 years of experience, was modest and self-confident. The teacher in the 2nd case study, PM, with 3 years, was stressed when teaching mathematics. The teacher in the 3rd, ES with 4 years experience replied in the questionnaire that she had a lot of self-confidence, contemporary beliefs and attitudes about mathematics and she had used the project method in the past. The teacher in the 4th case study, BK, had 20 years experience, average self-confidence and was conscientious in her duties and responsibilities.

Group of study	4th grade Karpenisi Primary school	4th grade Karpenisi Primary school	4th grade Karpenisi Primary school	5th grade Karpenisi Primary school

Place-material context	Small, full room. Desks facing forward and 'Π' formation	Classroom desks facing forward, computer lab, schoolyard, Traffic Safety Education Park.	Desks in a double 'Π' formation, groups of 2 facing each other, table for food preparation, schoolyard.	Spacious room, computer lab, schoolyard, Traffic Safety Ed Park, desks facing forward.
Time frame	School yr 2003-4, 4 weeks of two hour sessions during F.Z.	School yr 2004-5, 26 teaching hours.	School yr 2005-6, 22 teaching hours.	School yr 2005-6, 27 teaching hours.
Human resources	1 teacher, 13 students- 4 groups (3x3, 1x4)	1 teacher, 14 students – 7 pairs	1 teacher, 24 students,-6 groups of 4	1 teacher, 19 students- 8 groups (5 x2, 3x3)

Teaching material [6]	For the 1st and 3rd case studies: Old school mathematics book. 8-page handout with information, worksheets, bar charts and problems to solve. Worksheet: “Physical exercise time needed to burn calories of various snacks”. Drawing with the subject “Nutrition and physical exercise”. Calorie counter. For the 2nd and 4th case studies: Old school mathematics book. Pedagogical Institute book- Flexible Zone for Traffic Safety Education, children’s papers on traffic accidents, worksheets with mathematics problems, paper for parents, model of traffic lights & worksheet on “traffic signs”, 10-page handout entitled “Different Types of Ratios”

4. DESCRIPTION-ANALYSIS

Emerging from the classification and grouping of the notes were eleven higher order functional associative categories which will be projected in detail with indicative descriptive episodes and analytic notes. Following the description and analysis of learning episodes, before Flexible Zone (FZ), in the mathematics lesson, during the interdisciplinary approach (FZ), but also

6 In the few pages of this article it is difficult to be totally presented the interdisciplinary activities of the projects in “Nutrition” and “Traffic Safety Education”, that the classes implemented. The interdisciplinarity in the “nutrition project” concerns the combination of the following subject matters: physics (the energy that we take from nutrition, definition of the unit of calorie…), physical education (time needed to burn calories with various physical exercises and practice in the schoolyard), art (drawing), mathematics (bar charts, problems, worksheet with 5 kinds of Physical exercises and 8 kinds of snacks which produced 40 problems-combinations of snack- physical exercise) and health education (The children recorded in diaries for 3 days, the quantities and the types of food consumed and the time and the type of physical exercises practiced each day. They estimated for each day the total calories that they took from the foods and those they burnt doing physical exercises and they examined the balance of calories. They produced norms of proper, healthy nutrition). In the “Traffic safety education project” the interdisciplinarity concerns the combination of the following subject matters: Greek Language (Conceptual analysis of an article, Written expression “the advantages and the disadvantages of the use of vehicles”), mathematics (problems of percentages and ratios, geometrical problems, constructions, elaboration of statistical data, open problems), physics (problems of speed), physical education (timing of the students speed in the schoolyard), social and political education (discussion and reflection in social subjects), Inquiry learning (Simulation, field research), Traffic Safety Education (visit to the Traffic Safety Education Park), environmental study (pollution matters), geography (orientation in the city map), digital artwork (PC drawing of cars)…

after its completion again in the mathematics lesson, there appeared gradual changes in the attitudes of students and teachers.

- **Changes in attitudes of students towards mathematics**

1. In the 3rd case study, one student's answer in the questionnaire was impressive: "I liked it ...because we were a team and we solved problems we have in our daily lives. We analyzed whatever we ate and whatever we did, we did it with a smile and it wasn't forced on us by the teachers."

A positive assessment was spontaneous and unforced. The student's justification as to why he liked the unit includes many of the dimensions which we wanted to influence: cooperative learning, in context-realistic learning, a connection to daily life and a pedagogically sound climate, one which is pleasant, democratic and encourages intellectual autonomy.

2. In the 3rd case study, the teacher was asked if she had seen a change in the students' attitude towards mathematics. When she asked the students, "Has your opinion changed about maths after all of this?" 9 out of 24 students gave a positive answer.

3. In the 4th case study, the teacher detected positive attitude changes towards mathematics in four students. She mentions her students' comments: "Taksiarchis, who usually got upset when it was time for mathematics saying it was difficult and boring, showed an interest that I did not expect. Eleni began to take being corrected more calmly and panicked less. Athanassia tried to come up with various ways to solve mathematical problems. Kleanthis, who usually worked alone, was encouraged to participate in group work."

- **Conducting discussions with mathematical content**

In general during the mathematics lesson before 'Flexible Zone' was introduced, dialogue between students and teachers and among students themselves was rare. Teachers asking students questions in order to examine them, was the only form of communication. With the implementation of Flexible Zone, the quality of the dialogue improved significantly.

1. Teacher: And then you put down Area 2, since it was left over….?
   Alexis: But we put them in the order of less to more dangerous, first the 2nd and then the 4th.
   Magda: No, because Area 4 had four more traffic safety violations, while Area 2 had 22

(2^{nd} case study) Through dialogue and the students' disagreement, Magda was motivated to express her argument.

2. Marios: We found 148 calories in 37 minutes...
   Nefeli: We found 152 calories in 38 minutes...
   Nikos: It won't be either 37 or 38...I said since 150 is between 148 and 152 which the others found, the minutes will be between 37 and 38, so 37and a half.

(3^{rd} case study) Through mathematical dialogue, Nikos hears the solution and the results of the other groups and comes up with the exact result, finding the average.

- **Encouraging dialogue**[7]

The teachers gradually change their teaching behaviour, allowing more room for student participation in the dialogue. The discussion which is at first of non mathematical topics gradually shows increasing mathematisation.

Teacher: How can we reduce or limit pollution from cars?

Dimitris: With catalytic engines.

Elsa: If we take public means of transport.

(2^{nd} case study) The teacher continues the discussion developing critical

7 The NCTM at the Principles and Standards for school mathematics (2000) mentions the standards: Communication, Reasoning and Proof, Teacher's Role in Discourse, Students' Role in Discourse and Tools for enhancing discourse. Mathematical dialogue and mathematical discourse as theoretical notions are referred to in the bibliography (Duval 2000, Sfard 2000, Anderson et al. 2003, Morgan 2006).

thinking[8]

- **From reinforcing to reducing competitiveness among the students**

1. Angeliki and Tania: Miss, can we tell you the answer?
   Mara: Miss, we found the answer! (The teacher waits until all the students finish).

(2nd case study) There is a change in teaching behaviour. In the mathematics lesson before Flexible Zone, as we observed it in the class, the teacher used immediately to call on the first student who finished. Now she waits for all the students to finish.

2. Teacher: We have prepared a worksheet for you... (she divided the class into 8 groups of 3). Are you ready for us to announce each group's problems to the whole class?
   Ulysses: Miss ours too (all the groups presented the problems that they came up with).

(3rd case study) The teacher encourages group and cooperative learning. With group work teaching, competition is limited to the groups of students.

3. Nikos: There are some foods which several of us have and we can tell each other what the calories are when we find out.
   Teacher: Right! Your collaboration will help you gain time.

(3rd case study) Instead of cut throat competition on a personal level which was the case before Flexible Zone, the students themselves discover ways to cooperate, even in individual activities.

- **Dealing with mistakes**

1. (Nektarios made a mistake when he performed the subtraction 147.500 – 18.8.)

8 Critical thinking is a type of reasonable, reflective thinking that is aimed at deciding what to believe or what to do (Ennis 1987). The critical thinking philosophical frame traces its roots in analytic philosophy and pragmatist constructivism which dates back over 2,500 years, as in the Buddha's Teachings, as well as the Greek Socratic tradition in which probing questions were used.

Teacher: Why did you put the 1 under the 1 and the 8 under the 4?

Nektarios: I put them in order. I wrote one number under the other.

Teacher: What do 147.500 and 18.8 stand for?

Nektarios: Kilos!

Teacher: How many whole kilos are there in 147.500 and 18.8?

Nektarios: 147 in the one and 18 in the other.

Teacher: If you subtract 147- 18, what do you get?

(Nektarios writes out the subtraction of the integers correctly. Then the teacher asks him to compare this correct subtraction of the integers with the wrong subtraction which he did earlier with the decimals and to assess whether it was right. Nektarios answers “No! I should have...” Finally, he writes it out correctly again.)

(2nd case study) We noted a change of attitude. In mathematics lesson before Flexible Zone, as we observed it in the class, when a student made a mistake, either the student remained without help or explanation, or the teacher asked another student the same question. Now the teacher uses the mistake as “a window into the student’s mind”. She does not use a rule, but rather she helps Nektarios to correct himself.

2. Teacher: This is 100gr ... What were we saying the previous time? ...

   Nefeli: That’s not even enough for a baby to eat for lunch.

(3rd case study) Through experiential learning by weighing the food, the teacher leads the students to cognitive conflict. Previously, many students had written that they had eaten 100 grams of cooked spaghetti for lunch. Now that they have seen the actual quantities they had written about, they realise their mistake. It is easier to lead children to cognitive conflict by using tangible material and real situations.

3. Question in questionnaire: How was the interdisciplinary lesson different from the other lessons?

   Student answer: It was different because we weren’t nervous or afraid to make a mistake and we worked in groups.

(4th case study) In answering the questionnaire, the students registered the absence of anxiety about making mistakes and group work.

- **Emphasis on understanding**

1. Teacher: Nefeli, could you tell us how you thought about it and found 7? We don't only care about whether the answer is right or wrong. The important thing is how you got there.

(3rd case study) The teacher is interested in the procedure used for the solution.

2. Teacher: (Eva is reading something out) What are we going to do? (Eva answers "subtraction") Why? (Silence) Let's make a drawing. (She draws.) If Athens is here, and Patra is here, then....

(4th case study) The teacher asks 'why' so that Eva can justify her answer. When Eva cannot do it, the teacher intervenes and supports by drawing.

- **Emergence of strategies**

Teacher: You glued together 3 straws. How can we find the perimeter of the triangle?
Angeliki: We can measure the straws and add them up.
Magda: We don't have to measure all three of them since they are the same size. You just need one... It's 72 cm.
Teacher: How did you find it?
Magda: I measured one....24cm. 24+24+24=72
Nikos: Instead of adding, we could have multiplied 3x24.

(2nd case study) In finding the perimeter of the unilateral triangle and while one student adds to the other's thought, mathematical thinking is promoted.

- **Bringing out many ways of solving a problem - various answers**

Alexis: Miss, I solved it with cents... I thought, since we know that 1 euro is 100 cents, then 1.5 euro is 150 cents and the 0.75 euro is 75 cents. Therefore, I have 150-75=75 cents left. Teacher: Both ways are right. There isn't just one way.

(2nd case study in the mathematics lesson after Flexible Zone "subtraction of decimals") Alexis turns euros into cents and does the subtraction with integers instead of decimals. The teacher sends out the message that there are many ways to solve a problem.

- **Encouraging student autonomy**

1. (The teacher took the students to the Computer Lab, where they drew cars and traffic signs using the Windows drawing program).

(2nd case study) In the laboratory, groups replace traditional forward-facing seating arrangements. The teacher became a collaborator and co-researcher in learning.

2. Nefeli: Miss, which plate?
   Teacher: Odysseas who thought of it, will explain.

(3rd case study) The children are used to communicating indirectly through the teacher. However, she tactfully lets Odysseas explain.

- **Content and procedure expansion in school mathematics**

1. N: The vertical I think. (The teacher asks, "How can you make sure?") I will measure. (The teacher gives a ruler).

(2nd case study) The teacher demonstrates the need for measuring. In Flexible Zone she has the students do hands-on, real life activities, whereas before she did not.

2. Teacher: Which is more? The percentage of traffic offenders at 6.12 which was 2:6 or at 6.02 which was 1:2 or half?

(2nd case study) The teacher asks the students, even though it is a 4th grade class, to compare the ratios 2:6 and 1:2. The comparison of fractions with different denominators is usually taught in the 5th grade.

3. Teacher: How are we going to write it down? Either 37 ½ in mixed numbers or 37.5 in decimals? It's the same thing, children...which do you like best? Write it down any way you like.

(3rd case study) Within the framework of vertical intra-disciplinary integration, two units usually taught separately are combined, facilitating relational understanding (Van de Walle 2005).

4. Teacher: What do you think? Can you always form isosceles triangles with any size of straws that I give you? … (A construction activity follows)… Let's see what we found. When the sum of the 2 equal sides is larger than the other side, we can make a triangle. If it is smaller or equal cannot make an isosceles triangle.

(4^{th} case study: In the interdisciplinary project for Traffic Safety Education, the children brought papers on "traffic signs". Then the teacher asked the children to construct with straws the shapes of the signs). The teacher organises construction activities supplying the children with manipulatives. What follows is an investigation of geometric phenomena (not in the prescribed teaching material) as well as an investigation into the conditions necessary to construct an isosceles triangle.

5. Teacher: Four drivers passed through Area 4 within an hour and they were all traffic offenders, so 100%.
   Fotis: Yes, but just 4 drivers in 1 hour, that isn't very many, is it?
   Teacher: You're right, let's take it into consideration for the 2^{nd} question "What should the police report?" Is it worth spending money for traffic lights in Area 4?
   Fotis: No, rather in Area 2, because in Area 2 there are in total 42+20 = 62 cars.

(4^{th} case study) In the usual mathematics lesson, the children are given problems which need one correct solution so that it is easy to assess the solution. However, in this extract of the project we have enrichment of content with open problems with many possible answers. The comparison of possible choices fosters critical thinking.

- **Emphasis on the 'realistic'(real life) dimension in a mathematical context**

1. Anastasia: We saw different recipes like the ones we had done in the past...
   Theophilos: And we saw 3 math problems which looked like the recipes.
   Researcher: Tell me something, what was the most difficult thing for you?

Nikos: The problems were...The recipes were easier; we found them almost immediately.

(3rd case study) We see a parallel emerging between recipes with missing, irrelevant or superfluous ingredients and math problems with missing, irrelevant or superfluous data. Finally, Nikos compares the processes of investigating the recipes and math problems, concluding that the class had a hard time with the problems, while the answers were immediately found with the recipes. The advantage of learning in context emerges, as seen with the recipes.

2. Teacher: We are going to do a combination of things in an experiential way. The researcher giving out biscuits. (In the schoolyard, some children run for 7 minutes and others jump rope for 9 minutes)

(3rd case study) The teacher transforms the abstract mathematical context into an experiential, real life activity. The children calculate mathematically that in 7 minutes of running or 9 minutes of jumping rope they burn the 55 calories that a biscuit has. They then eat a biscuit and run 7 minutes or jump rope for 9 minutes to burn the calories.

3. Teacher: Let's look at some speeds. Kleanthis, for instance, ran 28 metres in 5 seconds.

   Iasonas: (figuring it out with a calculator) His speed is 28:5 = 5.6 metres per second. (Some children had a time of 5, 6 and some 7 seconds. As an exercise, the students found their own speeds.)

(4th case study) The students calculate their speeds for 28 metres in an experiential context of daily life, with increasing motivation.

4. Elsa: Regarding the activity where we discussed where traffic lights should be installed in our town, we passed by the crossroads at the swimming pool with my dad and we agreed that traffic lights are needed there for sure.

(4th case study) The reflections on activities from the pamphlet, is combined with experiences the children have and is taken out into the real world.

5. Fotis: The problem in the book asks us if we can find the diameter of the trunk of a tree with a 2.041 m of perimeter, without cutting down the tree.
   Teacher: We talked about Thales who measured the height of pyramids without climbing to the top of one. Geometers used to solve such everyday problems.

(4^{th} case study) With the example from the history of mathematics and the problem in the book, emerges the fact that we solve problems in everyday life with mathematics.

Finally, in the discussion of the findings, it is difficult for the whole picture of the data to be presented in its entirety in the few pages of this article. Concerning the comparison of the four teachers, the findings of the analysis emerged from the answers of the teachers to the questionnaires and from their general teaching attitude as it is recorded, before, during and after the project approach. If we relate the research findings to the characteristics of the four teachers, in order to observe the development in the teaching attitude of each one, beginning from different starting points, the data does not allow general and absolute results. We realize that even though the two experienced teachers TK, BK, differ from the novice teacher PM, certain characteristics of all three seem to be sources of resistance to change. For the former, their many years of applying traditional practices; for PM, the obstacles were insecurity in mathematics and unwillingness to depart from the beaten path. However, the other novice teacher ES showed little resistance, but also made fewer changes having begun at a more advanced starting point, as she had already used the project method before. It seems likely that there is a relation linking the amount/ kind of teaching experience and the self-confidence to resistance/ openness and to features of innovation processes.

5. CONCLUSIONS

Recapitulating the categories presented earlier, the following eleven categories emerge, which correspond to changes in the attitude of the students and of the teachers: 1) A change in students' attitudes regarding mathematics. 2) Conducting discussions with mathematical content. 3) Teachers ask “why’ and “how” encouraging dialogue. 4) Instead of reinforcing or rewarding competition, teachers try to reduce it. 5) They deal with students’ mistakes differently. 6) They focus their interest on students’ achieving understanding. 7) Teachers now elicit strategies from students. 8) They encourage children to formulate alternative solutions. 9) We detected a reduction in teacher guidance and an increase in autonomy in some students. 10) There was an expansion of content based procedures in the school mathematics lesson. 11) Teachers emphasize the real-life dimension in mathematical contexts.

Comparing the above changes in students’ and teachers’ attitudes in all four case studies during and after Flexible Zone in mathematics, we ascertained that the changes were not all stable. We classified the above 11 into three groups.

I) Those changes which occurred and remained unaltered until the end, were: 1, 2, 3, 6, 7, 8, 10 and 11.

II) Those which vacillated between contemporary pedagogical attitudes and traditional practices were 4, 5 and 9. Finally,

III) Those changes which met with the most resistance and were rejected were: the move of emphasis by the teacher on process rather than results and the reduction of guidance. They were both recorded in the 1st case study which lasted the least amount of time. Summing up we can claim that there was viability in the changes in teachers’ attitudes which was observed even after Flexible Zone in ‘regular’ mathematics lessons although the beneficial characteristics of being in Flexible Zone were absent. Through the project approach, teachers had the opportunity to get to know, apply, develop new teaching techniques from a contemporary developmental teaching approach and unconsciously perhaps to continue to apply some of these practices in mathematics lessons occasionally. Formation of progressive attitudes in the four teachers is a positive aspect of their self development. It is noteworthy that through the projects the teachers began to question and think about issues helped by feedback from reflection. To the research question of whether the interdisciplinary context using math based projects proved successful in changing traditional attitudes and practices, we can answer in the positive. According to Frey (1994) it is the quality of the tasks and group effort that determine the success of a project; also, in the assessment of a project, it is the extent to which previous negative attitudes changed that is of interest. Triangulation reinforces the reliability of conclusions concurring that changes truly occurred in the four classrooms.

Research has shown that teachers' perceptions change through their experience in class. Chapman (2008) writes that simply telling novice teachers to reflect does not necessarily lead to reflection. Instead, they must get involved in vital, conscious teaching, re-examining their way of thinking, their actions and their experiences so that their knowledge of teaching practice can be described, analyzed, assessed in order to ultimately be renewed. This process requires learning situations through which they can experience conflict, challenges and especially situations where changes occur. We tried to create such circumstances in our research undertaking, through the teaching of mathematics in the context of interdisciplinary projects. In all four case studies it was found that the context of interdisciplinary activities through contact with new, primary, original material, offers many opportunities for change and enrichment of traditional practices of teaching for teachers and of learning for students in the mathematics lesson. It was found that teachers, students and teaching material interacting with each other in an interdisciplinary context, change even temporarily and enrich their traditional structures. We can conclude that the contribution of the interdisciplinary approach and the project teaching method are necessary for the pursuit of a renaissance in mathematical education, which is emerging and progressively materializing globally.

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Kindergartners' performance on patterning

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Abstract: The aim of this article is to investigate the ability of the kindergarten children (5-6 years of age) to extend and reproduce different types of patterns which are constructed with a variety of materials, before formal teaching. From the record of their answers, a correlation was found between their actions in the different pattern constructs which formed six performance types: 1. Modification of the pattern. 2. Deficient extension (reproduction) of the pattern. 3. Random extension (reproduction) of the pattern. 4. Reverse extension (reproduction) of the pattern. 5. Sectional extension (reproduction) of the pattern. 6. Ordinary extension (reproduction) of the pattern. The results of the study showed that kindergarten children's performance on patterning was influenced from the pattern construct, from the complexity of the core unit and from the material type.

Key words: Patterning, Kindergartners Performance, Repeating Patterns, Growing Patterns, Physical Materials, Printed Materials.

INTRODUCTION

Many researchers mention the importance of pattern work in children's mathematical development. Also, guidelines, internationally, suggest the inclusion of patterning within the mathematics curriculum from the early years (ACARA, 2010˙ Depps, 2001˙ DfEE, 1999˙ NCTM, 2000).

Patterning begins before kindergarten (Synoś & Swoboda, 2007) when children are given opportunities to explore various materials; they structure their pattern-making using a variety of attributes, including shape, size, orientation, color and position (Garrick, Threlfall & Orton, 2005). Children also search for patterns while collecting, organizing, discussing and graphing data (Enright, 1998). An abstract understanding of patterns develops gradually during the early childhood years (Clements & Sarama, 2009).

It is assumed (Stalo, et al., 2006) that recognizing and coordinating different types and representations of patterns may cultivate students understanding of generalizations and develop their algebraic thinking. Recognizing a concept in a variety of representations and handling it within these representations proves that the concept has been understood.

But research has revealed that many young adolescents experience difficulties with the transition to patterns as functions and that these difficulties stem both from a lack of early experiences in the elementary school (Warren & Cooper, 2008) and from that early patterning experiences, if any, involve simple pattern tasks (such as AB) (Papic, Mulligan & Mitchelmore, 2011). It may also come from teachers' limited knowledge about students' performance and capabilities in pattern tasks. There is currently insufficient research or impetus from early childhood teachers to support mathematical patterning.

In order to contribute to this active field of inquiry, in this article is investigated the kindergartners' performance on patterning through the description of their actions when extending and reproducing different types of patterns using a variety of materials before formal teaching.

The importance of patterning in mathematics education

Researchers highlight the magnitude of pattern work either by suggesting how patterns can be used in teaching (Hendricks, Trueblood & Pasnak, 2006) or by suggesting what significance patterning has to later work and mostly in the area of (pre) algebra (Threlfall, 2005). A search for patterns is a common route to algebra because by recognizing, identifying and analyzing patterns children develop their intelligence and manage to bring order, cohesion, and predictability to seemingly unorganized situations (Clements, 2004˙ Clements & Sarama, 2009).

Work on patterning can be a rewarding activity offering a valuable context for the teacher to bring out mathematical perceptions. Patterning is fundamental for the children mathematical growth, for the development of mathematical reasoning, and for the abstraction of mathematical ideas and relationships (English, 2004˙ Mulligan, Prescott, & Michelmore, 2004). According to Ahmed, Clark-Jeavons and Oldknow (2004) children who are exposed to a rich variety of patterns, through which they can see regularity, sequencing and interconnections, grow mathematically. Patterning is vital for the development of spatial awareness, ordering, comparison and classification. It lays the basis for the development of counting and arithmetic structure, base ten concepts, units of measure, proportional reasoning and data exploration (Papic, 2005). Patterning tasks provides students with the opportunities to practice in the skills of searching for, identifying, recognizing, copying, extending and creating patterns as well as making generalizations (Billings, Tiedt, Slater, & Langrall, 2007/2008˙ Papic, Mulligan, & Michelmore, 2009˙ Waters, 2004). Patterning plays an important role also in the development of multiplicative reasoning (Mulligan & Michelmore, 1997˙ Papic, et al., 2011) as well as of analogical and inductive reasoning (English, 2004).

Patterns in the curriculum

There are recommendations, internationally, for the inclusion of patterning within the mathematics curriculum from the early years. NCTM (2000) states that in kindergarten through grade 2 children should focus on regularity and repetition of motion, color, sound, position, and quantity and be involved in recognizing, describing, extending, transferring, translating and creating patterns. Furthermore, to understand the mathematical concept of pattern, children must recognize the predictability and repetition that patterns imply when the core unit is repeated (like AABAAB), or grows (like ABAABAAAB).

The National Curriculum for England and Wales (DfEE, 1999) emphasizes for Level 1 of the Attainment Target for Number and Algebra, the use (of) repeating patterns to develop ideas of regularity and sequencing.

According to the Greek National Curriculum for Mathematics (Depps, 2001) in kindergarten (4-6 years of age), children should learn to recognize, reproduce and construct a variety of patterns. The main aim of the kindergarten program is the comparison and recognition of patterns. The types of patterning activities that Greek kindergarten usually offers to the children, in a mathematics class, are tasks that focus on copying an existing pattern, creating their own patterns or extending a pattern.

The Australian National Curriculum for Mathematics (ACARA, 2010) promotes early algebra in the elementary grades which incorporates patterning. Warren and Cooper (2008) maintain that in many early year classrooms in Australia they explore simple repeating and growing patterns using shapes, colours, movement, feel and sound. Young students are asked to copy and continue these patterns, identify the repeating part and find missing elements. But, although patterns are the heart and soul of mathematics, as Zazkis and Liljedahl mention (2002), pattern tasks, usually, do not lie on their own as a curriculum topic or activity.

Research on children's performance levels on patterning

There is a body of research into the process of patterning regarding teacher candidates (Elliott, 2005˙ Zazkis & Liljedahl, 2002) as well as kindergarten teachers (Economopoulos, 1998). There is also research for the contribution of patterning on the mathematical development of students of primary and secondary education (Barkley & Cruz, 2001˙ Billings et al., 2007/2008˙ Perry, 2000˙ Rivera, 2007˙ Rivera, 2010˙ Warren & Cooper, 2008). On the contrary it is harder to find research for kindergartners' performance on patterning even though, there are some relevant studies which investigate the spontaneous actions of young children to identify, repeat or continue a pattern (Τζεκάκη & Κούλελη, 2007).

Rustigian (1976) (as cited in Threlfall, 2005, p. 23) found a five level developmental hierarchy in 3-5 years-old responses at the extension of repeating patterns: 1. No reference to prior elements at all, and a random choice of new elements. 2. A phase of repeating the last element (perseverance). 3. Use of the elements used previously but in any order. 4. A symmetrical approach reproducing the sequence, but in reverse. 5. A deliberate continuation of the pattern, involving glances back to the start (to check).

Although in a study (Aubrey, 1993 as cited in Garrick et al., 2005) was found that fewer than half the group of children was successful in both copying and continuing simple repeating patterns, Klein and Starkey (2004) in their study found that pre-kindergartners can learn to duplicate simple concrete patterns and that kindergartners can learn to extend and create patterns. The "pattern duplication task" (p. 349) assessed children's ability to copy a linear repeating pattern and the pattern extension task assessed children's ability to complete a linear repeating pattern. In both tasks patterns were constructed with small colored blocks. Klein and Starkey (2004) noted some children's pattern duplication errors. The first error was that "no initial unit reflected the least developed knowledge of patterns" (p. 353). In attempting to duplicate the model pattern, children did not even begin their construction with the core unit AB (eg. BAAB and AABA). The second was that "only initial unit reflected a partial knowledge of repeating patterns" (p. 353). Children who made this type of error apparently understood that the pattern began with the core unit AB, but they did not systematically iterate this core unit throughout their construction (eg. ABBA and ABAAB). "Model pattern plus extra blocks" (p. 355) was the third error where children correctly constructed the core unit AB and then systematically iterated the core unit until they duplicated the model pattern. However, after duplicating the model pattern, they did not stop. Children incorrectly added one or two extra blocks from the source set (eg. ABABA and ABABAA). In "pattern extension tasks" (p. 349), children made the same three types of errors as they did on pattern duplication. Children made a greater number of errors on pattern extension than on pattern duplication. The findings suggest that many 4-year-old children at the beginning of the kindergarten year find it difficult to identify and to analyze the core unit of a repeating pattern. The majority of pre-kindergartners experienced difficulty at the beginning of the year with the core unit of the pattern.

Stalo and his colleagues (2006) attempt to synthesize Kyriakides &

Gagatsis (2003) pattern's structure complexity (a. repeating patterns in symbolic numerical form, b. repeating patterns with geometric shapes, c. developing patterns in symbolic numerical form, d. developing patterns with geometric shapes, e. patterns requiring simple numerical calculations, namely simple patterns, and f. patterns requiring more complex numerical calculations, namely complex patterns) and Lannin (2005) levels of students' understanding of patterns (Level 0: No justification, Level 1: Appeal to external authority, Level 2: Empirical evidence, Level 3: Generic example and Level 4: Deductive), and propose the following levels of the understanding of mathematical relations in patterns: Level 1: Empirical abstraction of mathematical relations. Pupils at this level are able to continue a pattern; Level 2: Implicit use of a general rule. Pupils at this level are in a position to predict terms in further positions of a pattern; and Level 3, Explicit use of a general rule. Pupils at this level are able to generalize the pattern giving a symbolic or a verbal rule.

Clements and Sarama (2009), present a learning trajectory for patterns which concerns the simple, typical case of sequential repeated patterns. The developmental progression for young children (2-7 years old) includes (p. 195-196): 'Pre-Explicit Patterner' (2 years old), who detects and uses patterning implicitly, but may not recognize sequential linear patterns explicitly or accurately. 'Pattern Recognizer' (3 years old), who recognizes a simple pattern. 'Pattern Fixer' (4 years old), who fills in missing elements of patterns (initially with ABAB patterns), who duplicates simple patterns (such as ABABAB and ABBABB), who extends AB repeating patterns. 'Pattern Extender' (5 years old), who extends simple repeating patterns (such as ABBABB). 'Pattern Unit Recognizer' (6 years old), who identifies the smallest unit of a pattern and can translate patterns into new media. 'Numeric Patterner' (7 years old), who describes a pattern numerically and can translate between geometric and numeric representation of a series.

Papic's, Mulligan's and Michelmore's (2011) findings show a link between patterning and multiplicative reasoning. In their 6-month intervention focusing on AB repeating patterns found that the intervention group of pre-schoolers outperformed the comparison group across a wide range of patterning tasks. The intervention group revealed greater understanding of the repeating unit and of the spatial structuring whereas most of the comparison group treated repeating patterns as alternating items and rarely recognized simple geometrical patterns. The strategies pre-schoolers used for repeating patterns were in increasing order of sophistication (p. 251-254): ‘Random arrangement’—children placed items in a random fashion when attempting to copy or construct patterns. ‘Direct comparison’—children copied a pattern by matching one item at a time. ‘Alternation’—children focused on items alternation regardless of the complexity of the unit of repeat. ‘Core unit of repeat’—children identified the unit of repeat independently of the number, type and complexity of items and of attributes such as size, shape, dimension, and orientation. ‘Advanced unit of repeat’—children transfer the same pattern to different materials or modes, or reconstruct the pattern in creative ways. The researchers also assessed children after the 1st year of schooling in growing patterns and they found four main categories of response: (p. 258-259): ‘Repeating pattern’—children interpreted the given pattern as a unit of repeat and simply replicated it. ‘Incorrect shape’—children extended the patterns using shapes that did not match the given squares or triangles. ‘Inaccurate shape’—children seem to form a holistic impression of the shape and do realize that it was increasing in size but were unable to construct the correct continuation. ‘Correct extension’—children indicated through their explanations that they were aware of both the numerical and spatial structure of the pattern.

Rustigian’s developmental hierarchy is related to the extension of repeating patterns. Klein’s & Starkley’s and Papic’s, Mulligan’s & Michelmore’s description is related to pre-schoolers (pre-kindergartners and kindergartners) errors and strategies on AB repeating patterns, after a teaching intervention. Clements’s and Sarama’s learning trajectory for patterns describes young children’s developmental progression in patterning. None of the above researchers deal with different pattern types and constructs neither with a variety of materials except Stalo and his colleagues. They depict levels of the understanding of mathematical relations in several pattern types and pattern constructs which derive from grade 1 to grade 6 students' answers and not from kindergartner's answers.

From the above mentioned, it seems that questions about kindergarten children's performance on different pattern types and constructs with the use of a variety of materials remain unanswered. The aim of this article is to investigate the ability of the kindergarten children to extend and reproduce different types of patterns which are constructed with a variety of materials, before formal teaching. Their actions are described in order to give answers in the following questions:

a. How kindergarten children extend and reproduce different types of patterns constructed with a variety of materials (physical materials and printed materials)? Is there any correlation between their actions in the different pattern constructs?
b. Is kindergarten children's performance influenced by the pattern construct, by the complexity of the core unit or by the material type?

METHOD

Individual task-based and standardized interviews were administered in three kindergarten classrooms (45 children: 23 girls and 22 boys, the rage of ages were the same, about 5 years old) at three public schools. The students in these schools represented a broad spectrum of socioeconomic backgrounds. All three classes had the same curriculum characteristics. Although pattern tasks are included within the mathematics curriculum none of the three kindergarten' teachers incorporate pattern tasks in their teaching.

The interviews were conducted at each kindergarten, by the researcher, in a room separate from the main classrooms. The researcher kept notes of the constructions of each child. The order of the tasks and the procedures remained the same for all interviews and all tasks were administered to all children. The data were collected in May.

The main pattern constructs, identified from the literature and from the early childhood mathematics curriculum guidelines, are repeating patterns where a core unit is repeated and growing patterns consisting of a sequence of elements that increase (or decrease) systematically. Repeating patterns that were used in this study were patterns with a recognizable repeating cycle of elements, referred as a core unit. A growth pattern that was used was a pattern that changes from one term to the next in a predictable way.

The pattern tasks were set not only with physical materials but also with printed materials because we wanted to observe the role that material type may play in children's actions on these tasks. However, the most complex pattern types (such as ABBC) and the growing pattern were presented only with physical materials in order to investigate the ability of kindergartners to manage such situations.

Tasks were designed to elicit children's abilities about patterns. There were three different pattern constructs incorporated in the interviews: extension of a repeating pattern, reproduction of a repeating pattern and extension of a growing pattern.

Extension of repeating patterns: Children were asked to extend different types of patterns constructed with different materials (table 1).They were asked to extend a pattern (AB) constructed with cube blocks for task 1, a pattern (ABC) constructed with beads for task 2, a pattern (ABBC) constructed with people figures for task 3, a pattern (AB) constructed with printed squares for task 4 and a pattern (ABC) constructed with printed circles for task 5. In these tasks the experimenter constructed two core units of a pattern in front of the child and then asked the child to extend it. The child was given more than the minimum number of materials needed to copy the pattern.

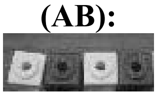
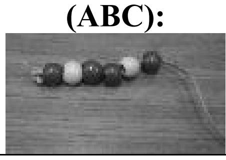
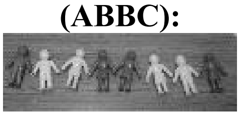
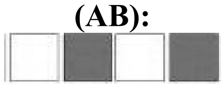
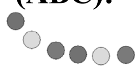
EXTENSION OF REPEATING PATTERNS **"Can you extend the pattern?"**				
Task 1 (AB): 	**Task 2 (ABC):** 	**Task 3 (ABBC):** 	**Task 4 (AB):** 	**Task 5 (ABC):** 

Table 1: Tasks 1-5

Reproduction of repeating patterns: Children were asked to transfer the same pattern to different materials. In these tasks the experimenter constructed two core units of the pattern in front of the child. The child was given a type of material that was different from the one used in the presented pattern and was asked to make a pattern that looks just like the presented (table 2). For task 6, a pattern (AB) with cube blocks was presented and its reproduction with people figures was asked. For task 7, a pattern (AAB) with cube blocks was presented and its reproduction with beads was asked. For task 8, a pattern (AB) with printed squares was presented and its reproduction with printed people shapes was asked. For task 9, a pattern (AAB) with printed squares was presented and its reproduction with printed circles was asked. The child was given more than the minimum number of materials needed to copy the pattern.

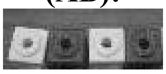
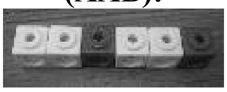
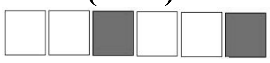
REPRODUCTION OF REPEATING PATTERNS "Can you make a pattern that looks just like mine?			
Task 6 (AB):  **(The child was given the people figures ).**	**Task 7 (AAB):**  **(The child was given beads ).**	**Task 8 (AB):**  **(The child was given printed shapes of people).**	**Task 9 (AAB):**  **(The child was given printed circles).**

Table 2: Tasks 6-9

Extension of a growing pattern: Children, in task 10, were asked to extend a growing pattern ('A AB ABA'), presented vertically with cube blocks (table 3). The experimenter constructed three growing units with colored blocks in front of the child and then asked the child to continue the pattern. The child was given more than the minimum number of materials needed to copy the pattern.

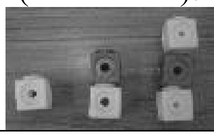
EXTENSION OF A GROWING PATTERN **“Can you continue the pattern?”**
Task 10 (A AB ABA)**:** 

Table 3: Task 10

FINDINGS

Children’s actions in the three pattern-constructs

The data analysis follows a qualitative description, according to the three pattern constructs, focused only on children’s actions. It is hard for young children to verbalize their observations effectively (Rawson, 1993) as well as to express verbally the mathematical features of patterns (Swoboda & Tatsis, 2009). Thus, the way children describe their actions in pattern tasks is not sufficient for reliable assessment of their understanding (Threlfall, 2005), and does not always validate the conjectures they create (Clements, 2004˙ Clements & Sarama, 2009). Kindergartners in our research were asked to justify their actions but their explanations did not contribute to understanding their actions, as well as to supporting our interpretations, because were limited in describing the type and the color of the object they had used without referring to mathematical features of the patterns.

<ins>Extension of repeating patterns</ins>

Concerning the extension of the repeating patterns children demonstrated different actions. The following types of actions were identified (table 4). Children put the elements of the patterns not only on the left side but also on the right side or they changed the place of the blocks in the presented pattern (type 1). Others extended the presented repeating pattern using only one element when the core unit consisted of two elements (e.g. AAA instead of ABAB for task 1 and 4) and using only two different elements when the core unit consisted of three different elements (e.g. ABA instead of ABC for tasks 2 and 5, as well as ABA instead of ABBC for task 3) (type 2). Others constructed a sequence with all the elements of the core unit of the presented pattern, but by placing them in other orders (e.g. AABBB instead of ABAB for task 1 and 4, ACB instead of ABCABC for tasks 2 and 5, as well as AACBCAC instead of ABBCABBC for task 3) (type 3). There were children who repeated the core unit in reverse (e.g. BABA instead of ABAB for tasks 1 and 4, as well as CBA instead of ABCABC for tasks 2 and 5); sometimes they constructed the symmetry of the pattern starting from the last element but usually they did not systematically iterate the reverse unit throughout their construction—after repeating the core unit in reverse they continued adding other elements from the core unit (e.g. BAABAB instead of ABAB for tasks 1 and 4, as well as CBABA instead of ABCABC for tasks 2 and 5) (type 4). Another action which was adopted from children was to repeat the core unit but only once and then incorrectly added elements from the source set (e.g. ABBBA instead of ABAB for task 1 and 4, as well as ABCBC instead of ABCABC for task 2 and 5) (type 5). Finally, there were children who extended the repeating pattern successfully, repeating the core unit once or twice (e.g. AB or ABAB for tasks 1 and 4, as well as ABC or ABCABC for tasks 2 and 5) (type 6).

EXTENSION OF REPEATING PATTERNS "Can you extend the pattern?"			
Types of children' actions / Tasks	**1 & 4 (AB)**	**2 & 5 (ABC)**	**3 (ABBC)**
Type 1	A**AB**B	A**ABC**C	A**ABBC**C
Type 2	**AB**AAA	**ABC**ABA	**ABBC**ABA
Type 3	**AB**AABBB	**ABC**ACB	**ABBC**AACBCAC
Type 4	**AB**BABA	**ABC**CBA	**ABBC**CBBA
Type 5	**AB**ABBBA	**ABC**ABCBC	**ABBC**ABBCABC
Type 6	**AB**ABAB	**ABC**ABCABC	**ABBC**ABBCABBC

Table 4: Different types of actions for extending repeating patterns

Reproduction of repeating patterns

Concerning the reproduction of the repeating patterns the following types of actions were identified (table 5). There were some children who placed the given materials on the presented pattern (type 1). Others used only some elements of the presented core unit, to reproduce the pattern (e.g. AAA instead of ABAB for task 6 and 8) (type 2). Or they constructed a sequence with all the elements of the core unit of the presented pattern, but by placing them in other orders (e.g. AABB instead of ABAB for task 6 and 8, as well as ABB or ABA instead of AAB for tasks 7 and 9) (type 3). Others constructed the core unit in reverse (e.g. BABA instead of ABAB for tasks 6 and 8 and BAA instead of AAB for tasks 7 and 9) (type 4) and sometimes after reproducing the core unit in reverse they continued adding other elements from the core unit (e.g. BAAA instead of ABAB for tasks 6 and 8 as well as BAAAA instead of AAB for tasks 7 and 9). There were some children who constructed the core unit but only once and then incorrectly added elements from the source set (eg. ABBBA instead of ABAB for task 6 and 8, as well as AABABAA instead of AAB for task 7 and 9) (type 5). Finally, there were some children who reproduced the given patterns, using the different material, successfully (type 6).

REPRODUCTION OF REPEATING PATTERNS “Can you make a pattern that looks just like mine?		
Types of children’ actions / Tasks	**6 & 8 (AB)**	**7 & 9 (AAB)**
Type 1	A**AB**B	A**AAB**B
Type 2	**AB**AAA	**AAB**AA
Type 3	**AB**AABB	**AAB**ABB
Type 4	**AB**BABA	**AAB**BAA
Type 5	**AB**ABBBA	**AAB**AABABAA
Type 6	**AB**ABAB	**AAB**AABAAB

Table 5: Different types of actions for reproducing repeating patterns

Extension of a growing pattern

For extending the growing pattern, children used several types of actions (table 6). Some children put cube blocks to complete the empty places of the presented pattern (type 1). Others just extended the pattern, without growing it either by copying the last unit using one color instead of two or by extending the growing pattern horizontally repeating the elements of the last unit in a random position (type 2). Some children grew the last unit but in reverse or grew the elements of the last unit unsystematically (type 3). Others extended the growing pattern by copying the last unit or copying the entire pattern (type 4). In this case they acted as for the extension of a repeating pattern. Finally, there were some children who constructed successfully the next term of the growing pattern and put some more elements (type 5) or not (type 6). None of the children extended the pattern more than once.

EXTENSION OF A GROWING PATTERN "Can you continue the pattern?"	
Types of children' actions / Task	**10** (A AB ABA)
Type 1	A**B**A **AB**A **ABA**
Type 2	**A AB ABA** AAA
Type 3	**A AB ABA** BAB
Type 4	**A AB ABA** ABA
Type 5	**A AB ABA** ABABB
Type 6	**A AB ABA** ABAB

Table 6: Different types of actions for extending a growing pattern

A framework for describing children's actions in pattern tasks

The different types of the children's actions on the patterning extension and reproduction form six categories. These categories can be ordered, as follows, from the more primitive to the more mature mathematically in regard with the recognition of the core unit:

1. MODIFICATION OF THE REPEATING/GROWING PATTERN. Children extended (reproduced) the pattern in both ways without using all the elements of the core unit. Children modified the pattern instead of extending or reproducing it. They changed the place of the elements of the presented pattern or they put elements in both sides of the presented pattern. This may mean that children had not understood what the core unit of the pattern was as well as what it means to extend or to reproduce a pattern.
2. DEFICIENT EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Children extended (reproduced) the pattern without using all the elements of the core unit. Children extended (reproduced) the pattern by repeating only a part of the core unit. Thus, they extended (reproduced) the patterns constructing a sequence with just some of the elements of the core unit of the presented pattern. Children who gave this type of answer seemed to have not identified the core unit of the presented pattern. They seemed to consider the pattern as a sequence of isolated elements with no relation among them. This type is partially in line with Rustigian's second level at which the children repeat the last element.
3. RANDOM EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Children extended (reproduced) the pattern using all the elements of the core unit in a random order. Children, extended (reproduced) the pattern by using all the elements of the core unit but placing them in a random position. Thus, they finally created a no pattern. Children may still see the pattern as a sequence of isolated elements, but they took into account all the elements of the core unit regardless of using them in unsystematic manner. This type of action is similar to the third level of Rustigian.
4. REVERSE EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Children extended (reproduced) the pattern using all the elements of the core unit in the reverse order. Some of them continued adding elements without order. Children extended (reproduced) the pattern by repeating the core unit in reverse, similarly to Rustigian's

forth level. They constructed the symmetry of the pattern starting from the last element but usually they did not systematically iterate the reverse unit throughout their construction. Sometimes after repeating the core unit in reverse they continued adding other elements from the core unit. Children, seemed to start recognizing the existence of the core unit but they may have not understood that the pattern began with the core unit. Thus they used it, but in reverse.

5. SECTIONAL EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Children extended (reproduced) the pattern using all the elements of the core unit in correct order once. Some of them continued adding elements without order. Children extended (reproduced) the pattern by repeating (constructing) the core unit once and then incorrectly added elements from the source set. According to Klein and Starkey (2004) children who make this type of error apparently understand that the pattern begins with the core unit, but they do not systematically iterate this core unit throughout their construction.
6. ORDINARY EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Children extended (reproduced) the pattern using all the elements of the core unit in correct order at least twice.

Children's actions in relation with the pattern type, the pattern construct and the material type

A quantitative depiction of children's actions in the three pattern constructs through the presentation of the percentages of their responses to each task is presented (table 7).

MODIFICATION OF THE REPEATING/GROWING PATTERN. The majority of these actions occurred with the growing pattern (42,2% for the task 10). But there were many actions (13,3%), concerning modification that were related to the reproduction task in which the core unit AAB was presented with printed squares and the children had to reproduce it with printed circles (task 9). The same task with the use of cube blocks (task 7) leaded only a child to the same action (2,2%). In the same construct (reproduce the pattern) there were few children who modified the pattern AB either with the use of physical materials (6,6% for the task 6) or with the use of printed materials (4,4% for the task 8). In extending the repeating pattern none of the children modified the pattern when the core unit was ABC.

DEFICIENT EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. The vast majority of the children's actions (60%), concerning deficient extension, were related to the pattern ABBC which was constructed with people figures (task 3). This pattern seemed to be difficult for the children of that age and they used only a part of the core unit when extending it. When the pattern with the core unit ABC consisted of three different elements it seemed to be easier for the children regardless of the material's type (4,4% for the task 5; 11,1% for the task 2). The same happened in the AB pattern (11,1% for the task 4; 15,5% for the task 1). There were very few children who adapted this action when reproduced the repeating pattern (6,6% for the task 8; 2,2% for the task 6; 2,2% for the task 9) and when extended the growing pattern (4,4% for the task 10).

RANDOM EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Most of the children reproduced AAB patterns with cube blocks (71,1% for the task 7) and printed squares (40% for the task 9) in a random manner. Fewer random extensions existed with AB patterns with cube blocks (4,4% for task 1) and printed squares (11,1% for the task 4) as well as with the growing pattern (11,1% for the task 10). This type of action is in line with Papic's, Mulligan's and Michelmore's Random arrangement strategy.

REVERSE EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Most of the children acted in this way when extended the growing pattern (15,5% for the task 10) as well as when they extended the ABC pattern using beads (15,5% for the task 2). Fewer were the children's reverse reproductions of the repeating patterns (4,4% for the task 6; 2,2% for the tasks 7, 8 and 9) (see tables 2, 3 and 4).

SECTIONAL EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. When the core unit was AB and constructed with cube blocks only two children (4,4% for the task 1) extended the pattern in this way. When the core unit was ABC and constructed with printed circles only four children (8,8% for the task 5) acted in that manner. The vast majority of the students' sectional reproduction occurred in AB pattern tasks with cube blocks (48,8% for the task 6) and printed squares (42,2% for the task 8).

ORDINARY EXTENSION (REPRODUCTION) OF THE REPEATING/GROWING PATTERN. Most children's ordinary extensions occurred both in AB patterns (66,6% for the task 1 and 57,7 for the task 4) and in ABC patterns (48,8% for the task 2 and 46,6 for the task 5) constructed with physical materials as well as with printed materials. The most ordinary reproductions occurred with printed materials both for AB patterns (37,7% for the task 8) and for AAB patterns (40% for the task 9). For the growing pattern the respective percentage was 17,7%.

Pattern Constructs	**Material Type**	**Tasks**	**Actions' Categories**					
			1	**2**	**3**	**4**	**5**	**6**
Extend repeating patterns	Physical	Task 1 (AB)	1 (2,2%)	7 (15,5%)	2 (4,4%)	3 (6,6%)	2 (4,4%)	30 (66,6%)
	Physical	Task 2 (ABC)	0	5 (11,1%)	11 (24,4%)	7 (15,5%)	0	22 (48,8%)
	Physical	Task 3 (ABBC)	2 (4,4%)	27 (60%)	8 (17,7%)	0	0	8 (17,7%)
	Printed	Task 4 (AB)	4 (8,8%)	5 (11,1%)	5 (11,1%)	4 (8,8%)	1 (2,2%)	26 (57,7%)
	Printed	Task 5 (ABC)	0	2 (4,4%)	14 (31,1%)	4 (8,8%)	4 (8,8%)	21 (46,6%)
Reproduce repeating patterns	Physical	Task 6 (AB)	3 (6,6%)	1 (2,2%)	6 (13,3%)	2 (4,4%)	22 (48,8%)	11 (24,4%)
	Physical	Task 7 (AAB)	1 (2,2%)	0	32 (71,1%)	1 (2,2%)	4 (8,8%)	7 (15,5%)
	Printed	Task 8 (AB)	2 (4,4%)	3 (6,6%)	3 (6,6%)	1 (2,2%)	19 (42,2%)	17 (37,7%)
	Printed	Task 9 (AAB)	6 (13,3%)	1 (2,2%)	18 (40%)	1 (2,2%)	1 (2,2%)	18 (40%)

Extend growing pattern	Physical	Task 10 (A AB ABA)	19 (42,2%)	2 (4,4%)	5 (11,1%)	7 (15,5%)	4 (8,8%)	8 (17,7%)

Table 7: *Frequency and Percentages of Children's Answers (Action Category, Pattern Construct, Pattern Type and Material Type)*

DISCUSSION - CONCLUSIONS

From the study which was focused in investigating kindergarten children's performance on patterning, it became apparent that kindergartners have the ability to extend and reproduce different patterns types and constructs that were constructed with a variety of materials before teaching. The results of the study showed a strong influence of pattern construct as well as of pattern type and the relatively smaller influence of material type on the children's performance on patterning. For the first construct, which concerned the extension of repeating patterns, the large majority successfully extended the repeating patterns that had simple core units (like AB or ABC). Although in Aubrey's study (1993 as cited in Garrick et al., 2005) less than half of the children were successful in both copying and continuing simple repeating patterns, in our study more than half of participants were successful in the extending condition when the core unit was AB and about half when the core unit was ABC. When the core unit got more complicated (like ABBC) children's successful actions were fewer. Unlike Sternberg (1974 as cited in Threlfall, 2005), we found that the core unit AAB was more difficult for young children to handle than ABC, as children tend to intuitive construct the pattern. The use of physical materials leads the children to more successful actions than the use of printed materials within this construct. In the second construct, in which the pattern had to be reproduced with the use of different materials, children were more successful when they used the printed materials rather than the physical materials, independently from whether the core unit was simple or not. The third construct concerned the extension of a growing pattern and led most of the children, to modify the pattern. However, little can be concluded from the evidence of the inclusion of a single task. In general, the extension of the growing pattern was more difficult for children. The reproduction of the repeating patterns was also difficult for them, especially with the use of physical materials, related to the extension of repeating patterns.

Kindergarten children’s actions on patterning differed and formed six categories which can be ordered according to the way the core unit was treated, regardless of the pattern construct and the type of the material: 1. Modification of the pattern. 2. Deficient extension (reproduction) of the pattern. 3. Random extension (reproduction) of the pattern. 4. Reverse extension of the pattern. 5. Sectional extension (reproduction) of the pattern. 6. Ordinary extension (reproduction) of the pattern.

The above categories enrich Clements’s and Sarama’s Pattern Extender and are in accordance with Rustigian’s (1976 as cited in Threlfall, 2005) developmental hierarchy of pre-schoolers responses at the extension of repeating patterns, as well as with Papic’s, Mulligan’s and Michelmore’s description of pre-schoolers strategies for AB patterns. They also have similarities with Klein’s and Starkley’s (2004) description of preschoolers’ AB pattern errors. In particular, the third and the fifth category are in accordance with the three types of Klein’ and Starkley’ pattern extension errors that are mentioned as “no initial unit reflected the least developed knowledge of patterns”, “only initial unit reflected a partial knowledge of repeating patterns” and “model pattern plus extra blocks”.

The above mentioned categories that describe children’s patterning performance could be valuable in providing kindergarten’s teachers with a useful background for student’s actions and capabilities in different pattern constructs and pattern types which are constructed with a variety of materials. It may support their instructional design informing them about the complexity of the patterns children can work with, the ways they treat the core unit and the different materials, as well as how can children cope with the extension and the reproduction of patterns.

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An Interactive Way to Reproduce Structured Matrices

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Abstract. This paper suggests a direction for computer experiments, aimed at students with an interest in Matrix Algebra. We propose a programming style in the environment of free computer algebra system Xcas, as an interactive tool to study and/or reproduce the structure of matrices of various forms. As the turtle visualizes Logolike programming commands, our programmed functions create matrices using the recursive formula that their elements obey. Writing or accessing computer codes that produce structured matrices ensures deep understanding of their structure architecture, which is traditionally explained using indices and numerals. Suggestively we present the codes for symmetric and skew-symmetric matrices, codes for centrosymmetric and skew-centrosymmetric matrices, for both theoretical and arithmetic formulation and, codes for hermitian and skew-hermitian matrices. Similar applications can be student projects or classroom instructions for Linear Algebra courses.

Keywords: matrix functions; structured matrices; functional programming

1. Introduction

The last 20 years it has been proved that the use of CAS is integrated successfully into mathematics education and is considered essential in teaching geometry, algebra or calculus. There is a distinct stream in the field technology enhanced mathematics, in which the teaching process emphasizes in computer-based instruction (Alpers, 2002, Artigue, 2002; Hoyles et al. 2004; Robyn and Kaye, 2010; Robyn and Kaye, 2001; Yerushalmy 1999; Zsolt Lavicza, 2010; Peschek and Schneider, 2002; Drijvers, 2002). Research and development in the use of computer algebra in mathematics education are matters of great interest as seen in the table of content of issues of international journals as Int. Journal of Computer Algebra in Mathematics Education, Journal of Mathematics Science and Technology Education, International Journal of Computers for Mathematical Learning, or in the topics of several recently organized scientific symposia, (e.g. TIME-2008 Symposium Technology and its Integration in Mathematics Education, 6th CAME Symposium: Improving tools, tasks and teaching in CAS-based Mathematics education).

In a typical Matrix Algebra course, several matrices of special forms are introduced and studied. In such context, the traditional teaching approach is based on definitions and formulas with indices and numerals, methods which are generally considered as unattractive for the students. In an integrated curriculum, the teaching process when introducing structured matrices and testing their properties should include a CAS environment, for either the instructor to present numerous theoretical or numerical examples or the students to generate automatically matrix structures on their own. As science experiments in chemistry and physics helps ground students' understanding, in the same way, a computer-based approach helps understanding of abstract matrix algebra formulations. Matrix theory applications using computer software are proposed in (Anton et al., 2003; Demmel, 1997; Halkos & Tsilika, 2011; Luszczek & Dongarra, 2011).

In this study, we propose a set of representations and functionalities using the environment of a user friendly, free of charges, powerful CAS software, Xcas. A symbolic language, the Xcas program editor is used to syntax computer codes for structured matrices creation. The programming style proposed uses recursive formulas that describe matrix elements.

When the matrices produced are filled with random input, the Xcas' random number generator is activated. In cases of symmetric, skew-symmetric, centrosymmetric, skew-centrosymmetric, hermitian and skew-hermitian matrices, the general algebraic structure is reproduced, in the desirable matrix size. Making the matrix out of a recursive formula is a good way to study matrix structures. An introduction to general programming philosophy is also achieved.

Our approach makes empirical research easily reproducible. The output produced can also be used

i) for verification of the properties of matrices of special forms i.e. the eigenvalues of a symmetric matrix are real numbers and the corresponding eigenvectors can always be assumed to be real, the eigenvalues of a skew-hermitian matrix are imaginary, up to advanced applications as in (Bai, 2011; Gallier & Xu, 2002)

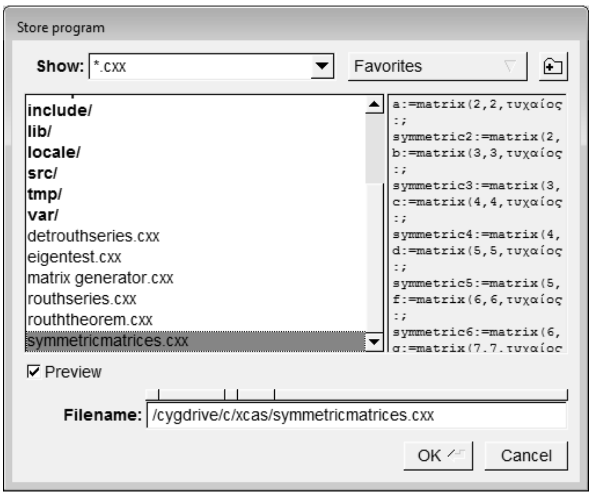
ii) prediction of the matrix form that results after matrix operations

iii) creation of the matrix needed, simplifying and abbreviating the law of formation.

The study is organized as follows. Section 2 discusses briefly the programming environment in Xcas. Sections 3,4,5 provide the codes for automatic construction of symmetric and skew-symmetric matrices, centrosymmetric and skew-centrosymmetric matrices and hermitian and skew-hermitian matrices. Several examples to perform computational adequacy of programmed functions are also given. Section 6 presents the choices in the design and indication of the tasks learning potential of our codes in a teaching sequence. Finally, section 7 concludes the paper.

2. Programming in Xcas

Programs in Xcas may be written in a separate program level, via Prg->New of Prg Menu. This will open an editor in a new level. The editor has its own menu, where we can import our computer codes of the following sections, save and export the current program as symmetricmatrices.cxx. Working in any session of Xcas, by writing in a commandline read("symmetricmatrices.cxx") we can use symmetric, skewsymmetric, centrosymmetric, skew-centrosymmetric, hermitian, skewhermitian functions.



A detailed user's guide of Xcas is available at http://www-fourier.ujf-grenoble.fr/~parisse/giac/tutoriel_en.pdf

3. Constructing Symmetric and Skew-Symmetric Matrices

In this section we consider two types of matrices, real symmetric and skew-symmetric matrices. The typical recursive formulas of a great variety of structured matrices can be found among others in (Strang, 1988; Demmel, 1997).

3.1 Computer Codes for Symmetric Matrices

Definition. A symmetric matrix A is a matrix which equals its own transpose; that is, it satisfies the equation: $A=A^T$. The matrix is necessarily square and each entry in one side of the diagonal equals its mirror image on the other side: $a_{ij}=a_{ji}$.

Our programmed function `symmetric(m)` presents theoretically the structure of a symmetric matrix of size m. The codes in Xcas are:

```
symmetric(m):=matrix(m,m,(j,k)-
>if(j<k&&j!=k)a(j+1,k+1);else a(k+1,j+1);)
```

For example, by writing in Xcas:

`symmetric(6)` the output is:

$$\begin{bmatrix} a(1,1), & a(1,2), & a(1,3), & a(1,4), & a(1,5), & a(1,6) \\ a(1,2), & a(2,2), & a(2,3), & a(2,4), & a(2,5), & a(2,6) \\ a(1,3), & a(2,3), & a(3,3), & a(3,4), & a(3,5), & a(3,6) \\ a(1,4), & a(2,4), & a(3,4), & a(4,4), & a(4,5), & a(4,6) \\ a(1,5), & a(2,5), & a(3,5), & a(4,5), & a(5,5), & a(5,6) \\ a(1,6), & a(2,6), & a(3,6), & a(4,6), & a(5,6), & a(6,6) \end{bmatrix}$$

Our programmed function `symmetric(m,a(m))` generates symmetric matrices of size m with elements 2-digit real non-negative random numbers, generated by Xcas function `rand(0..100)`. The codes in Xcas are:

```
a(m):=matrix(m,m,rand(0..100)):;
symmetric(m,a(m)):=matrix(m,m,(j,k)->if(j<k&&j!=k)
a(m)[[j,k]]; else a(m)[[k,j]];):;
```

We present below computational performance of symmetric(m,a(m)) function, creating random symmetric matrices of sizes 2, 4, 5:

`symmetric(2,a(2))`

$$\begin{bmatrix} 28.157, & 69.709 \\ 69.709, & 24.571 \end{bmatrix}$$

`symmetric(4,a(4))`

$$\begin{bmatrix} 65.228, & 85.618, & 40.738, & 78.415 \\ 85.618, & 22.712, & 30.422, & 75.673 \\ 40.738, & 30.422, & 60.732, & 87.919 \\ 78.415, & 75.673, & 87.919, & 49.611 \end{bmatrix}$$

`symmetric(5,a(5))`

$$\begin{bmatrix} 44.094, & 82.444, & 77.768, & 13.803, & 47.249 \\ 82.444, & 91.243, & 70.564, & 48.556, & 69.049 \\ 77.768, & 70.564, & 51.105, & 23.476, & 1.518 \\ 13.803, & 48.556, & 23.476, & 94.747, & 25.486 \\ 47.249, & 69.049, & 1.518, & 25.486, & 90.924 \end{bmatrix}$$

3.2 Computer Codes for Skew-Symmetric Matrices

Definition. A skew-symmetric (or antisymmetric or antimetric) matrix is a square matrix A whose transpose is also its negative; that is, it satisfies the equation A^T=-A. If the entry in the i th row and j th column is a_{ij}, then the skew-symmetric condition is a_{ij}=-a_{ji}. All main diagonal entries of a skew symmetric matrix must be zero.

Our programmed function `skewsymmetric(m)` presents theoretically the structure of a skew-symmetric matrix of size m. The codes in Xcas are:

```
skewsymmetric(m):=matrix(m,m,(j,k)-
>if(j<k&&j!=k)a(j+1,k+1); else if(j==k) 0;else -
a(k+1,j+1);;):;
```

For example, by writing in Xcas:
`skewsymmetric(5)`, the output is:

$$\begin{bmatrix} 0, & a(1,2), & a(1,3), & a(1,4), & a(1,5) \\ -(a(1,2)), & 0, & a(2,3), & a(2,4), & a(2,5) \\ -(a(1,3)), & -(a(2,3)), & 0, & a(3,4), & a(3,5) \\ -(a(1,4)), & -(a(2,4)), & -(a(3,4)), & 0, & a(4,5) \\ -(a(1,5)), & -(a(2,5)), & -(a(3,5)), & -(a(4,5)), & 0 \end{bmatrix}$$

Our programmed function `skewsymmetric(m,b(m))` generates skew-symmetric matrices of size m with elements 2-digit real random numbers, generated by Xcas function `rand(-100..100)`. The codes in Xcas are:

```
b(m):=matrix(m,m,rand(-100..100)):;
skewsymmetric(m,b(m)):=matrix(m,m,(j,k)-
>if(j<k&&j!=k) b(m)[[j,k]]; else (if(j==k) 0;else -
b(m)[[k,j]];););:;
```

The following examples present computational performance of skewsymmetric(m,b(m)) function, creating random skew-symmetric matrices of sizes 2, 4, 5.

```
skewsymmetric(2,b(2))
```

$$\begin{bmatrix} 0 & -79.05 \\ 79.05 & 0 \end{bmatrix}$$

```
skewsymmetric(4,b(4))
```

$$\begin{bmatrix} 0 & 23.101 & -76.333 & -62.671 \\ -23.101 & 0 & 66.153 & -38.614 \\ 76.333 & -66.153 & 0 & -59.642 \\ 62.671 & 38.614 & 59.642 & 0 \end{bmatrix}$$

```
skewsymmetric(5,b(5))
```

$$\begin{bmatrix} 0 & -51.132 & 41.298 & 10.194 & 21.907 \\ 51.132 & 0 & -37.65 & -1.1496 & 6.6907 \\ -41.298 & 37.65 & 0 & 17.617 & 71.746 \\ -10.194 & 1.1496 & -17.617 & 0 & 45.559 \\ -21.907 & -6.6907 & -71.746 & -45.559 & 0 \end{bmatrix}$$

4. Constructing Centrosymmetric and Skew-Centrosymmetric Matrices

In this section we consider two types of matrices, centrosymmetric and skew-centrosymmetric matrices.

4.1 Computer Codes for Centrosymmetric Matrices

Definition. A centrosymmetric matrix is a matrix which is symmetric about its center. More precisely, a square matrix is centrosymmetric when its entries in the i th row and j th column a_{ij} satisfy the condition $a_{ij}=a_{n-i+1,n-j+1}$.

Our programmed function `centrosymmetric(n)` represents the general algebraic structure of a centrosymmetric matrix of size n. The codes in Xcas are:

```
centrosymmetric(n):=matrix(n,n,(j,k)-> if(j>k) a(n-j,n-k); else if(j=k) a(min(n-j,j+1),min(n-k,k+1)); else a(j+1,k+1);;):;
```

For example, by writing in Xcas:

`centrosymmetric(5)` the output is:

$$\begin{bmatrix} a(1,1), a(1,2), a(1,3), a(1,4), a(1,5) \\ a(4,5), a(2,2), a(2,3), a(2,4), a(2,5) \\ a(3,5), a(3,4), a(3,3), a(3,4), a(3,5) \\ a(2,5), a(2,4), a(2,3), a(2,2), a(4,5) \\ a(1,5), a(1,4), a(1,3), a(1,2), a(1,1) \end{bmatrix}$$

Our programmed function `centrosymmetric(n,c(n))` generates centrosymmetric matrices of size n with elements 2-digit real random numbers, generated by Xcas function `rand(-100..100)`. The codes in Xcas are:

```
c(n):=matrix(n,n,rand(-100..100)):;
centrosymmetric(n,c(n)):=matrix(n,n,(j,k)-> if(j>k)
c(n)[[(n-j-1,n-k-1)]]; else if(j==k) c(n)[[min(n-j-
1,j),min(n-k-1,k)]];else c(n)[[(j,k)]];;):;
```

The following examples present computational performance of centrosymmetric(n,c(n)) function, creating random centrosymmetric matrices of sizes 2, 3, 7.

```
centrosymmetric(2,c(2))
```

$$\begin{bmatrix} 45.462, & 53.81 \\ 53.81, & 45.462 \end{bmatrix}$$

```
centrosymmetric(3,c(3))
```

$$\begin{bmatrix} -22.299, & 27.087, & -97.827 \\ 42.712, & 40.123, & 42.712 \\ -97.827, & 27.087, & -22.299 \end{bmatrix}$$

```
centrosymmetric(7,c(7))
```

$$\begin{bmatrix} -75.84, & 15.241, & -95.489, & 77.161, & -71.602, & -81.098, & -67.562 \\ 95.788, & 24.569, & 81.928, & -80.555, & 52.748, & 46.094, & -55.44 \\ 17.485, & -29.674, & 82.659, & -63.484, & -18.795, & 4.0797, & 74.267 \\ -7.3504, & 76.149, & -2.4769, & -89.831, & -2.4769, & 76.149, & -7.3504 \\ 74.267, & 4.0797, & -18.795, & -63.484, & 82.659, & -29.674, & 17.485 \\ -55.44, & 46.094, & 52.748, & -80.555, & 81.928, & 24.569, & 95.788 \\ -67.562, & -81.098, & -71.602, & 77.161, & -95.489, & 15.241, & -75.84 \end{bmatrix}$$

4.2 Computer Codes for Skew-Centrosymmetric Matrices

Definition. An n × n matrix A is said to be skew-centrosymmetric if its entries satisfy the condition $a_{ij}=-a_{n-i+1,n-j+1}$.

Our programmed function `skewcentrosymmetric(n)` represents the general algebraic structure of a skew-centrosymmetric matrix of size n. The codes in Xcas are:

```
skewcentrosymmetric(n):=matrix(n,n,(j,k)->if(j>k)-
a(n-j,n-k);  else  if(j=k&&j+1>(n+1)/2)  -a(min(n-
j,j+1),min(n-k,k+1)) ;else a(j+1,k+1);;):;
```

For example, by writing in Xcas:

`skewcentrosymmetric(4)` the output is:

$$\begin{bmatrix} a(1,1), & a(1,2), & a(1,3), & a(1,4) \\ -(a(3,4)), & a(2,2), & a(2,3), & a(2,4) \\ -(a(2,4)), & -(a(2,3)), & -(a(2,2)), & a(3,4) \\ -(a(1,4)), & -(a(1,3)), & -(a(1,2)), & -(a(1,1)) \end{bmatrix}$$

Our programmed function `skewcentrosymmetric(n,d(n))` generates skew-centrosymmetric matrices of size n with elements 2-digit real random numbers, generated by Xcas function `rand(-100..100)`. The codes in Xcas are:

```
d(n):=matrix(n,n,rand(-100..100)):;
skewcentrosymmetric(n,d(n)):=matrix(n,n,(j,k)-
>if(j>k)
-d(n)[[(n-j-1,n-k-1)]];  else  if(j==k&&j+1>(n+1)/2)
-d(n)[[min(n-j-1,j),min(n-k-1,k)]];            else
d(n)[[(j,k)]];;):;
```

The following examples present computational performance of skewcentrosymmetric(n,d(n)) function, creating random skew-centrosymmetric matrices of sizes 2, 3, 7.

`skewcentrosymmetric(2,d(2))`

$$\begin{bmatrix} 73.75, & -16.662 \\ 16.662, & -73.75 \end{bmatrix}$$

`skewcentrosymmetric(3,d(3))`

$$\begin{bmatrix} -95.44, & -99.458, & 75.966 \\ 96.636, & -52.593, & -96.636 \\ -75.966, & 99.458, & 95.44 \end{bmatrix}$$

`skewcentrosymmetric(7,d(7))`

$$\begin{bmatrix} 97.343, & 50.168, & 55.399, & 67.638, & 34.797, & -51.208, & 33.721 \\ 74.326, & -79.868, & 40.152, & -71.464, & 87.967, & 69.59, & 99.007 \\ -81.409, & -76.654, & 71.651, & 21.684, & -77.237, & -12.994, & -37.323 \\ 72.602, & 53.135, & 86.964, & -54.84, & -86.964, & -53.135, & -72.602 \\ 37.323, & 12.994, & 77.237, & -21.684, & -71.651, & 76.654, & 81.409 \\ -99.007, & -69.59, & -87.967, & 71.464, & -40.152, & 79.868, & -74.326 \\ -33.721, & 51.208, & -34.797, & -67.638, & -55.399, & -50.168, & -97.343 \end{bmatrix}$$

5. Constructing Hermitian and Skew-Hermitian Matrices

In this section we consider two types of matrices, hermitian and skew-hermitian matrices.

5.1 Computer Codes for Hermitian Matrices

Definition. A square matrix with complex entries A is said to be hermitian if it equals its conjugate transpose: $A=A^H$, that is a_{ij} =conj(a_{ji}). The entries on the main diagonal of any hermitian matrix are necessarily real.

Our programmed function `hermitian(n,h(n))` generates hermitian matrices of size n with elements having real and imaginary part 2-digit real random numbers, generated by Xcas function `rand(-100..100)()`. The codes in Xcas are:

```
h(n):=matrix(n,n,(j,k)->if(j==k)  rand(-100..100)();
else rand(-100..100)()+i* rand(-100..100)(););:;
hermitian(n,h(n)):=matrix(n,n,(j,k)->if(j<k&&j!=k)
h(n)[[j,k]];else conj(h(n)[[k,j]]););:;
```

The following examples present computational performance of hermitian(n,h(n)) function, creating random hermitian matrices of sizes 2,4,5.

```
hermitian(2,h(2))
```

$$\begin{bmatrix} 93.614, & -86.654-4.3438*i \\ -86.654+4.3438*i, & 86.507 \end{bmatrix}$$

```
hermitian(4,h(4))
```

$$\begin{bmatrix} -39.72, & 81.689+82.815\cdot i, & 13.299-62.651\cdot i, & 28.946+61.216\cdot i \\ 81.689-82.815\cdot i, & 98.246, & 31.463+4.0736\cdot i, & 41.349+31.437\cdot i \\ 13.299+62.651\cdot i, & 31.463-4.0736\cdot i, & 73.544, & -78.1-20.067\cdot i \\ 28.946-61.216\cdot i, & 41.349-31.437\cdot i, & -78.1+20.067\cdot i, & -53.274 \end{bmatrix}$$

```
hermitian(5,h(5))
```

$$\begin{bmatrix} 6.398, & 74.798-27.492\cdot i, & 66.678+56.486\cdot i, & -44.677-20.024\cdot i, & 93.836+84.269\cdot i \\ 74.798+27.492\cdot i, & -21.332, & -60.562+3.4298\cdot i, & 82.741+80.787\cdot i, & -65.133-55.502\cdot i \\ 66.678-56.486\cdot i, & -60.562-3.4298\cdot i, & 35.58, & 65.71-96.973\cdot i, & 66.104+14.291\cdot i \\ -44.677+20.024\cdot i, & 82.741-80.787\cdot i, & 65.71+96.973\cdot i, & 57.513, & 43.01+75.973\cdot i \\ 93.836-84.269\cdot i, & -65.133+55.502\cdot i, & 66.104-14.291\cdot i, & 43.01-75.973\cdot i, & -54.795 \end{bmatrix}$$

5.2 Computer Codes for Skew-Hermitian Matrices

Definition. A square matrix with complex entries A is said to be skew-hermitian or antihermitian if its conjugate transpose is equal to its negative: A^H=-A, that is a_{ij} =-conj(a_{ji}). All main diagonal entries of a skew-hermitian matrix is imaginary.

The following codes generate skew-hermitian matrices of size n, with elements having real and imaginary part 2-digit real random numbers, generated by Xcas function `rand(-100..100)()`:

```
s(n):=matrix(n,n,(j,k)->if(j==k) rand(-100..100)();
else rand(-100..100)()+i* rand(-100..100)(););:;
skewhermitian(n,s(n)):=matrix(n,n,(j,k)-
>if(j<k&&j!=k) s(n)[[j,k]]; else (if(j==k) i*
rand(-100..100)() ;else -conj(s(n)[[k,j]]););):;
```

The following examples present computational performance of skewhermitian(n,s(n)) function, creating random skew-hermitian matrices of sizes 2, 4, 5.

```
skewhermitian(2,s(2))
```

$$\begin{bmatrix} -24.234*i, & -95.786+2.441*i \\ 95.786+2.441*i, & 58.623*i \end{bmatrix}$$

```
skewhermitian(4,s(4))
```

$$\begin{bmatrix} -30.662*i, & -46.569-28.635*i, & -74.325+40.66*i, & 47.483+60.504*i \\ 46.569-28.635*i, & -62.746*i, & -8.7237-5.7876*i, & -40.09+22.512*i \\ 74.325+40.66*i, & 8.7237-5.7876*i, & 34.154*i, & 39.574+90.197*i \\ -47.483+60.504*i, & 40.09+22.512*i, & -39.574+90.197*i, & 26.978*i \end{bmatrix}$$

```
skewhermitian(5,s(5))
```

$$\begin{bmatrix} -55.44*i, & -94.482+52.653*i, & -68.655+53.001*i, & 13.157-85.609*i, & -44.722-34.432*i \\ 94.482+52.653*i, & -75.155*i, & -71.821+64.166*i, & -74.884-27.903*i, & 72.456+38.099*i \\ 68.655+53.001*i, & 71.821+64.166*i, & 18.551*i, & 8.9562+66.272*i, & -54.774-15.513*i \\ -13.157-85.609*i, & 74.884-27.903*i, & -8.9562+66.272*i, & 82.659*i, & -95.489+77.161*i \\ 44.722-34.432*i, & -72.456+38.099*i, & 54.774-15.513*i, & 95.489+77.161*i, & -63.484*i \end{bmatrix}$$

6. Potential Use of our Routine in Educational Practice

The routine symmetricmatrices.cxx could be used exclusively by the instructor, in order to perform visual representation of the structured matrices studied, of any size. If this is the case, Xcas could function as an interactive presentation environment. The instructor has the choices of visualizing matrices using a symbolic approach or through several numerical examples. The instructor could use projection technologies in the classroom or could upload the "symmetricmatrices.cxx" program file in the computer network of a laboratory. Then, the students by writing in a commandline of Xcas read("symmetricmatrices.cxx"), can use all programmed functions to generate the related output.

In a different learning scenario, students could be involved in the programming procedure; an advanced project could aim in the codes construction. After a typical explanation of the rule which elements of special matrices obey, the students could be asked to program the recursive formula that results to the matrix under study. This is a trial and error procedure, which helps the students realize how an inappropriate or deficient formula affects the matrix structure. In this direction, and depending on students' computer skills and programming qualification, potential student tasks could be to implement the codes in order to generate a structured matrix of dimension m, e.g.:

```
symmetric(m):=matrix(m,m,(j,k)->  …  ):;
skewsymmetric(m):=matrix(m,m,(j,k)->  … ):;
centrosymmetric(n):=matrix(n,n,(j,k)->…            ):;
skewcentrosymmetric(n):=matrix(n,n,(j,k)->   …   :;
```

or to syntax a structured matrix function from first to last code element.

7. Conclusion

In this study, computer codes that build structured matrices are given, written in the programming environment of free computer algebra system Xcas. Our codes present theoretically the low of formation and, alternatively, use random numbers as nonzero elements, allowing the creation of large matrices instantly. They are written in a user friendly program editor, being open source material to replicate or to be modified to the user's interest.

Our programming style helps the user, without serious programming skills, to program specialized matrix structures by examining closely their structure architecture. This is the reason why our computer codes can function as an educational tool for teaching special matrices.

Building special matrices with random input has also the advantage of infinite number of examples. In contrast with built-in functions of mathematical packages (e.g. MATLAB, wxMaxima, Linear Algebra), which either test whether a given matrix is symmetric, hermitian, etc. or, construct structured matrices with given entries, our functions construct structured matrices requiring only the matrix size as input. In addition, they could be programmed by the user himself.

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Use of fuzzy logic in students' assessment

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Abstract

The present paper proposes the use of fuzzy logic as a tool in assessing the students' knowledge and skills. In our model the students' characteristics under assessment (e.g. knowledge of the subject matter, problem solving skills and analogical reasoning abilities) are represented as fuzzy subsets of a set of linguistic labels characterizing the degree of their success and the possibilities of all student profiles are calculated. In this way a detailed quantitative/qualitative study of the students' performance is obtained. The centroid method is used as a defuzzification technique in converting our fuzzy outputs to a crisp number. According to this method the coordinates of the centre of gravity of the graph of the membership function involved provide a measure of a students' group performance. Techniques of assessing the individual students' abilities are also studied and examples are presented to illustrate the use of our results in practice.

Keywords: Fuzzy sets, Fuzzy logic, Centroid defuzzification method, students' assessment

1. Introduction

There used to be a tradition in science and engineering of turning to probability theory when one is faced with a problem in which uncertainty plays a significant role. This transition was justified when there were no alternative tools for dealing with the uncertainty. Today this is no longer the case. *Fuzzy logic*, which is based on fuzzy sets theory introduced by Zadeh (1965), provides a rich and meaningful addition to standard logic. This theory proposed in terms of the membership function operating over the range [0, 1] of real numbers. New operations for the calculus of logic were also proposed and showed to be in principle at least a generalization of classic logic (Zadeh 1965, 1968). Despite the fact that both operate over the same numerical range [0, 1], fuzzy sets theory is distinct from probability theory. For example, the probabilistic approach yields the natural language statement "there is an 85% chance that Mary is tall", while the fuzzy terminology corresponds to the expression "Mary's degree of membership within the set of tall people is 0,85". The semantic difference is significant: The first view supposes that Mary is or is not tall (still caught in the law of the Excluded Middle); it is just that we only have a 85% chance of knowing in which set she is in. By contrast, fuzzy terminology supposes that Mary is "more or less" tall, or some other term corresponding to the value of 0,85. Another immediately apparent difference is that the summation of probabilities of the single subsets (events) of the universal set must equal 1, while there is no such requirement for membership degrees. Further distinctions and differences between probability and fuzziness also exist, arising from the way that the corresponding operations are defined.

The applications which may be generated from or adapted to fuzzy logic are wide-ranging and provide the opportunity for modelling under conditions which are inherently imprecisely defined, despite the concerns of classical logicians. Sutable methods for teaching the basics of fuzzy logic have been recently appeared in the literature (Sobrino 2013, Yalcin and Kose, etc). A real test of the effectiveness of an approach to uncertainty is the capability to solve problems which involve different facets of uncertainty. Fuzzy logic has a much higher problem solving capability than standard probability theory. Most importantly, it opens the door to construction of mathematical solutions of computational problems which are stated in a natural language. In contrast, standard probability theory does not have this capability, a fact which is one of its principal limitations. All the above gave us the impulsion to introduce principles of fuzzy logic to describe in a more effective way a system's operation in situations characterized by a degree of vagueness and/or uncertainty (e.g. see Voskoglou 2011a, 2012). In Education they often appear such kind of situations in the cases of learning a subject matter, of problem-solving, of modeling, of analogical reasoning, etc. In fact, students' cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teacher's point of view there usually exists vagueness about the degree of success of students in each of the stages of the corresponding didactic situation. As a consequence, our fuzzy model mentioned above finds a lot of applications in the area of education (see Voskoglou 2011a, 2011b, 2011c, etc).

One of the main teachers' concerns is the assessment of their students' knowledge and aptitudes. In fact, our society demands not only to educate, but also to classify the students according to their qualifications as being suitable or not to carry out certain tasks or to hold certain posts. According to the standard methods of assessment, a mark, expressed either with a numerical value within a given scale (e.g. from 0 to 10) or with a letter (e.g. from A to F) corresponding to the percentage of a student's success, is assigned in order to characterize his/her performance. However, this crisp characterization, based on principles of the bivalent logic (yes-no), although it is the one usually applied in practice, it is not probably the most suitable to determine a student's performance. In fact, the teacher can be never absolutely sure about a particular numerical grade characterizing the student's abilities and skills. On the contrary, fuzzy logic, due to its nature of including multiple values, offers a wider and richer field of resources for this purpose.

The present paper proposes the use of fuzzy logic in assessing students' knowledge and skills. The text is organized as follows: In section 2 we adapt our general fuzzy framework mentioned above for use in assessing the students' performance. In section 3 we apply the method of the "centre of gravity" as a defuzzification method in converting our fuzzy outputs to a crisp number and we present an example illustrating our results in practice. In section 4 we study techniques of students' individual assessment and finally in section 5 we state our conclusions and we discuss our plans for future research. For general facts on fuzzy sets we refer freely to the book of Klir and Folger (1988).

2. The fuzzy model

Let us consider a class of n students, $n \geq 1$ and let us assume that the teacher wants to assess the following students' characteristics: S_1 = knowledge of a subject matter, S_2 = problem solving related to this subject matter and S_3 = ability to adapt properly the already existing knowledge for use in analogous similar cases (analogical reasoning)[1]. Denote by *a, b, c, d,* and *e* the linguistic labels (fuzzy expressions) of very low, low, intermediate, high and very high success respectively of a student in each of the S_i*'s* and set $U = \{a, b, c, d, e\}$.

We are going to attach to each students' characteristic S_i , i=1, 2, 3 , a fuzzy subset, A_i of U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of students that faced very low, low, intermediate, high and very high success with respect to S_i respectively, we define the *membership function* m_{Ai} for each x in U, as follows:

$$m_{A_i}(x) = \begin{cases} 1, & if \ \frac{4n}{5} < n_{ix} \leq n \\ 0,75, & if \ \frac{3n}{5} < n_{ix} \leq \frac{4n}{5} \\ 0,5, & if \ \frac{2n}{5} < n_{ix} \leq \frac{3n}{5} \\ 0,25, & if \ \frac{n}{5} < n_{ix} \leq \frac{2n}{5} \\ 0, & if \ 0 \leq n_{ix} \leq \frac{n}{5} \end{cases}$$

In fact, if one wanted to apply probabilistic standards in measuring the degree of the students' success at each stage of the process, then he/she should use the relative frequencies $\frac{n_{ix}}{n}$. Nevertheless, such an action would be highly questionable, since the n_{ix}*'s* are obtained with respect to the linguist labels of U, which are fuzzy expressions by themselves. Therefore the application of a fuzzy approach by using membership degrees instead of probabilities seems to be the most suitable for this case. But, as it is well known, the membership function is usually defined empirically in terms of logical or/and statistical data. In our case the above definition of m_{A_i} seems to be compatible with the common sense. Then the fuzzy subset A_i of U corresponding to S_i has the form:

$$A_i = \{(x, m_{Ai}(x)) : x \in U\}, i=1, 2, 3.$$

[1] Of course the teacher could choose students' characteristics different from those mentioned here. The model also works if the characteristics under assessment are more than three, but technically it becomes more complicated.

In order to represent all possible students' *profiles (overall states)* with respect to the assessing process we consider a *fuzzy relation*, say R, in U^3 (i.e. a fuzzy subset of U^3) of the form:

$$R=\{(s, m_R(s)): s=(x, y, z) \in U^3\}.$$

For determining properly the membership function m_R we give the following definition:

A profile $s=(x, y, z)$, with x, y, z in U, is said to be well ordered if x corresponds to a degree of success equal or greater than y and y corresponds to a degree of success equal or greater than z.

For example, (c, c, a) is a well ordered profile, while (b, a, c) is not.

We define now the *membership degree* of a profile s to be $m_R(s) = m_{A_1}(x)m_{A_2}(y)m_{A_3}(z)$, if s is well ordered, and 0 otherwise[2]. In fact, if for example the profile (b, a, c) possessed a nonzero membership degree, how it could be possible for a student, who has failed at the problem solving stage, to perform satisfactorily at the stage of analogical reasoning, where he/she has to adapt the existing knowledge for solving problems related to analogous similar cases?

Next, for reasons of brevity, we shall write m_s instead of $m_R(s)$. Then the *probability* p_s of the profile s is defined in a way analogous to crisp data, i.e. by $p_s = \frac{m_s}{\sum_{s\in U^3} m_s}$. We define also the *possibility* r_s of s to be $r_s=\frac{m_s}{\max\{m_s\}}$, where $max\ \{m_s\}$ denotes the maximal value of m_s for all s in U^3. In other words the possibility of s expresses the "relative membership degree" of s with respect to $max\ \{m_s\}$. From the above two definitions it becomes evident that $p_s < r_s$ for all s in U^3, which is compatible to the common logic. In fact, whatever is probable it is also possible, but whatever is possible need not be very probable.

Assume now that one wants to study the *combined results* of the performance of k different groups of students, $k\geq 2$[3]. For this, we introduce the *fuzzy variables* $A_1(t)$, $A_2(t)$ and $A_3(t)$ with $t=1, 2,\dots, k$. The values of

---

[2] It can be shown that this definition by the product rule satisfies the axioms that must be hold for the general aggregation operations among fuzzy sets (see [7], pp. 58-59 and 283), where the independence of the corresponding events is not required, as it happens in probability theory.

[3] This is, for example, useful if one wants to compare the students' performance of one school as a whole with respect to another school.

these variables represent fuzzy subsets of *U* corresponding to the students' characteristics under assessment for each of the *k* groups; e.g. $A_1(2)$ represents the fuzzy subset of *U* corresponding to the knowledge of a subject matter (characteristic S_1) for the second group (*t=2)*. Obviously, in order to measure the degree of evidence of the combined results of the *k* groups, it is necessary to define the probability *p(s)* and the possibility *r(s)* of each profile *s* with respect to the membership degrees of *s* for all groups. For this reason we introduce the *pseudo-frequencies* $f(s) = m_s(1)+m_s(2)+\ldots+m_s(k)$ and we define the probability and possibility of a profile *s* by $p(s) = \frac{f(s)}{\sum_{s\in U^3} f(s)}$ and $r(s)=\frac{f(s)}{\max\{f(s)\}}$ respectively, where *max{f(s)}* denotes the maximal pseudo-frequency. The same method could be applied when one wants to study the combined results of *k* different assessments of the same student group.

The above model gives, through the calculation of probabilities and possibilities of all students' profiles, a quantitative/qualitative view of their realistic performance.

3. The centroid defuzzification technique

Defuzzification is the process of producing a quantifiable result in fuzzy logic given fuzzy sets and corresponding membership degrees. A common and useful defuzzification technique is the method of the *centre of gravity*, usually referred as the *centroid method.* According to this method, given a fuzzy subset

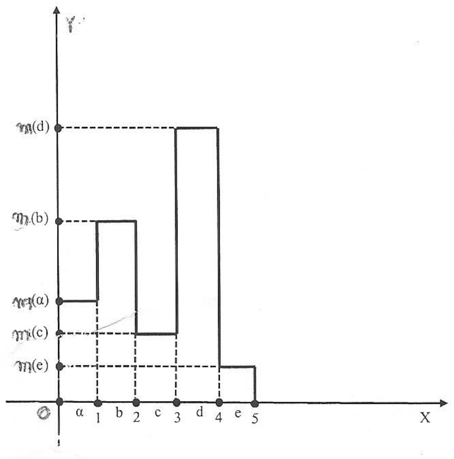
$A = \{(x, m(x)): x\in U\}$ of the universal set *U* of the discourse with membership function $m: U \to [0, 1]$, we correspond to each $x\in U$ an interval of values from a prefixed numerical distribution, which actually means that we replace *U* with a set of real intervals. Then, we construct the graph *F* of the membership function *y=m(x)*. There is a commonly used in fuzzy logic approach to measure performance with the pair of numbers (x_c, y_c) as the coordinates of the centre of gravity, say F_c, of the graph *F* (e.g. see van Broekhoven & De Baets, 2006), which we can calculate using the following well-known from the Mechanics formulas:

$$x_c = \frac{\iint_F x\,dxdy}{\iint_F dxdy}, \quad y_c = \frac{\iint_F y\,dxdy}{\iint_F dxdy} \qquad (1)$$

Subbotin et al. (2004, 2011) adapted the centroid method for use with our fuzzy model for the process of learning (Voskoglou 1999) and they have applied it on comparing students' mathematical learning abilities (2004) and for measuring the scaffolding (assistance) effectiveness provided by the teacher to students (2011) . More recently together with Prof. Subbotin we have applied this method in measuring the effectiveness of Case – Based Reasoning Systems (Subbotin & Voskoglou 2011) and of students' Analogical Reasoning skills (.Voskoglou & Subbotin 2012).

Here we shall apply the centroid method as a defuzzification technique for the students' assessment model developed in the previous section. For this, we characterize a student's performance as very low (*a*) if $x \in [0, 1)$, as low (*b*) if
$x \in [1, 2)$, as intermediate (*c*) if $x \in [2, 3)$, as high (*d*) if $x \in [3, 4)$ and as very high (*e*) if $x \in [4, 5]$ respectively[4]. Therefore, if $x \in [0, 1)$, then $y=m(x)=m(a)$, if
$x \in [1, 2)$ then $y=m(x)=m(b)$, etc. In this case the graph *F* of the corresponding fuzzy subset of *U* is the bar graph of Figure 1 consisting of 5 rectangles, say F_i, i=1,2,3,4,5, having the lengths of their sides on the *x* axis equal to 1.

Figure 1: Bar graphical data representation



[4] These characterizations are usually awarded on the basis of the reports prepared by the students during the course and the results of the progress exams (if any) and the final exam.

Here $\iint_F dxdy$, which is the total area of F, is equal to $\sum_{i=1}^{5} y_i$. We also have that $\iint_F xdxdy = \sum_{i=1}^{5}\iint_{F_i} xdxdy = \sum_{i=1}^{5}\int_{i-1}^{i}[\int_0^{y_i} xdy]dx = \sum_{i=1}^{5} y_i \int_{i-1}^{i} xdx = \frac{1}{2}\sum_{i=1}^{5}(2i-1)y_i$, and $\iint_F ydxdy = \sum_{i=1}^{5}\iint_{F_i} ydxdy = \sum_{i=1}^{5}\int_{i-1}^{i}[\int_0^{y_i} ydy]dx = \frac{1}{2}\sum_{i=1}^{5} y_i^2$. Therefore formulas (1) are transformed into the following form:

$$x_c = \frac{1}{2}\left(\frac{y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5}{y_1 + y_2 + y_3 + y_4 + y_5}\right),$$

$$y_c = \frac{1}{2}\left(\frac{y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2}{y_1 + y_2 + y_3 + y_4 + y_5}\right).$$

Normalizing our fuzzy data by dividing each $m(x)$, $x \in U$, with the sum of all membership degrees we can assume without loss of the generality that $y_1+y_2+y_3+y_4+y_5 = 1$. Therefore we can write:

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right),$$

$$y_c = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right) \quad (2)$$

with $y_i = \frac{m(x_i)}{\sum_{x \in U} m(x)}$.

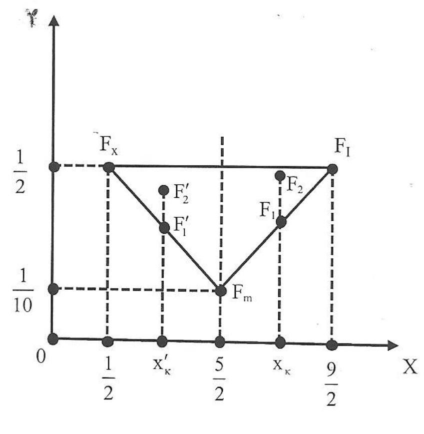
But $0 \leq (y_1-y_2)^2 = y_1^2+y_2^2-2y_1y_2$, therefore $y_1^2+y_2^2 \geq 2y_1y_2$, with the equality holding if, and only if, $y_1=y_2$. In the same way one finds that $y_1^2+y_3^2 \geq 2y_1y_3$, and so on. Hence it is easy to check that $(y_1+y_2+y_3+y_4+y_5)^2 \leq 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2)$, with the equality holding if, and only if $y_1=y_2=y_3=y_4=y_5$. But $y_1+y_2+y_3+y_4+y_5 = 1$, therefore $1 \leq 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2)$ (3), with the equality holding if, and only if $y_1=y_2=y_3=y_4=y_5=\frac{1}{5}$. In this case the first of formulas (2) gives that $x_c = \frac{5}{2}$. Further, combining the inequality (3) with the second of formulas (2) one finds that $1 \leq 10y_c$, or $y_c \geq \frac{1}{10}$ Therefore the unique minimum for y_c corresponds to the centre of gravity $F_m(\frac{5}{2}, \frac{1}{10})$.

The ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from formulas (2) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the centre of gravity in the ideal case is the point
$F_i\ (\frac{9}{2},\ \frac{1}{2})$. On the other hand the worst case is when $y_1=1$ and $y_2=y_3=y_4=y_5=0$. Then for formulas (2) we find that the centre of gravity is the point $F_w\ (\frac{1}{2},\ \frac{1}{2})$. Thus, the "area" where the centre of gravity F_c lies is represented by the triangle $F_w F_m F_i$ of Figure 2.

Then from elementary geometric considerations it follows that for two groups of students with the same $x_c \geq 2{,}5$ the group having the centre of gravity which is situated closer to F_i is the group with the higher y_c; and for two groups with the same $x_c < 2.5$ the group having the centre of gravity which is situated farther to F_w is the group with the lower y_c. Based on the above considerations we formulate our criterion for comparing the groups' performances as follows:

- *Among two or more groups the group with the greater x_c performs better.*
- *If two or more groups have the same $x_c \geq 2.5$, then the group with the higher y_c performs better.*
- *If two or more groups have the same $x_c < 2.5$, then the group with the lower y_c performs better.*

Figure 2: Graphical representation of the "area" of the centre of gravity



Next we give an example illustrating our results in practice.

EXAMPLE: The following data was obtained by assessing the mathematical skills (with respect to the characteristics S_1, S_2, and S_3 mentioned above) of two groups of students of the Technological Educational Institute of Patras, Greece being at their first term of studies:

First group (35 students from the School of Technological Applications, i.e. future engineers)

$$A_{11} = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e,0..25), A_{12} =$$
$$=\{(a,0),(b,0),(c, 0.5),(d, 0.25),(e,0)\},$$
$$A_{13} = \{(a, 0.25),(b, 0.25),(c, 0.25),(d,0),(e,0)\}$$

According to the above notation the first index of A_{ij} denotes the group ($i=1,\ 2$) and the second index denotes the corresponding students' characteristic S_j ($j=1, 2, 3$). We calculated the membership degrees of the 5^3 (ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles as it is described in section 2 (see column of $m_s(1)$ in Table 1). For example, for the profile $s=(c, c, a)$ one finds that $m_s = 0.5 \text{ x } 0 .5 \text{ x } 0.25 = 0.06225$. From the values of the column of $m_s(1)$ it turns out that the maximal membership degree of students' profiles is 0.06225. Therefore the possibility of each s in U^3 is given by $r_s=\frac{m_s}{0.06225}$. The possibilities of the students' profiles are presented in column of $r_s(1)$ of Table 1. One could also calculate the probabilities of the students' profiles using the formula for p_s given in section 2. However, according to Shackle (1961) and many other researchers after him, human cognition is better presented by possibility rather than by probability theory. Therefore, adopting this view, we considered that the calculation of the probabilities was not necessary.

Second group (50 students from the School of Management and Economics).

$$A_{21} = \{(a, 0),(b, 0.25),(c, 0.5),(d, 0.25),(e, 0)\},$$
$$A_{22}=\{(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e, 0)\},$$
$$A_{23}=\{(a, 0.25),(b, 0.25),(c, 0.25),(d, 0),(e, 0)\}.$$

The membership degrees and the possibilities of students' profiles are presented in columns of $m_s(2)$ and $r_s(2)$ of Table 1 respectively.

In order to study the combined results of the two groups' performance we also calculated the pseudo-frequencies $f(s) = m_s(1)+m_s(2)$ and the combined possibilities of all profiles presented at the last two columns of Table 1.

Table 1: Profiles with non zero membership degrees

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	$f(s)$	$r(s)$
b	b	b	0	0	0.016	0.258	0.016	0.129
b	b	a	0	0	0.016	0.258	0.016	0.129
b	a	a	0	0	0.016	0.258	0.016	0.129
c	c	c	0.062	1	0.062	1	0.124	1
c	c	a	0.062	1	0.062	1	0.124	1
c	c	b	0	0	0.031	0.5	0.031	0.25
c	a	a	0	0	0.031	0.5	0.031	0.25
c	b	a	0	0	0.031	0.5	0.031	0.25
c	b	b	0	0	0.031	0.5	0.031	0.25
d	d	a	0.016	0.258	0	0	0.016	0.129
d	d	b	0.016	0.258	0	0	0.016	0.129
d	d	c	0.016	0.258	0	0	0.016	0.129
d	a	a	0	0	0.016	0.258	0.016	0.129
d	b	a	0	0	0.016	0.258	0.016	0.129
d	b	b	0	0	0.016	0.258	0.016	0.129
d	c	a	0.031	0.5	0.031	0.5	0.062	0.5
d	c	b	0.031	0.5	0.031	0.5	0.062	0.5
d	c	c	0.031	0.5	0.031	0.5	0.062	0.5
e	c	a	0.031	0.5	0	0	0.031	0.25
e	c	b	0.031	0.5	0	0	0.031	0.25
e	c	c	0.031	0.5	0	0	0.031	0.25
e	d	a	0.016	0.258	0	0	0.016	0.129
e	d	b	0.016	0.258	0	0	0.016	0.129
e	d	c	0.016	0.258	0	0	0.016	0.129

(The outcomes of Table 1 are with accuracy up to the third decimal point).

We compare now the two groups' performance by applying the centroid method. For the first characteristic (knowledge of the subject matter) we have:

$$A_{11} = \{(a, 0),(b, 0),(c, 0.5),(d, 0.25),(e, 0.25),$$
$$A_{21}= \{(a, 0),(b, 0.25),(c, 0.5),(d , 0.25),(e, 0)\}$$

and respectively

$$x_{c11} = \frac{1}{2}(5 \text{ x } 0.5 + 7 \text{ x } 0.25 + 9 \text{ x } 0.25) = 3.25,$$
$$x_{c21} = \frac{1}{2}(3 \text{ x } 0.25 + 5 \text{ x } 0.5 + 7 \text{ x } 0.25) = 2.25$$

Thus, by our criterion the first group demonstrates better performance.

For the second characteristic (problem solving abilities) we have:

$$A_{12} = \{(a, 0),(b, 0),(c, 0.5),(d, 0.25),(e, 0)\},$$

$$A_{22} = \{(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e, 0)\}.$$

Normalizing the membership degrees in the first of the above fuzzy subsets of U

($0.5 : 0,.75 \approx 0.67$ and $0.25 : 0.75 \approx 0.33$) we get

$$A_{12} = \{(a, 0),(b, 0),(c, 0.67),(d, 0.33),(e, 0)\}.$$

Therefore

$x_{c12} = \frac{1}{2}(5 \text{ x } 0.67 + 7 \text{ x } 0.33) = 2.83$, $x_{c22} = \frac{1}{2}(0.25 + 3 \text{ x } 0.25 + 5 \text{ x } 0.25) = 1.125$

By our criterion, the first group again demonstrates a significantly better performance.

Finally, for the third characteristic (analogical reasoning) we have

$$A_{13} = A_{23} = \{(a, 0.25),(b, 0.25),(c, 0.25),(d, 0),(e, 0)\},$$

which obviously means that in this case the performances of both groups are identical.

Based on our calculations we can conclude that the first group demonstrated a significantly better performance concerning the knowledge of the subject mater and the problem solving, but performed identically with the second one concerning analogical reasoning.

4. Students' Individual Assessment

The outputs of our fuzzy model developed above can be used not only for assessing the performance of student groups', but also for the students' individual assessment. In fact, if $n=1$ (we recall that n denotes the number of students' of the group under study), then from the definition of the membership function m given in section 2 it becomes evident that in each A_i, i = 1, 2, 3, there exists a unique element x of U with membership degree 1, while all the others have membership degree 0. The centroid method is trivially applicable in this marginal case. For example, if

$A_{11} = \{(a, 0), (b, 0), (c,0), (d,1), (e,0)\}$ and $A_{21} = \{(a, 0), (b, 0), (c,1), (d,0), (e,0)\}$, then obviously the first student demonstrates a better performance with respect to the knowledge acquisition (characteristic S_1). This is crossed by the centroid method, since $x_{c\,11} = \frac{7}{2}$ and $x_{c\,21} = \frac{5}{2}$.

As a consequence of the above situation ($n=1$), there exists a unique student profile s with $m_s = 1$, while all the others have membership degree 0.

In other words, each student is characterized by a unique profile, which gives us the requested information about his/her performance. The students' profiles define a relationship of partial order among students' with respect to their total performance (see the below example).

A. Jones developed a fuzzy model to the field of Education involving several theoretical constructs related to assessment, amongst which is a technique for assessing the deviation of a student's knowledge with respect to the teacher's knowledge , which is taken as a reference (Jones et al. 1986, Espin & Oliveras 1997). Here we shall present this technique, properly adapted with respect to our fuzzy model.

Let $X= \{S_1, S_2, S_3\}$ be the set of the students' characteristics under assessment that we have considered in section 2. Then a fuzzy subset of X of the form

$\{(S_1, m(S_1)), (S_2, m(S_2)), (S_3, m(S_3)\}$can be assigned to each student , where the membership function m takes the values $0, 0.25, 0.5, 0.75, 1$ according to the level of the student's performance. The teacher's fuzzy measurement is always equal to 1, which means that the fuzzy subset of X corresponding to the teacher is

$\{(S_1, 1), (S_2, 1), (S_3, 1)\}$.

Then the *fuzzy deviation* of the student i with respect to the teacher is defined to be the fuzzy subset $D_i = \{(S_1, 1- m(S_1)), (S_2, 1-m(S_2)), (S_3,1-m(S_3)\}$ of X.

This assessment by reference to the teacher provides us with the ideal student as the one with nil deviation in all his/her components and it is actually equivalent with our fuzzy method. The following example illustrates all these in practice.

EXAMPLE: In assessing the students' individual performance of a class by applying the A. Jones technique we found the following types of deviations with respect to the teacher:

$D_1 = \{(S_1, 0.75), (S_2, 0.75), (S_3, 1)\}$, $D_2 = \{(S_1, 0.5), (S_2, 1), (S_3, 1)\}$,
$D_3 = \{(S_1, 0.5), (S_2, 0.75), (S_3, 1)\}$, $D_4 = \{(S_1, 0.5), (S_2, 0.75), (S_3, 0.75)\}$,
$D_5 = \{(S_1, 0.25), (S_2, 0.5), (S_3, 0.75)\}$, $D_6 = \{(S_1, 0.25), (S_2, 0.25), (S_3, 0.5)\}$
$D_7 = \{(S_1, 0), (S_2, 0.5), (S_3, 0.75)\}$, $D_8 = \{(S_1, 0), (S_2, 0.5), (S_3, 0.5)\}$,
$D_9 = \{(S_1, 0), (S_2, 0.25), (S_3, 0.5)\}$, $D_{10} = \{(S_1, 0), (S_2, 0.25), (S_3, 025)\}$
$D_{11} = \{(S_1, 0), (S_2, 0), (S_3, 0.25)\}$

On comparing the above types of students' deviations it becomes evident that the students possessing the type D_3 of deviation demonstrate a better performance than those possessing the type D_1, the students possessing the type D_4 demonstrate a better performance than those possessing the type D_3 and so on. However, the students possessing the type D_1 demonstrate a better performance with respect to problem solving than those possessing the type D_2, who demonstrate a better performance with respect to the knowledge acquisition. Similarly, the students possessing the type D_6 demonstrate a better performance with respect to problem solving and analogical reasoning than those possessing the type D_7, who demonstrate a better performance with respect to the knowledge acquisition. In other words, this type of assessment by reference to the teacher defines in general a relationship of partial order among students' with respect to their performance.

Notice that each deviation D_i corresponds to a student's profile s_i, $i = 1, 2, \ldots, 11$. For example, the deviation D_1 corresponds to the student $\{(A_1, 0.25), (A_2, 0.25), (A_3, 0)\}$, whose profile is $s_1 = (b, b, a)$. Applying the same argument one finally finds the following profiles characterizing the students' performance in our example:

$s_1 = (b, b, a)$, $s_2 = (c, a, a)$, $s_3 = (c, b, a)$, $s_4 = (c, b, b)$, $s_5 = (d, c, b)$, $s_6 = (d, d, c)$,
$s_7 = (e, c, b)$, $s_8 = (e, c, c)$, $s_9 = (e, d, c)$, $s_{10} = (e, d, d)$ *and* $s_{11} = (e, e, d)$

In other words, the A. Jones technique is actually equivalent to our method for the students' individual assessment. The only difference is that the former expresses the fuzzy data with numerical values, while the latter expresses it qualitatively in terms of the fuzzy linguistic labels of U.

Notice also that the teacher may put a target for his/her class and may establish didactic strategies in order to achieve it. For example he/she may ask for the deviation, say D, with respect to the teacher to be $0.25 \le D \le 0.5$, for all students and in all steps. In this case the application of the A. Jones technique could help the teacher to determine the divergences with respect to this target and hence to readapt his/her didactic plans in order to diminish these divergences.

5. Conclusions and Discussion

The following conclusions can be drawn from those presented in this paper:

- Fuzzy logic, due to its nature of including multiple values, offers a wider and richer field of resources for assessing the students' performance than the classical crisp characterization does by assigning a mark to each student.

- In this article we developed a fuzzy model for assessing student groups' knowledge and skills, in which the students' characteristics under assessment are represented as fuzzy subsets of a set of linguistic labels characterizing their performance.
- The coordinates of the centre of gravity of the graph of the membership function involved were used as a defuzzification method in converting our fuzzy outputs to a crisp number. In this way one can compare the performances of different student groups'.
- Techniques of assessing the students' individual performance were also discussed and examples were presented illustrating the use of our results in practice.

Our model is actually a proper adaptation of a more general fuzzy model developed in earlier papers to represent in an effective way a system's operation in situations characterized by a degree of vagueness and/or uncertainty. Our plans for future research include among the others the possible extension of this model for the description of more such situations in Education and in other human activities (human cognition, artificial intelligence, management, etc).

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