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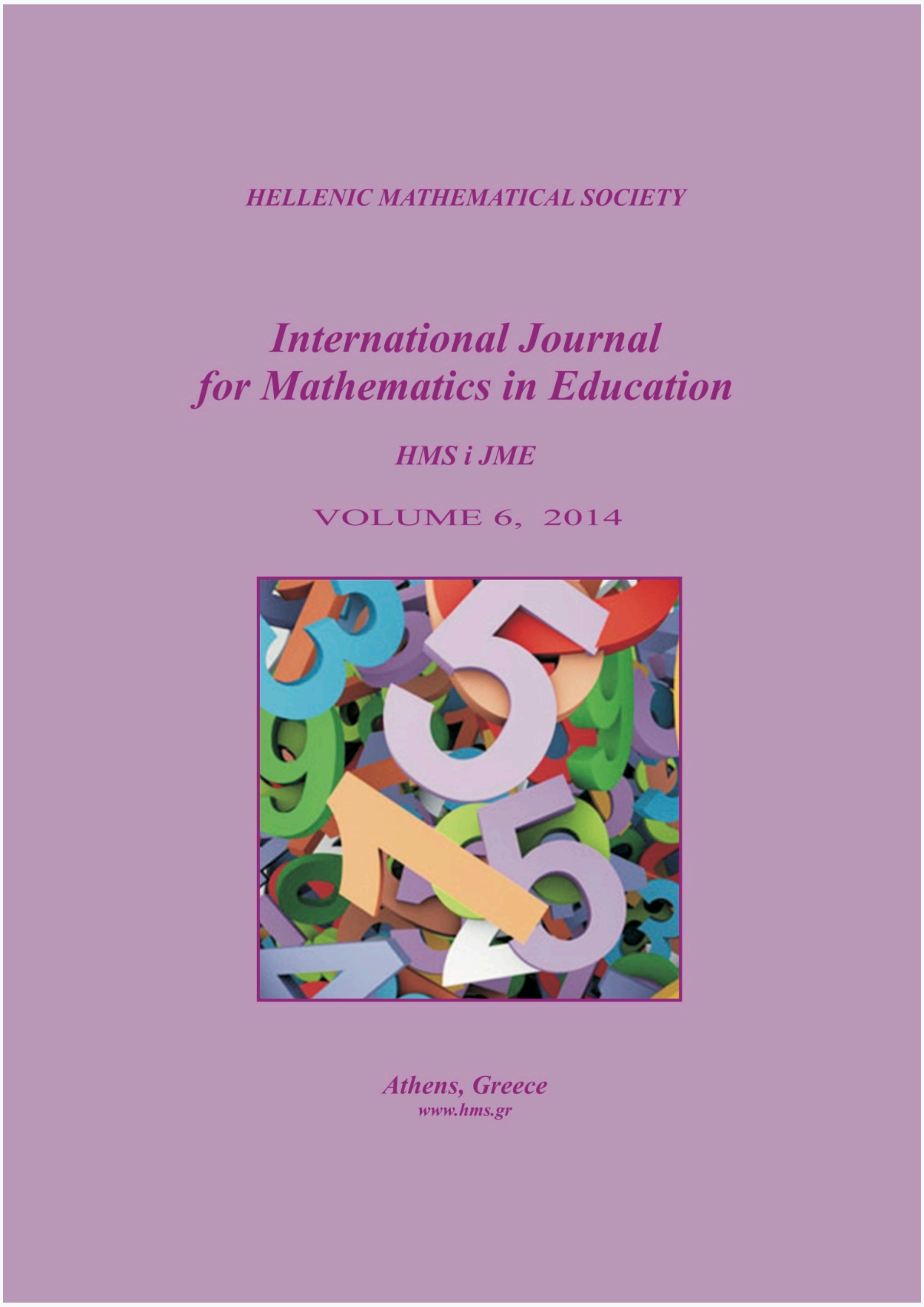

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GREEK MATHEMATICAL SOCIETY
34, Panepistimiou (Eleftheriou Venizelou) Street, GR. 106 79 - Athens - HELLAS
Tel. +30 210 3616532 - +30 210 3617784 - Fax: +30 210 3641025
e-mail : info@hms.gr - www.hms.gr

International Journal for Mathematics in Education (*HMS i JME*)

The Hellenic Mathematical Society (HMS) decided to add this Journal, the seventh one, in the quite long list of its publications, covering all aspects of the mathematical experience. The primary mission of the HMS International Journal for Mathematics Education (***HMS i JME***) is to provide a forum for communicating novel ideas and research results in all areas of Mathematics Education with reference to all educational levels.

The proposals must be written almost exclusively in English but may be admitted, if necessary, also in French, in German or perhaps in Spanish.

The proposals could concern: research in didactics of mathematics, reports of new developments in mathematics curricula, integration of new technologies into mathematics education, network environments and focused learning populations, description of innovative experimental teaching approaches illustrating new ideas of general interest, trends in teachers' education, design of mathematical activities and educational materials, research results and new approaches for the learning of mathematics.

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Address: *Hellenic Mathematical Society*
34, Panepistimiou (Eleftheriou Venizelou) Street
GR. 106 79 - Athens Greece e-mail : info@hms.gr
Tel. (0030210) 3616532 - (0030210) 3617784 - Fax: (0030210) 3641025

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NOTES FOR AUTHORS

Authors are asked to submit their manuscript in electronic form via e-mail to: **info@hms.gr; kalabas@rhodes.aegean.gr; kafoussi@aegean.gr,** with the indication **for hms i jme.**

The first page should contain the author's e-mail address and keywords. Papers should be written in English but also, if necessary, in French, in German or in Spanish.

The format of the manuscript: Manuscripts must be written on A4 white paper, double spaced, with wide margins (3 cm), max 20 pages, Times New Roman 12pt. Each paper should be accompanied by an abstract of 100 to 150 words. References cited within text (author's name, year of publication) should be listed in alphabetical order. References should follow the APA style (http//:www.apastyle.org/pubmanual.html).

The submitted papers are reviewed by two members of the International Scientific Committee (blind review). Two positive reviews are necessary for a paper to be accepted.

Authorship

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Hellenic Mathematical Society
34, Panepistimiou (Eleftheriou Venizelou) Street
GR. 106 79 - Athens - Greece
Tel. 3616532 - 3617784 - Fax: (0030210) 3641025
e-mail : info@hms.gr

Hellenic Mathematical Society
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Introduction

As the mathematical culture of every man and every culture is constructed and organized differently in the various seasons and regions of the globe, in the same diversified way are being built the directions of mathematical education and research on the particular circumstances of each country.

Nowadays this organization is dynamically emerging from the interaction among the internationalized processes and the local processes. On the one hand the comparative studies, the international medals or the controversial private financial supports and in the other hand individual acts are conducing to universal mathematical discoveries and local learning activities to teaching innovations. The digital bridge of the network creates a new topos for the mathematical culture and for the mathematical education.

In this area, between globalised standards for the mathematics education and the historicity of local mathematical communities and educational structures, the dialogue is open and intense while the teaching practices are now developed within the influences of the growing and antagonistic market of parallel educational services and products.

Our journal tries to contribute in the persistence of the variety, in the coexistance and the exchange of different ideas about mathematics and their education.

We believe that the improvement of mathematics teaching and learning in the entire world cannot be realized in a homogenous way. We have to organize and explore in a different mode the international comparative studies, profiting of the alternative views, the ecosystemic educational research, the digital and networking environment as well as the ethnomathematical approach.

The International Journal for Mathematics in Education (HMS i JME) wishes to contribute in this orientation and are inviting our colleagues of the international community to send us their contributions.

The sixth volume of the HMS International Journal for Mathematics in Education includes six research papers, four of them are respectively corresponded to four plenary speeches realized during the 31st Hellenic Conference of Mathematics Education on November 2014 titled “Challenges and Perspectives of Mathematical Research and Education in the Internationalized and Network Era” and two research papers.

The article of Ferdinando Arzarello titled “From Socrates to Sherlock Holmes: cultural, cognitive and didactical tools for pursuing the logic of inquiry in the classroom” concerns the introduction of activities using new technologies and if/how these relate with proving activities. Every activity in the classroom is shaped by cultural, epistemological, and cognitive analysis and the results one gets strongly depend on a combination of them, which vary with circumstances. He discussed such issues and illustrated them introducing some concrete examples from teaching experiences.

The article of Jean Dhombres titled “Réflexions mathématico-historiques à l’âge du Net sur les réformes dans l’enseignement et les réactions à ces réformes” makes an overview on several mathematics textbooks concerning the evolution of history of mathematics and its relationship with mathematics education.

The contribution of Karl Gustafson titled “The Future of Mathematics: From the Pure-Applied Debate to Reality” looks at Mathematics from Past to Present to Future, using case-studies approach, selecting key examples to illustrate and from which to learn.

The contribution of Marianna Tzekaki titled “Mathematics Education for a new era” aims to investigate the following questions: In what society do we estimate that today's students will live, what will they face in the future and what disposition will they need to meet the requirements of the new era? Can mathematics education play a role in the development of this disposition and what goals need to be set, what content and what teaching approaches to meet these requirements? Isn’t it critical for researchers, programs’ designers and teachers to pursue systematic answers to these questions and bring in new advances to the teaching of Mathematics?

Sonia Kafoussi, in her article “Can preschool children collaborate in mathematical tasks productively?”, investigates how pairs of preschool children collaborated productively during their efforts to solve mathematical tasks.

Finally, Michael Voskoglou, in his article “The APOS/ACE Instructional Treatment for Mathematics: A Fuzzy Approach”, introduces principles of fuzzy logic on comparing the performance of two student groups concerning the comprehension of the real numbers in general and of the irrational numbers in particular.

Hellenic Mathematical Society
International Journal for Mathematics in Education

From Socrates to Sherlock Holmes: cultural, cognitive and didactical tools for pursuing the logic of inquiry in the classroom

Ferdinando Arzarello
Dipartimento di Matematica
Università di Torino-Italia
ferdinando.arzarello@unito.it

Introduction

For the second time in the last sixty years, some historical and cultural circumstances offer the opportunity to develop a global perspective of reform of mathematics education all over the world.

Fifty years ago, the necessity of meeting the challenge of the Sputnik (a strong historical and political motivation), the crucial role taken by mathematics in science and technology, and the extraordinary cultural enterprise of Bourbaki and its encounter with a leading personality in psychology, like Piaget, found in the OCSE-OCDE context the appropriate environment to launch the New Mathematics reform as the main road to increase mathematical (and reasoning) skills of the population of leading western countries. In the reality, the New Mathematics movement spread not only in the OCSE-OCDE countries (with different influences on national curricula and practical implementations in school), but also in several African, Asiatic and South American countries.

In recent years, globalization of economy, universality of technological development and related needs for manpower skills play the role of strong historical motivations for a reform that should bring to unified standards for mathematics in school not only in the 34 OCSE-OCDE countries, but in several other countries as well (keeping into account the political and cultural leading role of OCSE-OCDE countries).

As a consequence, in the international debate, many scholars, teachers and policy makers speak now of the *21st century competencies* and consider important items like: “critical and inventive thinking; communication, collaboration and information skills; and civic literacy, global awareness and cross-cultural skills." (Ministry of Education, Academy of Singapore Teachers, 2014; see http://www.academyofsingaporeteachers.moe.gov.sg/professional-networks).

In many countries people are working on the so called "21st century competencies framework", in order to guide the development of the national curriculum and to design school-based programmes to nurture these competencies.

Like 55 years ago (Royamont meeting, the launch of the New Mathematics reform), OCSE-OCDE can play again a leading role in this new "reform movement", thanks to a very important cultural circumstance: the promotion of the PISA initiative, already so influential in many countries as concerns the change of local curricula and standards in order to meet the challenge of behaving well in that comparative assessment. Indeed PISA is not a neutral test to assess the 15-years olds students' competencies; gradually it becomes a reference for a universal reform of the teaching of mathematics, thanks to a more and more coherent, systematic and explicit framing of PISA test in a discourse on mathematical competences needed in today globalized society.

PISA stresses the role of *mathematical literacy* as a central goal in mathematics education, because it improves the life chances of most students, and justifies why mathematics is essential to describe, explain and predict the world. According to PISA, "mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens."
(see: http://www.oecd-ilibrary.org/education/pisa-2012-assessment-and-analytical-framework_9789264190511-en). In their last publication (PISA 2012 Results:

Creative Problem Solving Students' skills in tackling real-life problems, Volume V) they elaborate further this concept and speak of an "innovative concept of "literacy", which refers to students' capacity to apply knowledge and skills in key subjects, and to analyse, reason and communicate effectively as they identify, interpret and solve problems in a variety of situations;" (p.20).

However, the failure of the New Mathematics reform suggests reflecting on the requirements for a new curriculum, suitable to escape the causes of the complete, or partial, rejection of the New Mathematics reform in so many countries.

In particular, a reform movement should take into account:
I) the existence of different epistemological and cultural positions concerning mathematics and its relevance in the culture (the New Mathematics reform was refused by several outstanding mathematicians in the world for incompatibility of epistemological and cultural positions);
II) the possible, cultural distance of the proposed reform from the mathematical culture of the different countries (as an example of the importance of this issue, we can consider how in Italy the leading position of Dieudonné on geometry, so relevant in the design of the "new geometry", was in contrast not only with the positions of many mathematicians, but also with a widespread tradition in school of relevance for figural aspects, intuitive visual geometry, synthetic methods; another example concerns the difficulties met by the Modern Mathematics movement in the UK, due to the different cultural orientation of the teaching of mathematics in that country);
III) the relationships with the culture and the personal contributions brought by the students in the classroom, so relevant to avoid the students' alienation from their cultural environment and to allow students to engage in learning in a productive way.

A consequence of these reflections is that the communities of researchers, teachers and policy-makers must be aware that defining mathematical literacy raises a lot of issues, both from a scientific and from a cultural point of view. The word literacy itself is slippery not only linguistically but also epistemologically and politically: as someone pointed out (Jablonka, 2003, p. 77), trying to translate it into different languages is always a difficult, sometimes an impossible task; and also in the literature one finds more than one definition and many changes during the years. This is clearly shown in the review paper about this topic written by Mogens Niss and Eva Jablonka in the *Encyclopedy of Mathematical Education*. For example, it is there argued that ML is "*a socially and culturally embedded practice, and …[its] conceptions … vary with respect to the culture and values of the stakeholders who promote it*". Moreover in that paper it is summarized a review of literature made by Jablonka (2003, pp. 75-102), which "*identifies five agendas on which conceptions of mathematical literacy are based. These are: developing human capital (exemplified by the conception used in the OECD-PISA), maintaining cultural identity, pursuing social change, creating environmental awareness, and evaluating mathematical applications*"

(Niss & Jablonka, *ibid.*): the differences in approach are directly linked with the goals that are pursued in mathematics education in individual countries.

Also the PISA definition of ML has changed in time and the current one is the most known because of its important testing studies; but no one can say that tomorrow it will not change again (e.g. see the fresh elaboration pointed out above). And in many national frameworks there are different formulations, according to their cultural traditions. Moreover, if one consider the concrete items of PISA, this wide definition is somewhat reduced, as it is natural when one has to constrain it into items to which the students (at the age of 15 years) are asked to answer in a few minutes. A typical example is the notion of mathematical proof, which was practically absent up to 2009[1] and starts appearing in the last items.

It is clear that today (much more than sixty years ago!) a partial unification of standards for mathematics teaching all over the world is necessary, in order to enable citizens to acquire a common mathematical toolkit to deal with technology, quantitative and graphical information provided by media, problem solving and decision making in the workplace and in ordinary life. Also keeping the worldwide experience of the New Mathematics reform into account, the crucial issues to be dealt with in order to avoid the dangers of cultural refusal of the reform, and of cultural alienation and of loss of cultural richness if it not refused, are:

- the cultural and epistemological openness of the reform movement;
- the space left to cultural differences in the elaboration of the national standards and in the related educational aims;
- the space ensured to the dialogue with the students' cultural experience and personal contribution in the teaching and learning of mathematics;
- the development of teachers' and students' awareness about the multiplicity of the mathematical experience (from mathematical games to modelling of social and natural phenomena – the last PISA documents define also a *financial literacy*; from the construction of new algorithms, to the discovery of new properties).

For all these reasons it is important that teachers have clear such a landscape and keep a critical approach in designing their programs and tasks for the students. A typical touchstone for this issue is the introduction of activities using new technologies and if/how these relate with proving activities.

[1] Organisation for Economic Cooperation and Development, (1999). *Measuring Student Knowledge and Skills: A new Framework for Assessment*. Paris: OECD.

Here it is easy losing one's way and going behind fresh proposals only because these are or seem new, without submitting them to "the court of reason", namely forgetting to check their validity with a careful *cultural*, *epistemological,* and *cognitive* analysis. In fact, every activity in the classroom is shaped by these three components and the results one gets strongly depend on a combination of them, which vary with circumstances. For example it is easy to find statements, which speak of the death of proof because of the introduction of computers in mathematical practice, hence pushing towards its elimination from mathematical activities.

I wish to discuss exactly such issues, but instead of presenting only theoretical reflections I prefer to illustrate them introducing some concrete examples from teaching experiences I coached in my country. In these the three components have been carefully taken in consideration both in the process of designing the tasks and during the development of the teaching activities in the classrooms.

My aim is to contribute in this way to the debate about the theme of the 2014 Hellenic Mathematical Society Congress "Challenges and Perspectives of Mathematical Research and Education in the internationalized and Network Era".

Tools and proving activities in mathematics

As recalled above, in the PISA definition of mathematical literacy we find the capability of "using […] tools to describe, explain, and predict phenomena". In fact many national curricula at all grades suggest involving students in the use of (concrete or virtual) tools to model phenomena and to enter into mathematical ideas. This is not a novelty at all: the links between mathematics, natural sciences and technology, as well as the role of basing mathematics teaching on intuitive and empirical stances are in the foreground from the early documents of the International Commission on Mathematical Instruction (Bartolini Bussi et al., 2010; Ruthven, 2008; see also Smith, 1913). This represented and still represents a foundational theme for ICMI: for example, the role of new technologies in mathematics education have been the focus of two ICMI Studies within the last 30 years and the International Community of Teachers of Mathematical Modeling and Applications (ICTMA, http://www.ictma.net/) has been an ICMI affiliated study group since 2003 (see http://www.icmihistory.unito.it/ictma.php#9).

"In mathematics education, the availability of Information and Communication Technologies (ICT) has changed the landscape, including the belief that digital objects can substitute for the references to the concrete world where we live" (Bartolini Bussi et al., 2010, p. 20; see also the website http://nlvm.usu.edu/en/nav/vLibrary.html). However, these changes in the landscape do not mean that we have to throw away all the past: we should risk throwing the baby out with the bathwater. In other words, modelling and applications can be pursued within "an approach that does not neglect, but rather emphasizes, the cultural aspects of mathematics, going back to the prominent founders of modern mathematics and taking advantages of the ICT support" (*ibid*.). This program is widely present in many researches all over the world (for a summary see: Bartolini et al., 2010). My claim is that in order to design suitable learning situations in the classroom, where manipulative materials and instruments can be used to support learning, it is necessary carefully investigating the cultural, epistemological, and cognitive roots of mathematical concepts (Tall, 1989; Boero & Guala, 2008). This investigation will clarify how manipulative materials, instruments and ICT, suitably combined together in real and virtual environments, can help students to grasp those concepts, basing learning on what today, grounding on fresh research results (see Hall & Nemirovski, 2012), is called an embodied approach to mathematics learning. This was also present in the old documents of early ICMI (see: Ruthven, 2008) and in other coeval researches (e.g. in Enriques, 1906-1914), generally supported more by pure speculations than by scientific research or empirical evidence.

Using instruments in mathematics classes immediately poses a problem: how do they link with the rigorous formal aspects of the discipline, in particular with the teaching of proof? Answering this question in a proper way is crucial in order to avoid misunderstandings, some of which are diffuse in several research articles: they are at the origin of a sort of comedy of errors about proof, which I think absolutely necessary to avoid.

In these last twenty years there have been heated discussions in mathematics and mathematics education that have called into question the role of proof. In the nineties, a number of recent developments in mathematical practice, most of them reflecting in some way the growing use of computers, caused some mathematicians and others to call into question the continuing importance of proof or indeed to announce its imminent death (Horgan, 1993). One of the developments that prompted Horgan's announcement was the use of computers to create or validate enormously long proofs, such as that of the four-colour theorem by Appel and Haken, or of the solution to the party problem by Radziszowski and McKay.

These speculatiions had strong consequences on some curricula: in fact thay caused a serious turn away from proof:

> "Over the past thirty years or so proof has been relegated to a less prominent role in the secondary mathematics curriculum in North America.
> This has come about in part because many mathematics educators have been influenced by certain developments in mathematics and in mathematics education to believe that proof is no longer central to mathematical theory and practice and that its use in the classroom will not promote learning in any case.
> As a result many appear to have sought relief from the effort of teaching proof by avoiding it altogether.
> …
> The influence of these developments in mathematics has been strongly reinforced by the claims of some mathematics educators, inspired in part by the work of Lakatos, that deductive proof is not central to mathematical discovery, that mathematics is fallible in any case, and that proof is an authoritarian affront to modern social values and even hinders learning among certain cultural groups." (G. Hanna, 1996)

For example, in 1989 NCTM Standards proof is explicitly de-emphasized. They cite the difficulty of teaching and learning to do proofs, underlining that that:

- the amount of time proof takes up, which is out of proportion to its benefit;
- the fact that proof is really unnecessary for the majority of students, including many of those who are college-bound;
- proof's tendency to convey a picture of a static subject in which the students simply re-hash geometrical facts which have been known for 2500 years.

By the time the Standards was published (1989) by the National Council of Teachers of Mathematics (NCTM) in the United States, the concept of proof had almost disappeared from the curriculum (Greeno, 1994) or shrunk to a meaningless ritual (Wu, 1996).

This de-emphasis on proof in the 1989 version of the Standards created a tension within the document. The 2000 version of the Standards ameliorated this tension: moving away from the standard idea of proof, as a purely formal object they made explicit fresh functions of proof that should feature its teaching in the school, and stated goals of treating and teaching mathematics "as reasoning" and "as communication."

This revised outlook on proof is, no doubt, a response to a great deal of literature generated in the decade after the publication of the first Standards document. This body of work sprang up partly in explicit defence of proof (e.g. Epp, 1994; Greeno, 1994; Hanna, 2000), and partly to support and fill in the NCTM's overall picture of what mathematics education should be. In light of that research, the NCTM was able to revise and hone its aims. But even the 2000 Standards are rather vague about just what the value of proof is.

Assurance of truth is only one of proof's roles in mathematics, in the classroom or in professional practice. We should note that presenting justification as the sole reason to do proof has a few weaknesses, and some argue that because of these weaknesses, there is no need to teach proof in schools. For example: as it turns out, most students are not as convinced by a rigorous proof as they are by a number of examples or "empirical" evidence.

They may be happy to take a teacher's—or Euclid's—word for the truth of the Pythagorean theorem or by their own investigation of examples.

Another weakness is that if students are presented with a picture of proof which makes it seem as though proof's value lies in confirming facts, they may easily be turned off.

Proving things we already know to be true, they are likely to see no point in the exercise of proving.

The reply to this objection is twofold; there are so to say, ethical arguments. First, of course it doesn't matter that they're convinced by empirical arguments; they shouldn't be, and part of the point of education is to teach them not to be. They should value rationality over authority. Second, students can be led astray from intuition and perceived patterns. But there are also more concrete arguments: proofs can become the essential part of mathematical activities in the classroom, provided they become an integrated process in the process of discovering mathematical concepts and truths: a classroom climate can be created so that the students themselves bocome the mathematical authority. They can argue about problems and solutions, bringing reasons to bear on the problem, and accepting a proof only when they themselves are convinced by it. In a word, teachers can create a classroom climate, according to which students enters into what I call the *logic of inquiry*, which will be made explicit below.

When proof is introduced in such a fashion, its value in the classroom is quite apparent: students learn to rely on arguments and reasoning rather than authority, they make use of their factual knowledge, and they come to a deeper understanding of the way mathematical facts are related. Experimental and theoretical features will not be seen any longer as contrasting but as complementing components of processes that coach students to investigate, conjecture, and prove.

The manifesto of this approach is featured by the following passwords:

- Finding **if**
- Establishing **that**
- Ascertaining **why**
- Settling why **not**
- Investigating **what if**

They can guide the following steps in students' processes:

- if you can **prove**,
- then you can **explain**,
- which means you have active **metacognitive** processes,
- so you can **solve** problems,
- and therefore you **understand** mathematics.

These steps can be enormously facilitated by the use of tools and particularly of ICT: in fact old and new technologies − from the old rule and compass to nowadays i-phones − allow a crucial mediating role for the introduction of mathematical concepts and for triggering and supporting the steps of the logic of inquiry listed above.

Using instruments introduces an "experimental" dimension into mathematics:

> "Experimental mathematics is the use of a computer to run computations — sometimes no more than trial-and-error tests— to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search. Like contemporary chemists —and before them the alchemists of old— who mix various substances together in a crucible and heat them to a high temperature to see what happens, today's experimental mathematicians put a hopefully potent mix of numbers, formulas, and algorithms into a computer in the hope that something of interest emerges." (Borwein & Devlin, 2008, p. 1)

For an amazing example see also the discussion in the Devlin's angle:
http://www.maa.org/devlin/.

It is so generated a productive dynamic tension between the *empirical nature* of activities with instruments, which encompasses perceptual and operational components and the *deductive nature* of mathematics, which entails a rigorous and sophisticated formalization. From the one side, there is a strong historical and epistemological ground for such an approach: in fact, from straight-edge and compass to a variety of computational and drawing

tools, throughout history instruments have been deeply intertwined with the genesis and development of abstract concepts and ideas in mathematics (Ruthven, 2008). From the other side, the current great diffusion of ICT in all aspects of everyday life pushes towards a massive use of such tools in the school. The learning landscape is deeply and rapidly changing because of them: sometimes it seems that digital objects can substitute for the references to the concrete world where we live, which in fact is a questionable and delicate issue. I will show the pedagogical possibilities offered by the tension between these two aspects with some emblematic examples taken from teaching experiments I coached in Italy.
The first two show how a concrete instrument can help to digging into deep mathematics ideas.

The second part of the talk will show examples where ICT are used to face elementary geometry problems in the first years of secondary school: specifically I will show two different uses of Dynamical Geometry Software (DGS) in the classroom, depending on the available technology in the school. In fact, the evolution of technology makes possible different practices, specifically related to the way users can interact with the screen: from the *drag* and *drop* actions with the mouse to the *tap, drag,* and *flick* with one or more fingers on the screen of multi-touch devices and from the one-to-one interactions of the former to the multiple simultaneous interactions that the latter makes possible (Park *et al.,* 2011). These different technological features allow designing different tasks, which can change the cognitive processes of students and deeply modify their mathematical inquiries. The main result of such changes consists in a different and possibly better approach to proving activities in the classroom. Roughly speaking, as we have had a first shift and improving passing from paper and pencil environments to DGS with drag and drop activities (e.g. Cabri géomètre, Sketchpad, etc.: Arzarello et al., 2012), now we have a further shift and improvement with the transition to multi-touch environments (e.g. Geometric Constructor, SketchPad Explorer, Sketchometry, etc.: Arzarello *et al*., 2014) and to the variety of simultaneous fingers' actions they allow. In the talk I will underline analogies and differences between the two, discussing the advantages that each of them make possible.

Digging into the concept of space with instruments

The geometry of the Greeks was essentially a science of figures; with Riemann it became a "science of space". Poincaré went even further; he showed that it is the movement to create the concept of space:

> “un être immobile n'aurait jamais pu acquérir la notion d'espace puisque, ne pouvant corriger par ses mouvements les effets des changements des objets extérieurs, il n'aurait eu aucune raison de les distinguer des changements d'état » [Poincaré, 1902, p. 78] ... « localiser un objet en un point quelconque signifie se représenter le mouvement (c'est-à-dire les sensations musculaires qui les accompagnent et qui n'ont aucun caractère géométrique) qu'il faut faire pour l'atteindre.” (Poincaré, 1905, p. 67).

For Poincaré, it is the presence of the body, especially our body, and of movements, our movements, to generate the notion of space. For Poincaré, as for Riemann and unlike Kant, there is no a priori geometric theory of the world. Instead, this is built from the material world, even though our « sensations musculaires ... n'ont pas de nature géométrique ».

Today, advances in mathematics and logic, on the one hand, and in neuroscience and cognitive science, on the other hand, allow us to deal with the problem of the relationship between the geometry and the material world with more accuracy and to understand why certain choices are meaningful.

As a result, the ideas of Poincaré, but also of others, such as F. Enriques, H. Weyl, J. Piaget have a scientific basis more actual than ever. This issue is proved by several studies: for example, from researches conducted in recent years by the group *Gèomètrie et Cognition* at the *École Normale Supérieure* in Paris, coordinated by G. Longo, J.L. Petitot and B. Teissier. They illustrate the possibility and nature of approaching geometry (and mathematics in general) according to a genetic stance. For example, studies of A. Berthoz, a distinguished physiologist at the *College de France*, who actively contributes to the group, highlight that when one catches a ball, she/he realizes the multi-sensory integration of her/his different reference systems (Berthoz, 1997, p.90), which can "simulate" the space of perception. What we call the position, velocity and acceleration of the ball is represented in the various systems of representation of the retina to the arm muscles. This is where comes from our "geometric intelligence" as human beings. It is built as a network of encodings and/or of analogic representations, which are obtained through the practices of our actions in the world. It is the invariance of these representations and encodings to generate the invariance of our conscious representations, such as those of language (Gallese and Lakoff, 2005), and finally the space of the most invariant representations: those of mathematics.

It is important to consider these studies in order to designe suitable learning trajectories for geometry. In fact, its epistemological foundation reveals its deep cognitive roots (D. Tall, 1989): this is what H. Weyl called sufficient conditions for the emergence of a theory, namely those conditions which require exactly this theory and make it possible. For this reason, the geometry must be addressed in the same context according to which we act in the world: indeed, the objectivity of geometric conceptualization derives from its own constituent processes.

It is therefore necessary to develop a teaching method based both on the epistemological basis of the discipline, as well as on the cognitive aspects of its learning.

In fact, we can distinguish two modes of learning (Antinucci, 2001): the symbolic-reconstructive and the sensory-motor way.

In a nutshell, the symbolic reconstructive way:

- is based primarily on the interpretation and exchange of symbols (language, mathematics, logic);
- reconstructes in the mind the objects and their meaning through mental representations from the symbols themselves;
- is the most sophisticated and evolved way, through which we learn;
- its work takes place entirely in the mind and exchanges with the outside world are mediated by linguistic symbols;
- is conscious and very tiring.
- The sensory-motor way instead:
- takes place in a continuous exchange of perceptual inputs and motor outputs with the outside;
- often occurs at an unconscious level, so it is a very less tiring work.

The knowledge that comes from the symbolic reconstruction is always and only verbally expressible and does not occur spontaneously. What comes from the perceptual-motor way tends to be internalized and contextualized in a spontaneous manner. Thus, the human beings take it, whenever it is possible.

The sensory-motor approach must be considered when designing teaching situations: in fact the students, when exposed to such situations, can spontaneously develope ideas, making sense of them, basing exactly on this approach. This means that we need to introduce students to the cognitive and cultural roots of concepts (Tall, 1989; Guala & Boero, 2008) in an appropriate manner. It is the responsibility of the teacher to push this personal feelings, spontaneously produced by the students, towards the scientific meaning of concepts, supporting them towards the symbolic reconstructive path. To get this aim, appropriate tools and materials can be used.

It is interesting noticing that traditional mathematics instruction tends to be transmissive and based almost exclusively on a symbolic reconstructive method. On the contrary, the didactic use of various technological tools (not just the computer), internet, etc.. tends to produce a perceptual and motor learning, opposite to what happens using just books.

The slogan of this teaching method that inspires my presentation is the following quotation from Simon Papert (1980): "We learn best by doing, we learn even better, provided we connect our doing to a discussion and a reflection on what we did".

The "psychological genesis" of geometric concepts (and mathematical ones in general) is a problem that can not be avoided in the school. A careful selection of experiences, from which we can start our interventions, is essential. They must be consistent with the concepts to teach both from a cognitive and a cultural point of view. Any educational project for geometry therefore requires a substantial critical analysis of its fundamental concepts.
I will show what I mean with an example, which is fundamental for geometry: the notion of a straight line. In Euclid's Elements (Def. 4) we find the following definition:

Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται

[A straight line is "a line that lies evenly with the points on itself", translation of Heath, 1956].

Texts in modern elementary geometry (e.g. Hilbert, 1899), as we know, do not give any explicit definition of straight line, since its meaning is conveyed implicitly by the axioms: they distill its intuitive sense in a formal system (which is typical of a symbolic reconstruction). Often this seems meaningless to students. Also the modern approach to geometry using linear algebra gives similar results (Dorier, 1997).

What experiences can really make sense and be understandable for our students?

Euclid seems to refer to the concept of symmetry. Experiences related to this idea can consist in folding the paper: whatever way you fold a sheet of paper, you always get a (part of) a straight line. You can either use a "visual" approach, following the idea of Enriques that projective geometry is linked to visual sensations. Another approach is to ask how to draw (a part of) a straight line: a ruler is fine, but the question arises whether the ruler is "right" or "wrong". Here, you can use a "mechanical" control (following Lobachevsky): we make two identical copies of the ruler and put each of them exactly on the top of another in all possible pairs: if all the three pairs fit, it is sure that we have a right ruler.

Basically, if one makes a critical analysis of the concept of straightright line, she or he finds the following cognitive and cultural roots:

a) symmetry;

b) walking straight on;

c) the shortest line.

We note that all three aspects are useful when one feels immersed in an unknown space and tries to understand how she/he can produce a straight line. In fact, the three aspects can produce a perceptual motor learning.

The idea is not new: Enriques (1906, § 11) points out that, in order to introduce the curvature of a surface "Gauss put forward a suggestive argument, which was subsequently taken over by Helmholtz and Clifford, and usually goes under the name of the first of these two philosophers. Imagine the existence of small animals on a surface, which are free to move by crawling on it. We equip these imaginary beings of spatial intuition, which can help them to direct their movements in the surface forming their own space. Two similar animals, one of which moves in a plane, the other over an area slightly curved, could also be driven by one and the same geometric intuition, namely imagining their own space like a plan".

Translated in another way, if I imagine to be a small animal, how can I imagine to produce a straight line? Walking stright on (idea b). What does that mean? I could actually be on a curved surface, and have no perception of it. Then I have to move my feet ideally drawing a line where my feet are arranged symmetrically with respect to this (idea a), and I have not to curve (idea b), nor making my way longer (idea c). Using the language of D. Tall (1989), it is the cognitive root of the concept of geodesic. However the root is not only cognitive.

There are several tools that have been historically used to generate straight lines: ropes stretched by the “Arpenodaptes” (= ropes stretchers) of ancient Egypt; the articulated mechanisms of Watt and of Peaucellier; folding sheets of paper. These are not only sensory-motor activities: the intertwining with the symbolic and reconstructive component is experienced and constantly stimulated by them. And this activity entails not only a cognitive behaviour, consistent with our biological being; it entails also cultural aspects of us as social beings. Indeed, the practices mentioned above have a cultural significance that the historical-critical analysis reveals (Radford, 2003; Guala & Boero, 2008).

In researches pursued in these last years in Italy my research team has designed teaching experiments for an approach to geometry in secondary school based on these pedagogical and epistemological principles. I will present two of them.

The first concerns the introduction of the notion of geodesic as a basic concept of geometry introducing it from a variety of perspectives (cognitive, epistemological and pedagogical) and within different rich geometric environments: sphere, cone, cylinder, plan and finally (a little less simple) pseudo-sphere and using an instrument, the so called South-seeking chariot (fig. 1), coming from the Chinese culture (Santander, 1992), which embodies important theorems of geometry (essentially the Gauss-Bonnet theorem) through the use of a differential gear, a mechanical device crucial for allowing cars to steer without going outside the roadway and today massively present in each SUV. I will show how it can be used in the last years of secondary school to help students to think about the geometry of the universe and the notion of curvature of a surface.

Figure 1. The South-seeking chariot (指南车, Zhǐ nán jū)

The second concerns an approach to the notion of the area of a surface as a "swept area": the project is developed from the ideas of Kepler about the nature of planets rotation around the sun (first and second law) and arrives even to calculate the area of irregular surfaces. Another key instrument will be used, the *planimeter* (fig. 2), to approach the notion of area in the school: it embodies the Guass divergence theorem in dimension 2, and shows the possibility of using old technology intertwined with ICT to make students enter again into important mathematical ideas (Arzarello & Manzone, 2013).

Figure 2. The Amsler planimeter

In all these cases, the use of appropriate materials and tools can help students making the transition from intuitive concepts to their more formal aspects.

Instruments such as those presented here, which force us to question what it means to "go straight" in a context different from the ordinary Euclidean plane, or to deal with the notion of area in a manner different from the usual one, can push students to consider some classically “immutable” truths from different and unusual points of view. This approach can engendre a critical attitude in students and encourage them to bring all truths in front of the “reason’s tribunal”, so following a well known suggestion by E. Kant (2008).

From drag and drop with the mouse to finger manipulations on multi touch devices: how ICT practices can foster mathematical inquires

In the second part of the talk I will sketch how the dynamic tension empirical-deductive regulates the actions of students who are asked to solve geometrical problems using Dynamic Geometry Software (DGS) to make explorations, formulate conjectures, and prove them.

I will do that presenting some short video clips from the classroom life, where the use of new technologies in proving activities makes the dynamic tension palpable. A careful analysis of students’ procedures while using DGS will allow me to introduce and discuss some theoretical frameworks that explain how that tension can be used to design suitable didactical situations. Within these, students can learn and internalize specific practices, which become *psychological tools* (Vygotsky, 1978, p. 52 ff; Kozulin, 1998) for supporting their transition from the empirical to the theoretical side of mathematics. Such processes are strongly marked by the complex interactions between inductive, abductive and deductive modalities in their productions.

Specifically, I will illustrate how the evolution of technology made available different practices, specifically related to the way users can interact with the screen: from the drag and drop actions with the mouse to the tap, drag, and flick with one or more fingers on the screen of multi-touch devices and from the one-to-one interactions of the former to the multiple simultaneous interactions that the latter makes possible (Park et al., 2011). These different technological features allow designing different tasks, which can change the cognitive processes of students and deeply modify their mathematical inquiries. The main result of such changes consists in a different and possibly better approach to proving activities in the classroom. Roughly speaking, as we have had a first shift and improving passing from paper and pencil environments to DGS with drag and drop activities (e.g. Cabri géomètre, Sketchpad, etc.), now we have a further shift and improvement with the transition to multi-touch environments (e.g. Geometric Constructor, SketchPad Explorer, etc.) and to the variety of simultaneous fingers' actions they allow (Arzarello et al., 2014a). In the talk I will underline analogies and differences between the two, discussing the advantages that each of them make possible.

As it is well known, DGS makes available many geometric constructions using the buttons of the toolbar; but what makes DGS so interesting compared to the classic world of paper and pencil figures, is not only the construction facility but also the direct manipulation of its figures, conceived in terms of the embedded logic system of Euclidean geometry (Laborde & Straesser, 1990; Straesser 2001), namely the relational aspects between the built figures. DGS figures possess an intrinsic logic, as a result of their construction, placing the elements of a figure in a hierarchy of relationships that corresponds to the procedure of its construction according to the chosen tools and in a corresponding hierarchy of properties.

This relationship is made evident in the "dragging" mode: it preserves the intrinsic logic of the DGS figures, that is the logic of their construction. The DGS figure is the complex of these elements, incorporating various relationships, which can be differently referred to the definitions and theorems of geometry. The presence of the dragging mode introduces in the DGS environment a specific criterion of validation for the solution of the construction problems: a solution is valid if and only if the figure on the screen is stable under the dragging test. Thus, solving construction problems in DGS means not only accepting all the facilities of the software but also accepting a logic system within which to make sense of them.

The DGS's intrinsic relation to Euclidean geometry makes it possible to interpret the control 'by dragging' as corresponding to theoretical control 'by proof and definition' within the system of Euclidean Geometry. In other words, there is a correspondence between the world of DGS constructions and the theoretical world of Euclidean Geometry.

The analysis of dragging modalities allows entering into cognitive processes of students while solving geometric open problems (Arzarello et al., 1998; Antonini, & Mariotti, 2009; Baccaglini-Frank, 2010; Arzarello & Sabena, 2011). In the talk I will illustrate how suitably designed tasks can help students to face and possibly to overcome the obstacles between their empirical mathematical tasks and the discipline's theoretical nature. When integrated in the teaching of proofs, DGS triggers a network of interactive activities among different components that can be categorised at two different epistemological levels:

(1) The convincing linguistic logical arguments that explain WHY according to the specific theory of reference;

(2) The artefact-dependent convincing arguments that explain WHY according to the mathematical experimentation facilitated by an artefact.

Approaching proof in school consists in promoting a network of interactive activities in order to connect these different components.

I will illustrate this point discussing how the analysis of dragging modalities can give reason of the transition from empirical to theoretical strands in students action and productions, while solving geometrical problems within DGS environments. I will show that in this case the notion of *abduction* is a major analysis tool for the researcher. Abduction is a way of reasoning pointed out by Peirce, who observed that abductive reasoning is essential for every human inquiry, because it is intertwined both with perception and with the general process of invention: abduction becomes part of the process of inquiry along with induction and deduction. For example, abductive processes can support interactions between (1) and (2) components above, namely the transition to proof within experimental mathematics, a transition with novel and specific features compared to the transition to proof within more traditional approaches. Hence the distance between arguments and formal proofs (Balacheff, 1999; Pedemonte, 2007) produced by students can diminish because of the use of technologies according to a precise pedagogical design.

One of the deepest results of the cognitive and epistemic analysis of students behaviours while solving problems within a DGS environment is that it shows the structure of their processes of inquiry and the dynamics of the transition from the empirical to the theoretic strand of their arguments. This has important consequences for the task design (for this issue see ICMI Study 22 forthcoming volume): I will sketch some examples from Arzarello et al. (1998, 2002) and Baccaglini-Frank & Mariotti (2010).

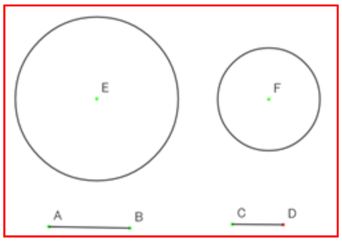
The evolution of technology, and particularly the actions allowed by multi-touch devices, allow deepening further the aspects (1) and (2) above and their mutual links. It is so possible designing fresh tasks, which support the transition from the empirical to the theoretical side of geometrical properties investigated within DGS.

In the second part of my talk I will exemplify them, basing from teaching experiments with multitouch devices made in Italy and Brasil, using DGS software like Geometric Constructor (designed by Yasuyuki Iijima at Aichi University of Education[2]), SketchPad Explorer[3] and Sketchometry[4] (Arzarello et al., 2013, 2014a, 2014b). The new technology, allows having more than one subject simultaneously operating on the screen of a tablet using as many fingers as they wishes: this facility, not possible within the mouse click-and-drag modality of DGS, makes it possible to design tasks where geometrical properties are introduced in a problematic way according to a *game theoretical transposition*. I will illustrate it with an example: the property "two circles in a plane intersect if and only if the sum of their radiuses lengths is lesser or equal to the distance of their centres" becomes the following (full-information) two-players game on a tablet (Fig. 3), which students must solve.

[2] http://souran.aichi-edu.ac.jp/profile/en.7RRZ6p1fkRx0afMM47vMnA==.html

[3] https://itunes.apple.com/en/app/sketchpad-explorer/id452811793?mt=8

[4] http://www.sketchometry.org/



<u>Rules of the game:</u>

- 2 players Z and Y
- player Z controls the radius of circumference E through the endpoints of the segment AB; player Y controls the radius of circumference F through the endpoints of segment CD
- Y's goal is to 'catch' Z; Z's goal is to run away.

Figure 3. A game within multitouch environments

We transform so mathematics investigations into game theoretical investigations: this design has at least two advantages. First, students are more engaged in the activity: the game theoretical formulation is more appealing than a task like 'explore and prove', even if in DGS environments: interactions between the players are more 'natural'. Second, and more important, the game theoretical transposition of geometrical problems introduces students into what I call the *Interrogative Model of Inquiry*. It is based on the logical researches of J.Hintikka (1998), who conceptualized inquiry in the sense of investigation, or search for truth, namely inquiry in the sense of a process of questioning, according to the well-known Socratic approach (for a modern example in the classroom see:
http://www.corndancer.com/tunes/tunes_print/soccirc.pdf).

Students, while solving the game-theoretical situations, ask pertinent questions to themselves or to their game-mates in order to elaborate suitable strategies, which allow them to "win the game". Such strategies consent them to discover, validate and possibly proof geometrical theorems. Students are so introduced into what I call the *logic of inquiry cycle*, which I will describe in the talk. Suitable task designs with tablets allow its effective instrumentation (Verillon-Rabardel, 1995; Rabardel, 2002; Trouche, 2005): as the dragging test is one of the most remarkable educational products of DGS with click and drag practices, the activation of the Interrogative Model of Inquiry is another very useful product made possible by DGS within multi-touch devices.

Discussion

In the paper I have discussed four teaching experiments, whose design is developed according to a careful cultural, epistemological, and cognitive analysis. I have showed how the main issue consists in elaborating tasks, which can satisfy all these three types of analysis. Fulfilling all of them shows a methodological proviso, which teaching situations should satisfy in order to design adequate didactical tasks. The discussion of the four concrete examples introduces a possible way to face the challenges that the "Internationalized and Network Era" poses to mathematics teachers.

In fact, observing such a proviso has two main didactical consequences with respect of the use of tools in the classroom:

(i) the (possible) introduction of instruments depends on the developed analysis and not vice versa;
(ii) the nature of the mediation is carefully designed as a consequence of the analysis.

More precisely,

- the introduced instrument must be
    a) consonant with the content to be taught, since its choice has been done in accordance with *epistemological analysis* of such a content;
    b) consonant with the learning processes that it triggers and supports in the students, according to the developed *cognitive analysis*.
- the teaching situation is adequate to be developed in that classroom with that mediating instrument according to a certain didactical design, because of
    c) the developed *cultural analysis*, which makes explicit the intertwining among the two components a) and b).

We have shown this way of designing a task in four different situations. In two of them the tool is a concrete instrument and manipulating it the students can discover its incorporated mathematical knowledge. In the other two the tool is a virtual environment, with which the students interact in a different way so discovering mathematical properties.

In all four cases the students are introduced into the logic of inquiry, which constitutes the strong cultural feature of their mathematical investigation, namely accomplishing such tasks as: Finding if, Establishing that, Ascertaining why, Settling why not, Investigating what if.

All these experiences involve both sensory-motor and highly symbolic activities: the mediation of the artefact allows intertwining the two so that the one can constantly be built on the other. It is worthwhile observing that such activities entail not only a cognitive behaviour, consistent with our biological being, but also cultural aspects of us as social beings. Indeed, the practices mentioned above show a deep intertwining between our cultural and cognitive components.

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Réflexions mathématico-historiques à l'âge du Net sur les réformes dans l'enseignement et les réactions à ces réformes

Jean Dhombres
Centre Alexandre Koyré,
CNRS/EHESS, Paris

Parce que cela ne nous concernerait pas directement *a priori*, il ne faudrait pas oublier que l'éducation est devenue un marché mondial en cette deuxième décennie du XXI^e^ siècle. Et en particulier celle qui forme en mathématiques et pour tous les niveaux dont celui de la recherche qui n'a jamais été aussi riche et multiforme. Il se manifeste un nouvel engouement pour les mathématiques, après des années de déclin, et il est vraisemblable qu'on le doive à l'action médiatique efficace de quelques jeunes mathématiciens titulaires notamment de médailles Fields promoteurs de mathématiques étonnamment évolutives et diverses, mais aussi au renom quelquefois douteux mais riche d'aventures de mathématiciens dans des banques d'affaires[1]. Aussi l'éducation mathématique est-elle un enjeu dans la concurrence manifeste qu'est la globalisation, entre les Etats et au-delà peut-être, entre des groupes financiers précisément dits "mutinationaux", et forcément entre des genres de formation, jusqu'à la formation à la recherche. Dans le cadre ainsi très largement défini, même si personne aujourd'hui après les abominations des "mathématiques nazies" n'ose plus trop glorifier telle ou telle mathématique nationale, il faut prendre conscience que l'affirmation, non discutée, d'une "mathématique internationale" comporte des risques. Je voudrais justement les discuter, d'autant que le Net peut paraître soutenir l'idée qu'existerait une

[1] Cette atmosphère est assez bien décrite dans un film récent réalisé par Olivier Peyon, Comment j'ai détesté les math, disponible en dvd.

mathématique internationale uniforme, c'est-à-dire lisse et sans changements, donc une seule école qui dise la norme à tous les niveaux, les figeant même dans l'immobilité. Il sufiit de mentionner la formation russe, issue du monde soviétique, pour se convaincre qu'autre chose est en jeu. Ici je voudrais en discuter en utilisant des moyens d'histoire. Pourquoi? Je crois que ce sont les risques d'uniformité des "mathématiques modernes" des années 1960 qui ont entraîné des réactions dont on se remet mal. Mon propos ici n'est pas de proposer des remèdes, mais il est que pour mesurer ces effets de réaction, l'histoire me paraît particulièrement utile, ne serait-ce que pour comprendre l'état même que nous vivons aujourd'hui.

Je me souviens en effet de revendications de différences, et ainsi à l'extrême, d'avoir entendu un "terroriste" français affirmer que la raison de son échec scolaire tenait à ce que les mathématiques qu'on lui avait enseignées n'étaient pas les "bonnes". Il disait préférer celles de l'âge d'or du monde de l'Islam. Et qu'importe qu'il ne les connaissait pas plus ! La revendication était une différence, et le ressenti d'adaptation nécessaire aux mentalités et aux valeurs d'une société. On a aussi entendu, il n'y a pas si longtemps et avant même la crise des subprimes, des étudiants et des enseignants en économie fustiger des mathématiques " à la française" . Car elles viendraient par leur embarras théorique cacher une défense du libéralisme sans frein. Sous la neutralité apparente d'un aspect structurel, donc irréel selon ces critiques, puisque ne prenant en compte que les "aspects quantitatifs". Cette dernière critique retrouve une démarche antimathématique toujours à l'œuvre et dans toutes les sociétés, en tout cas déjà portée par Aristote contre Platon. Ce qui témoigne, aujourd'hui, d'une méconnaissance profonde des mathématiques pratiquées et enseignées, et en particulier des probabilités qui n'en posent pas moins des questions fondamentales quant à notre conception du monde. Mais n'y a-t-il pas des dérives singulières dans l'emploi médiatique des outils statistiques, dont les enseignants ont du mal à prévenir les mauvais effets ? La contradiction épistémologique est que d'autres, sinon les mêmes, jettent le mépris sur les mathématiques financières en ce qu'elles apprécient trop bien les comportements réels des investisseurs professionnels, au point de savoir créer la même panique que les bourses à l'ancienne qui étaient dénuées d'ordinateurs et d'experts mathématiciens.

Ce contexte bizarre quoique tout actuel que je voulais rappeler me permet d'introduire une intéressante lettre publiée dans *The Guardian* du 5 mai 2014 demandant la suspension des notations "nationales" tous les trois ans du Program of International Student Assessment (PISA), donc concernant directement l'enseignement des mathématiques.

> En matière de politique de l'éducation, PISA, avec son cycle d'évaluation de trois ans, a provoqué un déplacement de l'attention vers des solutions à court terme, conçues pour aider rapidement un pays à grimper dans le classement, en dépit de recherches qui montrent que les changements durables dans la pratique éducative prennent des décennies pour se concrétiser, et non quelques années. Par exemple, nous savons que le statut des enseignants et le prestige de la profession enseignante, ont une forte influence sur la qualité de l'enseignement, mais ce statut et ce prestige varient fortement selon les cultures et ne sont pas facilement influencés par une politique à court terme. [...] Le nouveau régime PISA, avec son cycle continu de test global, nuit à nos enfants, appauvrit nos salles de classe car il implique inévitablement des batteries de plus en plus longues de tests à choix multiples, plus de leçons vendues en ligne dûment écrites et scénarisées, et moins d'autonomie pour les enseignants. De cette façon, PISA a encore augmenté dans les écoles le niveau de stress déjà élevé, ce qui met en danger le bien-être des élèves et des enseignants[2].

Cette lettre réunit les aspects assez disparates que je viens de mentionner : l'international et les cultures nationales, le court et le long terme dans l'éducation mathématique, les enjeux financiers de la concurrence des système éducatifs, la restriction supposée des

[2] In education policy, PISA, with its three year assessment cycle, has caused a shift of attention to short-term fixes designed to help a country quickly climb the rankings, despite research showing that enduring changes in education practice take decades, not a few years, to come to fruition. For example, we know that the status of teachers and the prestige of teaching as a profession have a strong influence on the quality of instruction, but that status varies strongly across cultures and is not easily influenced by short-term policy.[…] The new PISA regime, with its continuous cycle of global testing harms our children and impoverishes our classrooms, as it inevitably involves more and longer batteries of multiple-choice testing, more scripted "vendor"-made lessons, and less autonomy for teachers. In this way PISA has further increased the already high stress in schools, which endangers the wellbeing of students and teachers.

mathématiques au seul quantitatif mesurable ignorant bien des aspects de l'humain, et la domination de l'information non contrôlée des réseaux. Cette citation s'inscrit aussi dans les réactions "naturelles" des médias dans chaque pays, promptes à comptabiliser les médailles Fields nationales au mépris d'une conception internationale affichée de la recherche, oubliant de préciser à quoi tiennent des "écoles nationales", voire des "styles", au plus haut niveau de la recherche mathématique, comme à celui des enseignements. Les Olympiades, à un autre niveau, procèdent de même.

Sur toutes ces questions, je voudrais prendre le temps de la réflexion. Et je remercie vivement la Société hellenique de mathématiques de son invitation, car elle me donne cette occasion que je n'aurais pas cherché seul, à la fois en tant que mathématicien spécialisé en analyse fonctionnelle, en tant qu'historien des sciences mathématiques, et en tant qu'enseignant de mathématiques, même si ce fut seulement à l'Université. Je crois que cette triple familiarité, et je ne veux pas parler d'expertise, permet de mieux appréhender l'idée même des changements que généralement nous ne percevons pas comme répartis aussi bien dans la science que dans sa diffusion par l'enseignement scolaire, et qui proviennent aussi de l'imprégnation aux mathématiques dans la vie sociale dont témoigne le Net. Les changements ne sont pas de même nature, et justement je voudrais, par l'histoire, manifester des temps de réformes et des temps de réactions.

Les changements dans l'enseignement des mathématiques peuvent être aussi révolutionnaires que dans les mathématiques elles-mêmes

Si la didactique des mathématiques naquit en France dans les années 1960, grâce à Guy Brousseau[3] en bénéficiant notamment des réflexions du Genevois Jean Piaget[4], le but était d'envisager précisément les conditions de la mise en place de ce qui était présenté comme une révolution. La nature

[3] Nicolas Balachef, Martin Cooper, Rosamund Sutherland, (éd.), *Théorie des situations didactiques, 1970-1990*. Hommage à Guy Brousseau, La pensée sauvage, Grenoble, 1998.

[4] Il convient de signaler le rôle qu'a eu le livre de Jean Piaget dans la collection la Pléiade voulue par Raymond Queneau, *Logique et connaissance scientifique*, Paris, Gallimard, 1976, juxtaposant une épistémologie des mathématiques fondée sur les structures et la conception génétique d'appréhension de ces structures selon les âges de l'enfant de Jean Piaget.

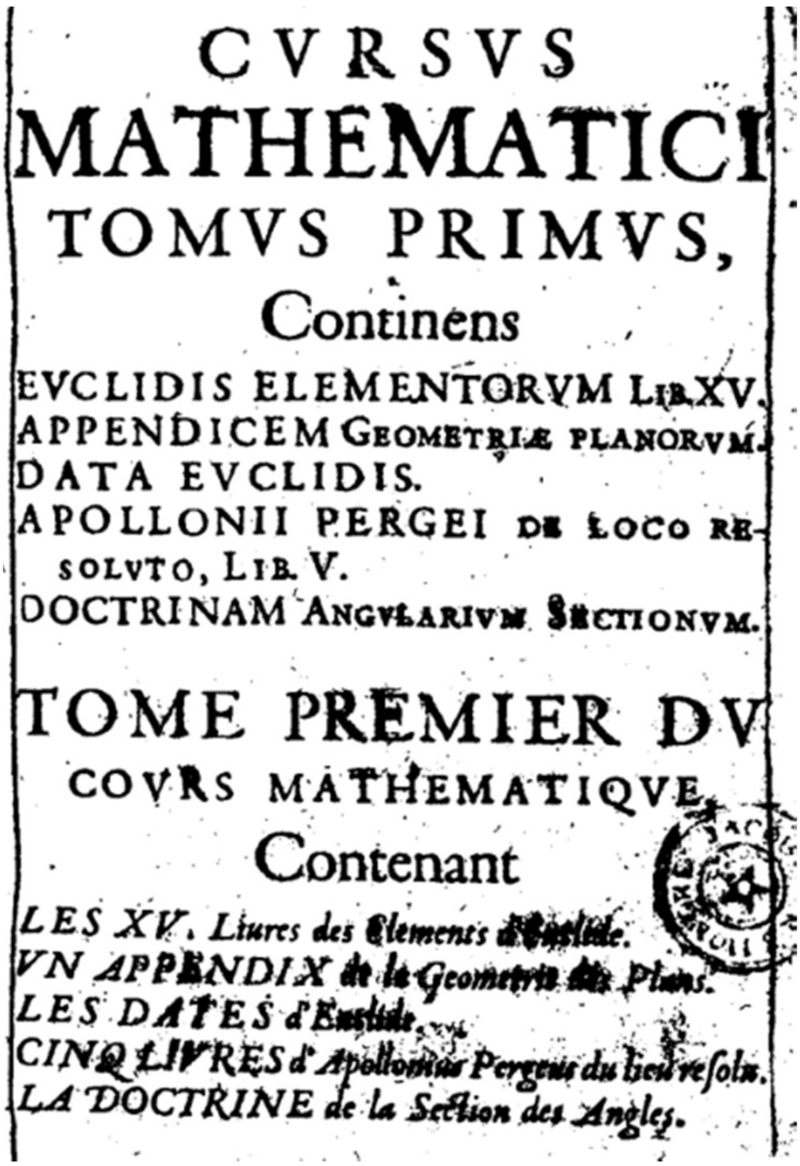
profondément politique de celle-ci, politique au sens aristotélicien généralisé de gouvernance des affaires publiques, dont aujourd'hui l'éducation, ne limite en rien la juste prétention de scientificité de la didactique. Cette discipline accompagnait indissolublement un mouvement pédagogico-scientifique désigné du nom de "mathématiques modernes", faisant un net appel au sens positiviste du progrès, à la volonté de passage à l'ère post-industrielle, et au sens d'un inéluctable changement auquel la querelle des Anciens et des Modernes donnait une lointaine mais active référence. Malheureusement, à ce discours politique qui ne colle pas à l'expression anglaise de *New math*, était en quelque sorte opposé un contenu qui paraissait découpé au sein d'un monde idéal et a-historique, toujours appelé mathématiques. Dans l'intention de lui donner un nouvel ordre, sans plus. Il s'agissait de travailler sur les structures, avec l'idée forte que les structures les plus fondamentales devaient venir en premier, donc étaient élémentaires au sens pédagogique cette fois. Cela voulait dire que la géométrie affine passait avant la géométrie métrique dans l'enseignement du collège, mais aussi que la topologie générale devait venir avant l'analyse des équations différentielles et des équations aux dérivées partielles dans l'enseignement universitaire, ou encore que la logique formelle devait passer avant tout exposé algorithmique. Le nouvel ordre n'était pas un choix, mais un impératif. Le mot d'ordre simpliste "à bas Euclide" symbolisa cette révolution, pourtant déconnectée de l'enjeu industriel et menée par les seuls mathématiciens dans une forme d'autisme aux autres sciences notamment, et peut-être aussi aux restructurations de leur science elle-même.

En chaussant les lunettes de l'historien, tant pour relativiser que pour spécifier, je voudrais faire prendre conscience des enjeux de tout changement dans les mathématiques, sans vouloir assurer qu'ils s'accompagnent toujours du type de révolution provoquée par les "mathématiques modernes", ni bien sûr qu'ils entraînent quasiment toujours les effets de réaction. Ce sont en tout cas ces effets qui me préoccupent ici. De sorte que si j'évoque la "réforme des mathématiques"[5] du XVII^e^ siècle et pour éviter une trop longue histoire, ce sera

[5] L'expression figure chez de nombreux auteurs du XVII^e^ siècle, même chez Thomas Hobbes, un Anglais comme toujours sensible aux effets de la tradition.

pour constater la naissance du livre que nous considérons à tort comme ayant de tout temps existé : le cours écrit de mathématiques. L'expression est toujours utilisée, mais avec le sens aujourd'hui d'un manuel. Ce "cours" fut plutôt le récit par les mathématiciens eux-mêmes de ce qu'ils voulaient changer. Ne peut-on dire qu'il correspond aux "leçons vendues en ligne", discutées dans la citation par laquelle j'ai commencé ? Le "Cursus mathematicus/Cours mathematique", premier ouvrage au monde à prendre un tel titre est dû à Pierre Hérigone en 1634: il atteindra six volumes huit ans plus tard. Le cours est bilingue, donc saute la barrière du latin qui était la langue réservée des universités ; le cours se veut aussi un accompagnement possible pour les "autodidactes", jusqu'à ranger sous ce nom ceux que les mathématiques universitaires ne satisfaisaient pas. A elle seule cette suggestion d'un public large change la donne mathématique. Le cours ne prétend pourtant pas contraindre un ordre de lecture, afin que "chacun puisse avoir séparément la partie qu'il jugera lui être le plus nécessaire", quoique retenant d'Euclide qu'il faille mettre en premier, mais au sein de chaque volume, "les parties nécessaires à l'intelligence de celles qui sont aux suivants"[6]. Cette versatilité en elle-même est encore un changement. Il ne sera pas définitif ! Il me suffit de commenter le contenu des cinq premiers livres pour que l'on perçoive la révolution sur le plan institutionnel, et je donne à voir cinq pages de titres pour les différents volumes, car elles explicitent le contenu (ill. 1 à 5).

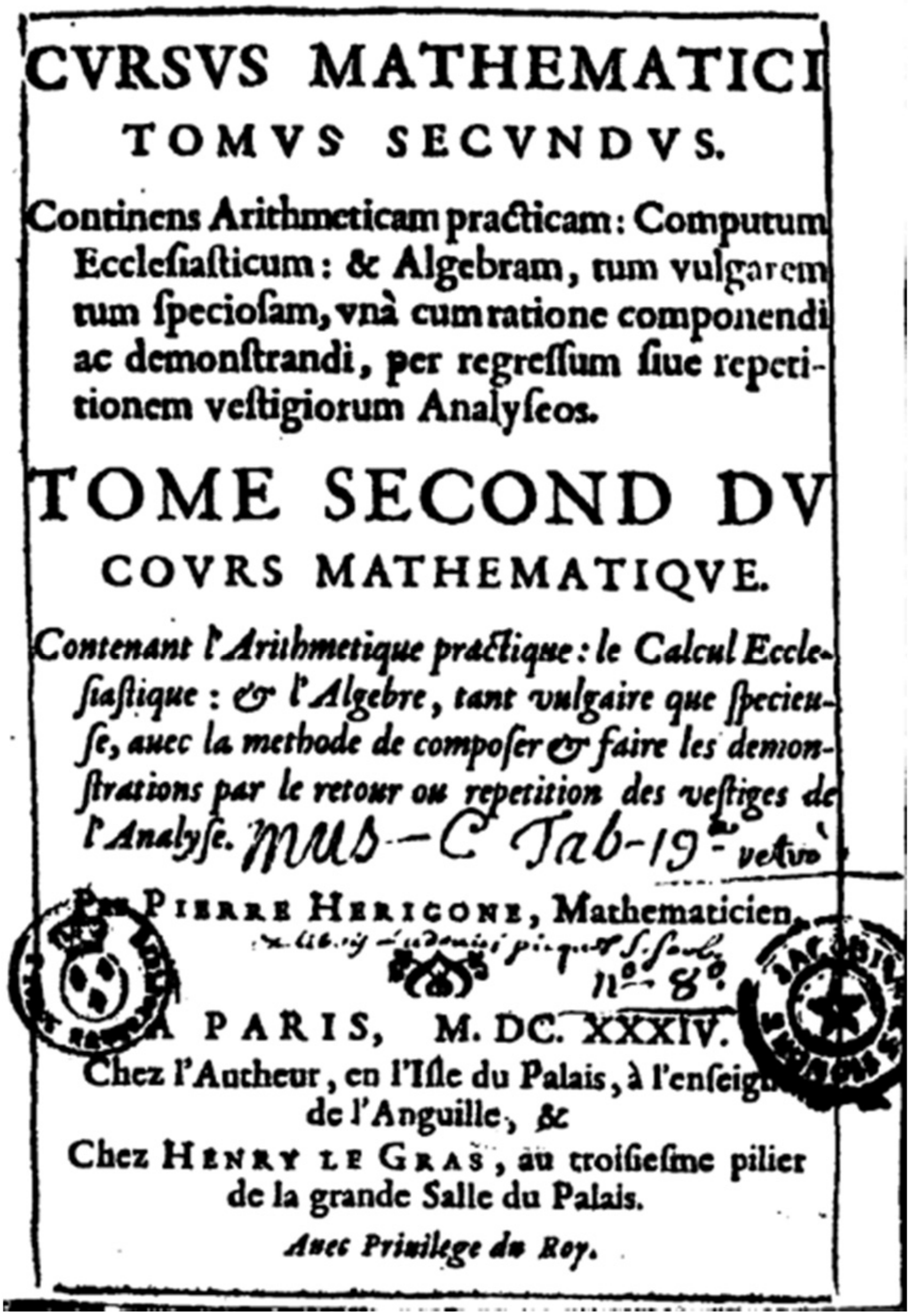
[6] Pierre Hérigone, *Cursus mathematicus/Cours mathematique*, Paris, 1634, tome I, Prolégomènes.



CVRSVS
MATHEMATICI
TOMVS PRIMVS,
Continens
EVCLIDIS ELEMENTORVM LIB. XV.
APPENDICEM GEOMETRIÆ PLANORVM.
DATA EVCLIDIS.
APOLLONII PERGEI DE LOCO RESOLVTO, LIB. V.
DOCTRINAM ANGVLARIVM SECTIONVM.

TOME PREMIER DV
COVRS MATHEMATIQVE,
Contenant
LES XV. Liures des Elements d'Euclide.
VN APPENDIX de la Geometrie des Plans.
LES DATES d'Euclide.
CINQ LIVRES d'Apollonius Pergeus du lieu resolu.
LA DOCTRINE de la Section des Angles.

Page de titre du premier volume du Cours de mathématiques de Pierre Hérigone en 1634. Les éléments d'Euclide sont traités par notes, comme on le verra plus loin, et suivis par des textes généralement ignorés des cours universitaires de cette époque. La théorie de Viète vient en dernier donnant le développement de *sin nx* et *cos nx* en *sin x* et cos x.



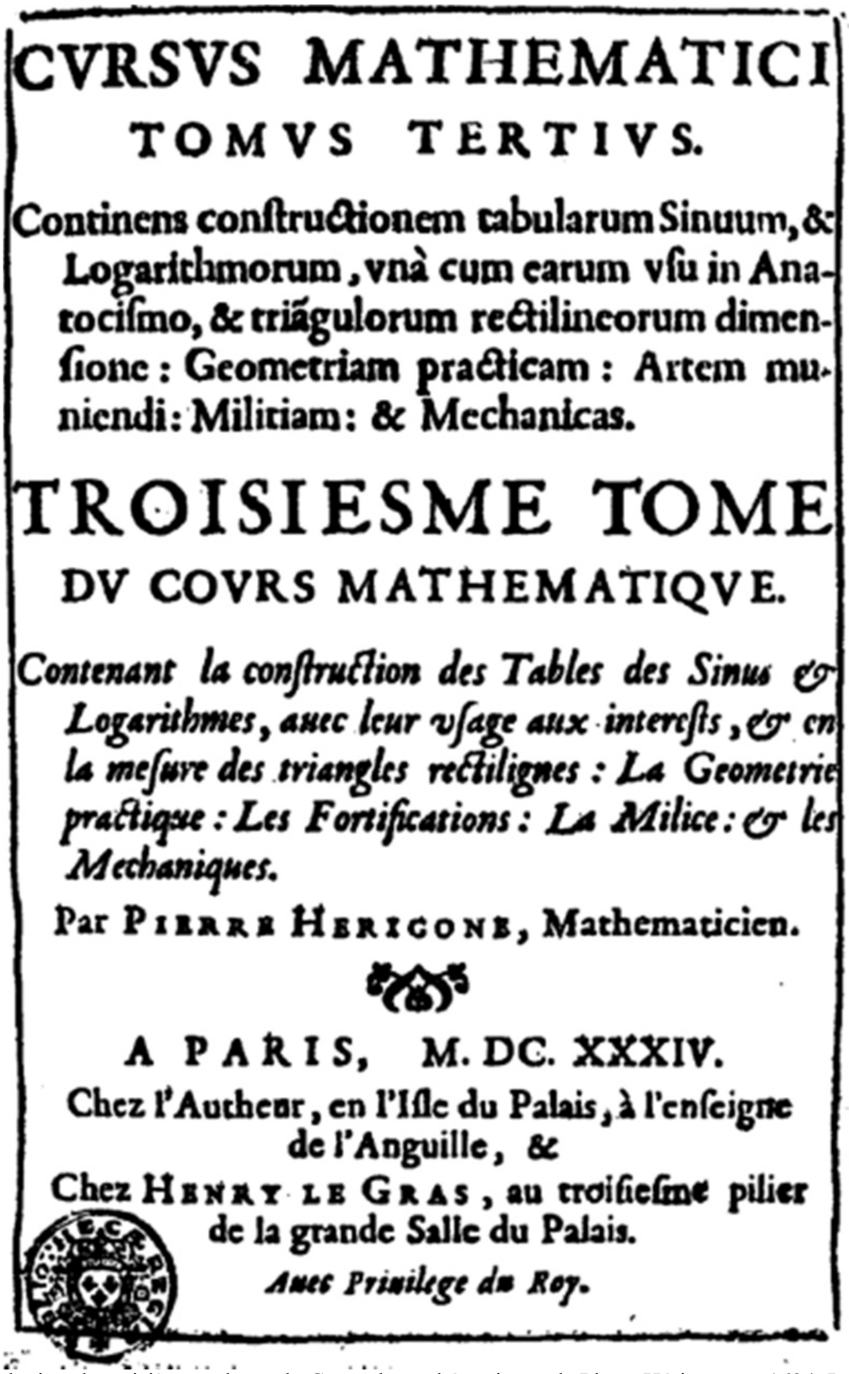
CVRSVS MATHEMATICI
TOMVS SECVNDVS.
Continens Arithmeticam practicam: Computum Ecclesiasticum: & Algebram, tum vulgarem tum speciosam, vnà cum ratione componendi ac demonstrandi, per regressum siue repetitionem vestigiorum Analyseos.

TOME SECOND DV
COVRS MATHEMATIQVE.
Contenant l'Arithmetique practique: le Calcul Ecclesiastique: & l'Algebre, tant vulgaire que specieuse, auec la methode de composer & faire les demonstrations par le retour ou repetition des vestiges de l'Analyse.

Par Pierre Herigone, Mathematicien.

A PARIS, M.DC.XXXIV.
Chez l'Autheur, en l'Isle du Palais, à l'enseigne de l'Anguille, &
Chez Henry le Gras, au troisiesme pilier de la grande Salle du Palais.
Auec Priuilege du Roy.

Page de titre du deuxième volume du Cours de mathématiques de Pierre Hérigone en 1634. Le calcul ecclésiastique est celui du calendrier, et l'on voit l'apparition de l'algèbre, avec indication de la méthode d'analyse.



CVRSVS MATHEMATICI

TOMVS TERTIVS.

Continens constructionem tabularum Sinuum, & Logarithmorum, vnà cum earum vsu in Anatocismo, & triãgulorum rectilineorum dimensione: Geometriam practicam: Artem muniendi: Militiam: & Mechanicas.

TROISIESME TOME

DV COVRS MATHEMATIQVE.

Contenant la construction des Tables des Sinus & Logarithmes, auec leur vsage aux interests, & en la mesure des triangles rectilignes: La Geometrie practique: Les Fortifications: La Milice: & les Mechaniques.

Par PIERRE HERIGONE, Mathematicien.

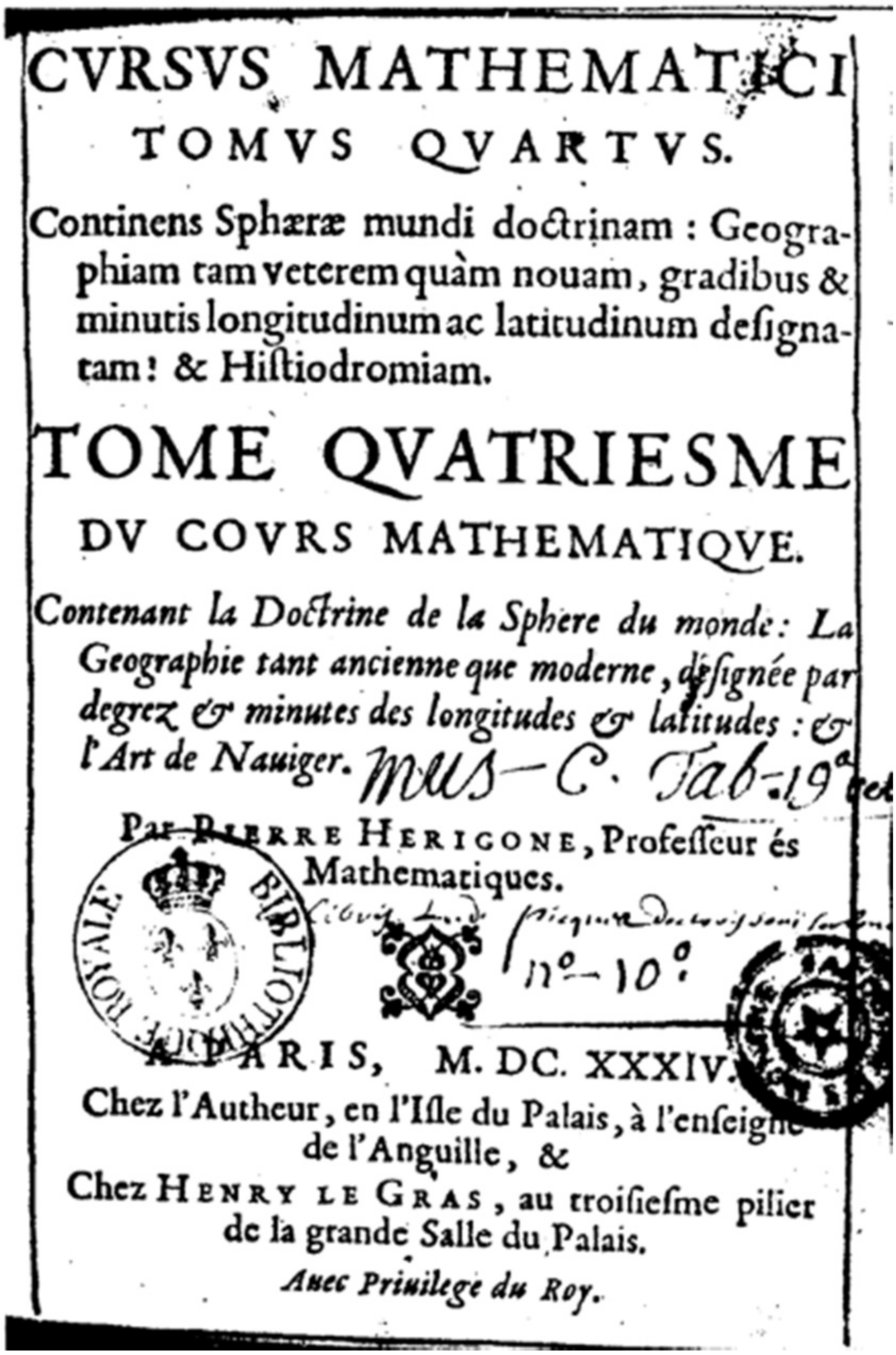
A PARIS, M.DC. XXXIV.

Chez l'Autheur, en l'Isle du Palais, à l'enseigne de l'Anguille, &

Chez HENRY LE GRAS, au troisiesme pilier de la grande Salle du Palais.

Auec Priuilege du Roy.

Page de titre du troisième volume du Cours de mathématiques de Pierre Hérigone en 1634. Les tables numériques viennent avant la géométrie pratique et les textes sur la mécanique.



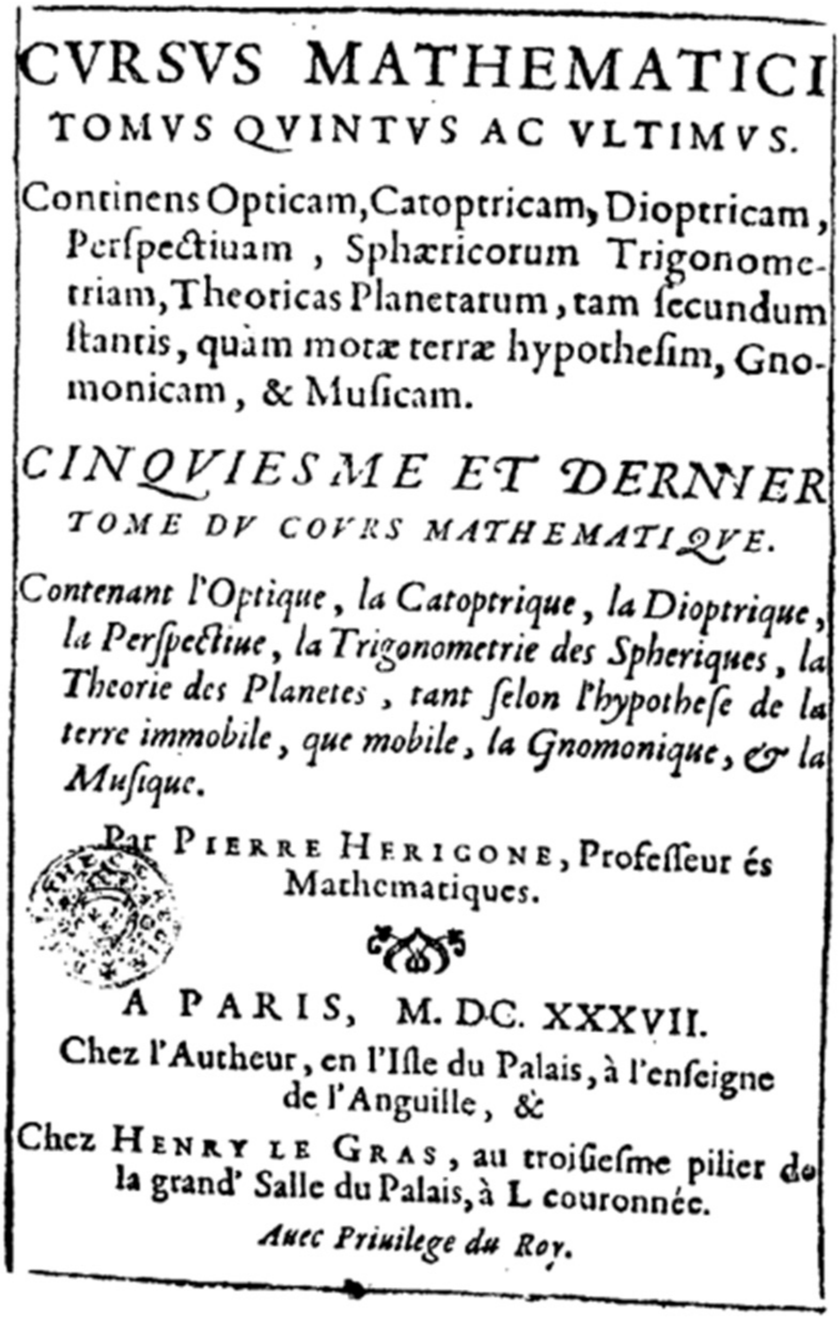
CVRSVS MATHEMATICI
TOMVS QVARTVS.
Continens Sphæræ mundi doctrinam: Geographiam tam veterem quàm nouam, gradibus & minutis longitudinum ac latitudinum designatam: & Histiodromiam.

TOME QVATRIESME
DV COVRS MATHEMATIQVE.
Contenant la Doctrine de la Sphere du monde: La Geographie tant ancienne que moderne, designée par degrez & minutes des longitudes & latitudes: & l'Art de Nauiger.

Par PIERRE HERIGONE, Professeur és Mathematiques.

A PARIS, M. DC. XXXIV.
Chez l'Autheur, en l'Isle du Palais, à l'enseigne de l'Anguille, &
Chez HENRY LE GRAS, au troisiesme pilier de la grande Salle du Palais.
Auec Priuilege du Roy.

Page de titre du quatrième volume du Cours de mathématiques de Pierre Hérigone en 1634.



CVRSVS MATHEMATICI
TOMVS QVINTVS AC VLTIMVS.
Continens Opticam, Catoptricam, Dioptricam, Perſpectiuam, Sphæricorum Trigonometriam, Theoricas Planetarum, tam ſecundum ſtantis, quàm motæ terræ hypotheſim, Gnomonicam, & Muſicam.

CINQVIESME ET DERNIER
TOME DV COVRS MATHEMATIQVE.
Contenant l'Optique, la Catoptrique, la Dioptrique, la Perſpectiue, la Trigonometrie des Spheriques, la Theorie des Planetes, tant ſelon l'hypotheſe de la terre immobile, que mobile, la Gnomonique, & la Muſique.

Par PIERRE HERIGONE, Profeſſeur és Mathematiques.

A PARIS, M.DC.XXXVII.
Chez l'Autheur, en l'Iſle du Palais, à l'enſeigne de l'Anguille, &
Chez HENRY LE GRAS, au troiſieſme pilier de la grand' Salle du Palais, à L couronnée.
Auec Priuilege du Roy.

Page de titre du cinquième volume du Cours de mathématiques de Pierre Hérigone en 1637, trois ans après les quatre précédents. Un dernier volume paraîtra quelques années plus tard.

Est détruit le robuste *quadrivium* universitaire, limité à l'arithmétique et à la géométrie selon Euclide, à la sphère selon Ptolémée et à la théorie des proportions propre à la musique, qui étaient les bases du savoir depuis Boèce au moins. Voici en effet qu'après les Eléments d'Euclide, sans gêne aucune, Hérigone ajoute des choses reconstituées d'Apollonius sur les problèmes des lieux, donc une préparation à ce qui deviendra la géométrie analytique avec Descartes et Fermat, et l'expression polynomiale des sinus et cosinus d'un multiple d'un angle à partir des mêmes fonctions de cet angle. C'est une formule mathématique qui fait jouer le développement du binôme, mais sans que l'on puisse alors expliquer pourquoi. Ce résultat provenait d'un travail de Viète publié en 1615 seulement par Anderson. Vingt ans plus tard, Hérigone l'insérait en ce premier volume[7], malgré son apparente complication, et surtout malgré son manque de lien avec le reste des mathématiques ! Personne alors n'avait l'idée des formules d'Euler qui firent ce lien, et apportèrent une autre révolution au siècle suivant, d'abord cantonnée dans des mathématiques très spécialisées[8]. Le second volume débute par l'arithmétique pratique, et il fournit des données sur le calendrier, en quelque sorte une actualité puisque le calendrier grégorien avait été établi en 1582 et ne convainquait toujours pas les pays protestants, ni les pays orthodoxes d'ailleurs. Cette arithmétique est suivie par l'algèbre, et c'est une grande nouveauté pour ce qui se présentait comme un programme d'études encyclopédiques. Aux volumes suivants apparaissent les logarithmes (dont la première publication, tables et théorie, remontait à 1614 seulement), les tables trigonométriques, la gnomonique, etc. Le mouvement numérique, largement absent des textes antérieurs mais caractéristique de la révolution marchande, est donc remarquablement représenté, de sorte qu'il n'y a aucune distinction faite entre des

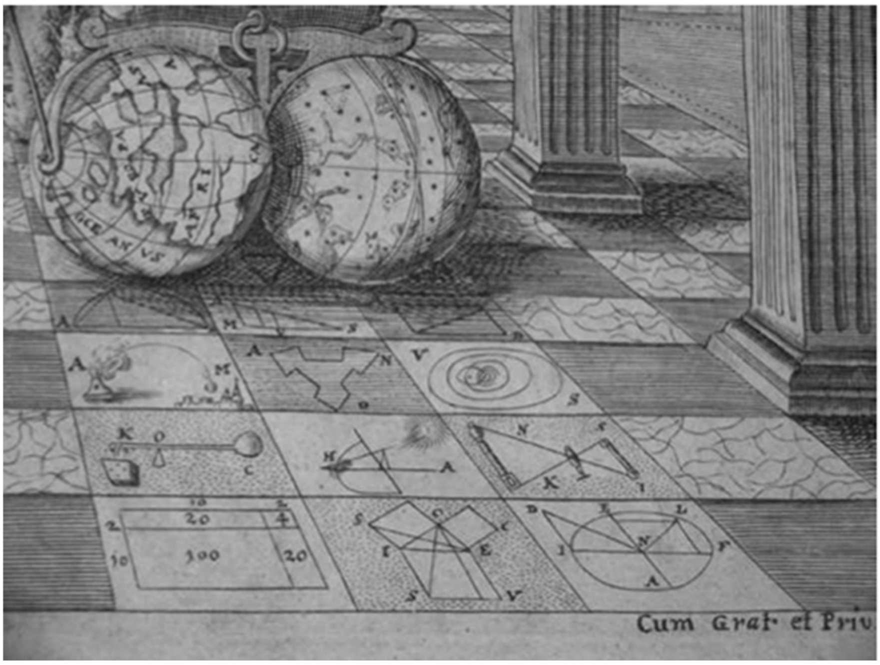
[7] L'algèbre posait un problème d'insertion, bien plus que la trigonométrie qui se coulait naturellement dans les Sphériques, ou les logarithmes qui trouvaient place plus ou moins naturelle dans l'évolution de la théorie des proportions. L'algèbre posait aussi une question de culture nationale, en rompant avec l'uniformité européenne du *quadrivium*, forcé par la *peregrinatio academica*, celle des étudiants comme celle des professeurs, uniformité qu'a décrite L.W.B. Brockliss en 1996 dans son article « Curricula » pour le volume publié à Cambridge sur *A History of Universities in Europe,1500-1800*, dirigé par H. de Ryder-Symoens. Il y avait de fait plusieurs algèbres, outre celles du monde arabe, celle des cossiques, celle des algébristes italiens du XVIe siècle. L'algèbre polynomiale de Descartes a tout balayé.

[8] Je résume par une formule qui n'est toujours pas au programme des lycées : $\cos nx + i\sin nx = (\cos x + i\sin x)^n$. La force de persuasion d'un d'Alembert fut insuffisante pour faire accepter cette formule en physique.

mathématiques pures et des mathématiques qui seraient seulement utiles. Pas plus qu'entre les mathématiques pour les collèges et les mathématiques pour les professionnels. La trigonométrie sphérique est traitée avec les fonctions de la trigonométrie plane, indispensable en navigation qui porte les Européens sur toute la surface océanique du globe. La “réforme” de Pierre Hérigone se résume symboliquement dans l'emploi du mot “mathématiques” ; il désigne concrètement un ensemble et non une seule voie linéaire à la manière des Eléments d'Euclide. Le mot « réforme », au XVII^e siècle, ne pouvait qu'avoir comme connotation la Réforme religieuse du siècle précédent, qui avait durablement divisé l'Europe, mais aussi favorisé l'idée d'une « république des lettres » et d'une solidarité dans laquelle purent s'insérer les tenants de la révolution scientifique. Comment penser à une possible Contre–Réforme ?

Si c'est en raison de la place donnée à l'algèbre qu'Hérigone dut inventer un tout nouvel ordre laissant la liberté au lecteur, il ne fut pas suivi et on peut justement parler d'une réaction, précisément dans le genre du cours, puisque le second cours publié, le *Cursus mathematicus* du jésuite Kaspar Schott en 1661, plaçait l'algèbre à la toute fin de l'ouvrage (ill. 6). Tout en ayant fait comprendre, de façon baroque, que cette finalité rendait vain tout le projet ! Dans les collèges jésuites, on préféra s'en tenir aux seuls Eléments d'Euclide, édulcorés en quelque sorte par omission du livre 10, et d'ailleurs en général des livres arithmétiques (7 à 9). Je rappelle que les 13 livres d'Euclide portèrent longtemps le seul titre d'éléments (sans que l'on spécifie de quelle science ils étaient les éléments), puis au XVII^e siècle, brusquement, et à la suite d'un professeur que l'on considère aisément comme un « réactionnaire », Clavius, on parla pour le même ouvrage seulement des « Eléments de géométrie »[9]. Le mot qui nous paraît si familier avait aussi pour fonction de gommer la présence de l'algèbre.

[9] Voir Jean Dhombres, *Les savoirs mathématiques et leurs pratiques culturelles*, Paris, Hermann, à paraître.



Ill. 6. Figure extraite du frontispice du Cours de Schott en 1661, exhibant par des figures cette fois les différentes disciplines mathématiques, les plus à jour en tout cas, puisque l'on voit les trajectoires elliptiques de planètes. L'algèbre est de fait reléguée en dernier dans cet ouvrage, donc n'y joue pratiquement pas de rôle.

La révolution de Hérigone fut pourtant bien perçue par quelques enseignants, mais ils eurent alors à se justifier d'être "modernes". Le mot sous-entendait une perte de rigueur, contre laquelle bien sûr ces enseignants devaient s'expliquer. Nous sommes donc à fronts renversés par rapport aux mathématiques modernes ! Et voilà qu'intervinrent, en contradiction avec le genre des Eléments d'Euclide, la nécessité ressentie par les mathématiciens de dire l'utilité de leur science par rapport aux autres sciences et non par rapport à la seule philosophie. De sorte que le XVII^e^ siècle vit la floraison moins de cours d'algèbre que de discours sur l'algèbre. Celui de John Wallis en 1685, A *Treatise of Algebra both historical and practical shewing the original, progres, and advancement thereof, from time to time, and by what steps it hath attained to the heigth it is now*, traduit en latin quelques années plus tard comme pour mieux le faire entrer dans la pratique universitaire, présente un titre positiviste avant la lettre. Mais si l'historienne Jacqueline A. Stedall a judicieusement intitulé son ouvrage, *A Discourse Concerning Algebra*, dans sa justification de la gloire donnée à

Harriot aux dépens de Descartes, ou encore par le sous-titre donné à son ouvrage, *English algebra to 1685*, elle manque assurément un mouvement en réseau bien plus largement européen. Elle opère, mais après John Wallis, une captation de type nationaliste, pour ce qui était en fait un mouvement de réforme. Ce qui est d'autant plus étrange que ce n'est pas l'algèbre anglaise qui a fait l'histoire ! Le *Traité d'algèbre* de Michel Rolle en 1690 fut le premier sans doute à éliminer ce genre de discours sur l'algèbre. Pour proposer l'organisation d'une discipline, qui naturellement incorporait bien des aspects d'analyse, notamment le théorème de Rolle pour l'étude des variations des fonctions. La méthode de Rolle fut poursuivie avec nonchalance par Newton dans son livre le plus souvent édité, parce que livre d'enseignement, *l'Arithmetica universalis*, un livre qui justement ne porte pas le nom d'algèbre.

La nouveauté n'en était pas moins dans l'ordre de la pensée : la conduite analytique du raisonnement devenait la garante du progrès, et en quelque sorte on pourrait lire le *Discours de la méthode* de Descartes en 1637 comme un dépassement des discours d'algèbre. Une image me fascine quant à l'enseignement : c'est celle d'une thèse de la toute fin du XVII^e^ siècle. C'est un exercice donné à des jeunes gens, en l'occurrence le travail d'un professionnel de la marine n'en suivant pas moins les cours des jésuites. On est bien dans l'esprit du Cours d'Hérigone, et loin de celui des "mathématiques modernes". Alors même que la modernisation de l'enseignement était indéniable. On voit sur l'image une formule d'algèbre (c'est en fait une citation de Descartes, et allusion aussi bien à la proportion dorée comme on peut le constater) dans un lieu scolaire (manifesté par des *putti*) pour dire la possibilité de gouverner la construction navale[10]. Ce qui est une propagande injustifiée à cette date, avant le *Traité du navire et de ses mouvemens* de Pierre Bouguer de 1746 ou la *Scientia Navalis* de Leonhard Euler de 1749 où le Calcul différentiel et intégral permettait enfin de traiter le problème de la stabilité ou du mouvement du navire. Sur ce dessin, le triangle didactique cher à bien des théoriciens est représenté avec trois composantes. Deux sont usuelles : les *putti* qui symbolisent le lieu scolaire, et les dessins et formules qui symbolisent le savoir théorique de l'enseignement de type nouveau. Mais le troisième est inédit, puisqu'il s'agit des applications, et sur cette image on voit même des ouvriers qui travaillent sur les lisses du navire en construction. On voit aussi la trace

[10] Cette gravure se trouve dans des thèses passée au séminaire de Toulon, et conservées à la Bibliothèque municipale de Lyon (BM,)

parabolique à la Galilée des boulets de canon. Le corps professoral, le troisième sommet usuel du triangle didactique, a disparu ! Cette thèse manifeste donc une réaction à la réaction du cours réduit au seul Euclide, formaté par Clavius. Si l'intérêt considérable du mouvement intellectuel à l'origine de notions comme celle de triangle didactique a été la prise en compte de l'acte d'enseignement des mathématiques, jusque dans ses aspects contractuels[11] avec comme raccourci langagier la transposition didactique[12], il apparaît que cet acte tient aussi aux circonstances historiques et épistémologiques, et notamment à l'imaginaire d'une société, sans doute en l'occurrence façonné par Descartes, qui pensait que l'on peut opérer beaucoup par les mathématiques, à condition que celles-ci ne soient pas tournées vers elles-mêmes.

Forcément des esprits se sont opposés à l'algèbre, et le témoignage de Blaise Pascal est ici majeur, qui évite l'algèbre alors que celle-ci lui aurait permis bien davantage dans ses constructions infinitésimales et de calcul intégral. Pour les enseignants, la difficulté était de laisser l'algèbre s'infilter dans le *quadrivium*, car elle prenait facilement un aspect totalitaire, se faufilant partout et détruisant les distinctions de disciplines auxquelles tenaient ces enseignants. Formidablement, l'invention du Calcul, différentiel et intégral, allait permettre donner une place à l'algèbre comme connaissance en prolégomènes, établie en vue des mathématiques dites sublimes, et aussi bien de leurs applications à la science navale par exemple[13]. Par ce mouvement de mise en préliminaires, mais de façon inattendue, l'élémentaire devint complètement cloisonné[14] : il avait sa vie propre, et forcément celle-ci devenait

[11] La notion très intéressante de contrat didactique correspond à la mort du cours répété parce que c'est ainsi, dont Lucienne Felix se moque avec une rare puissance dans ses mémoires publiées par L'Harmattan en 2005.

[12] Yves Chevallard, *La transposition didactique. Du savoir savant au savoir enseigné*, La pensée sauvage, Grenoble, 2e édition, 1991.

[13] Il me semble que l'on pourrait expliciter cette forme de lutte en faveur de l'algèbre en historicisant le reproche souvent fait au début du Calcul, avec le besoin de tout rapporter à la géométrie. Ce que l'algébriste André Weil manifestait en disant qu'il fallut l'extirper de la « gangue géométrique » (Article Calcul infinitésimal dans l'Histoire des mathématiques de Nicolas Bourbaki).

[14] Un deuxième exemple de révolution, que je traite au galop, est celui de la construction des nombres réels, inventée par Dedekind aussi bien que par Cantor. L'essentiel sans aucun doute est le procédé diagonal, permettant de distinguer le continu dans R et le dénomblable dans Q, mais aussi la possibilité de montrer que R est unique à un isomorphisme près de corps valué totalement ordonné complet. Ce qui fut fait par David Hilbert dans les *Grundlagen der Geometrie* de 1899. Dès lors, tout était prêt pour la définition du mètre

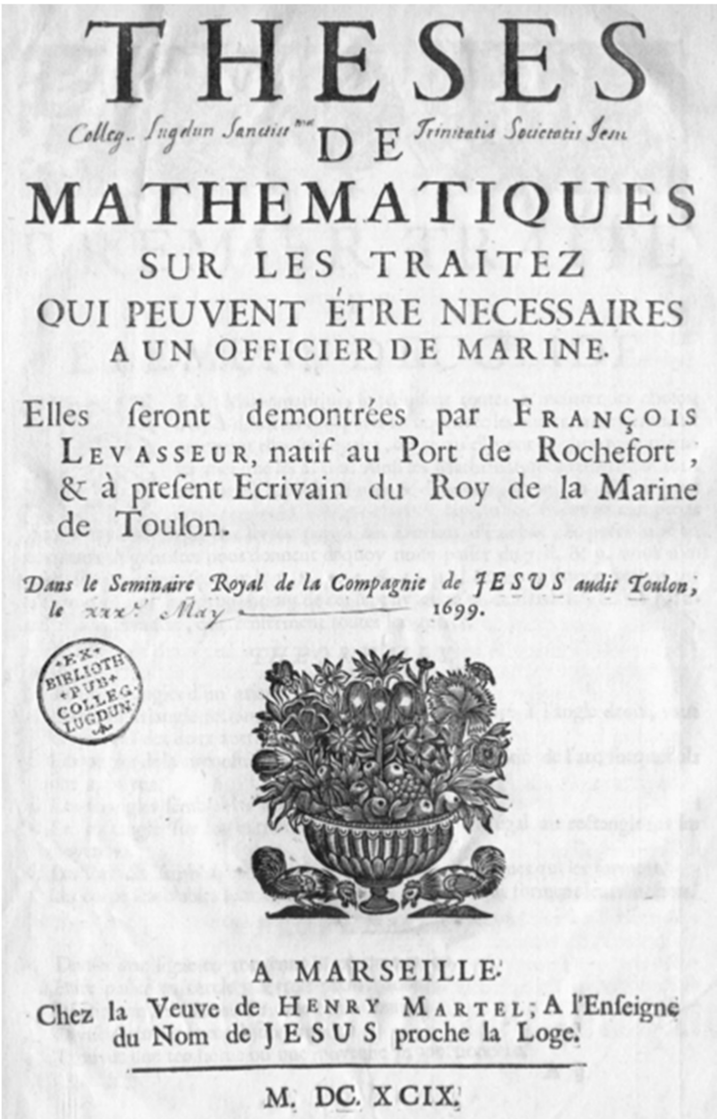
celle du collège, et non l'université. Dans ce cadre, bien des collèges se fixèrent sur les seuls Eléments d'Euclide, devenus symptomatiquement Eléments de géométrie. C'était éviter l'analytique, et en forgeant une “mathématique internationale”, alors dite jésuite en référence aux collèges de cette Compagnie, répartis dans le monde jusqu'en Chine ou en Amérique latine. Elle réduisait les mathématiques à un savoir assez étroit, mais ciselé dans ses détails, avec un accent particulier sur la géométrie. A tel point que l'on refusa d'utiliser la trignométrie pour la construction des cadrans solaires – l'objet le plus visible de l'influence mathématique dans une cité européenne à cette époque – au profit d'une construction géométrique, certes très jolie[15]. La considérer comme une réaction n'est pas la dénigrer, mais faire prendre conscience des choix, jusqu'à celui de l'esthétisme[16].

souple de géométrie affine de la classe de quatrième dans les premières années de la réforme des mathématiques modernes vers 1970.

15 Jean Dhombres, What images from the seventeenth century in the European cities may tell about the visibility of the mathematical sciences including astrology, in Vittoria Feola (éd.), *Antiquarianism and Science in Early Modern Urban Networks*, Sciences et Techniques en Perspective, vol. 16, fasc. 2, 2014, pp. 158-181.

16 Certains évitèrent l'algèbre par le biais des indivisibles. En témoigne le remarquable livre d'Ignace-Gaston Pardies qui fut traduit jusqu'en mandchou pour l'information de Kang Xi.

Ill. 7. Frontispice d'une thèse soutenue à Toulon, avec apparition (une première pour une gravure) d'une formule algébrique. Elle est en fait tirée de la *Géométrie* de Descartes et proche aussi bien du nombre d'or.



THESES

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DE

MATHEMATIQUES

SUR LES TRAITEZ

QUI PEUVENT ÊTRE NECESSAIRES

A UN OFFICIER DE MARINE.

Elles seront demontrées par FRANÇOIS LEVASSEUR, natif au Port de Rochefort, & à present Ecrivain du Roy de la Marine de Toulon.

Dans le Seminaire Royal de la Compagnie de JESUS audit Toulon, le xxx.e May 1699.

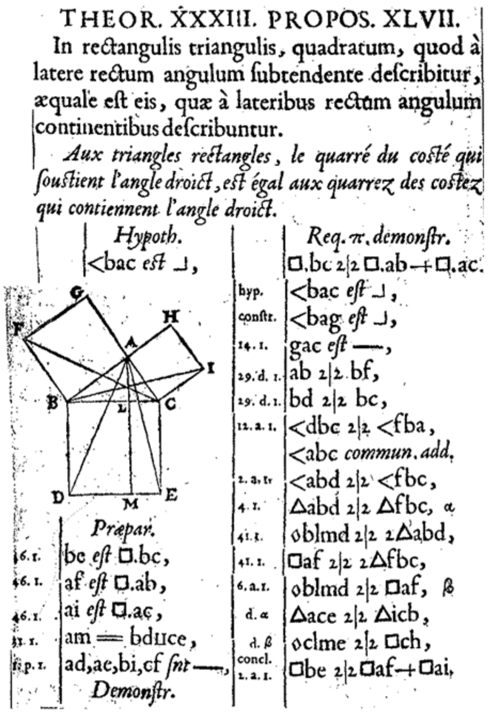
A MARSEILLE.

Chez la Veuve de HENRY MARTEL; A l'Enseigne du Nom de JESUS proche la Loge.

M. DC. XCIX.

Ill. 8. Page de titre de la thèse soutenue à Toulon, et publiée sous forme de livre

La révolution ratée des symboles pour le raisonnement



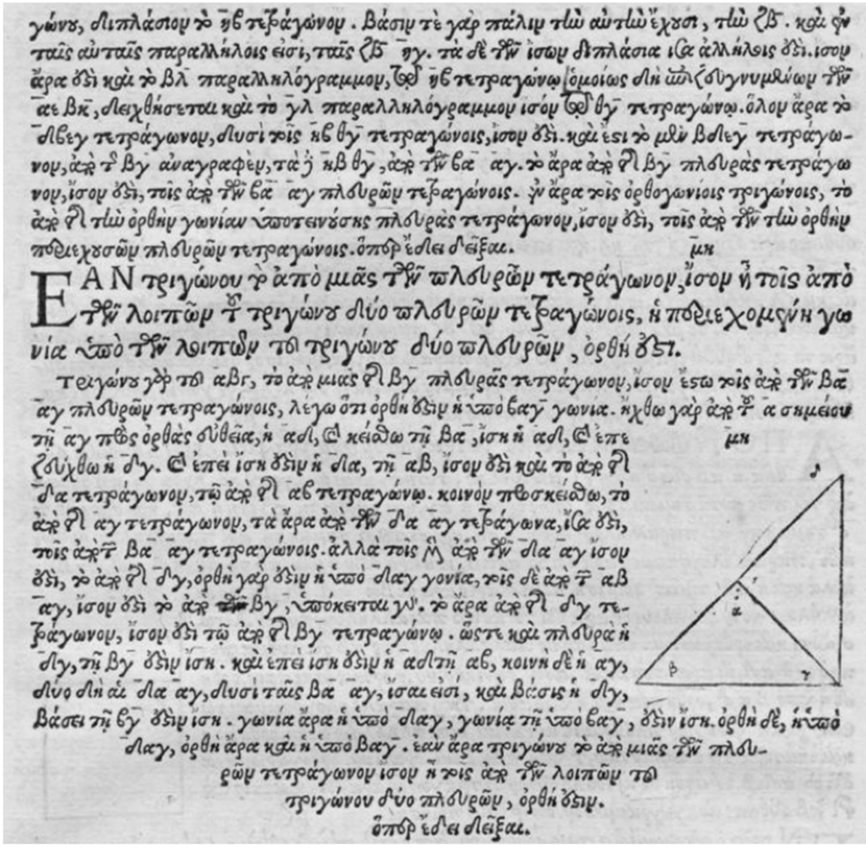
THEOR. XXXIII. PROPOS. XLVII.

In rectangulis triangulis, quadratum, quod à latere rectum angulum subtendente describitur, æquale est eis, quæ à lateribus rectum angulum continentibus describuntur.

Aux triangles rectangles, le quarré du costé qui soustient l'angle droict, est égal aux quarrez des costez qui contiennent l'angle droict.

Ill. 9. Le théorème de Pythagore chez Pierre Hérigone.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις, τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις. Ἔστω τρίγωνον ὀρθογώνιον, τὸ ΑΒΓ, ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ ...



Ill. 10. Les propositions 47 et 48 au livre I dans l'*editio princeps* de 1533, soit le théorème de Pythagore et sa réciproque

Je n'en ai pas fini avec le Cours de Pierre Hérigone, car il donne aussi à réfléchir sur la façon dont on peut procéder pour modifier l'enseignement. Je ais prendre pour m'expliquer le cas du théorème de Pythagore, ou plutôt la proposition 47 du premier livre des « Eléments d'Euclide », telle qu'elle est « écrite » par Pierre Hérigone dans *Cursus mathematicus/Cours mathématique* (ill. 9). L'imprimeur Henry le Gras n'a pas pu faire tenir le théorème sur une seule page, et du coup, comme le donne à voir le montage ci-dessus, n'a pas respecté l'alignement sur la gauche en passant à la page suivante. [17] Le résultat est suffisamment célèbre pour que l'on n'ait aucun

[17] Jean Dhombres, La preuve mathématique en tant qu'elle est épreuve de mémoire, in Rafael Mandressi (dir.), *La preuve*, PUF, Paris, 2009, pp. 59-84 ; Jean Dhombres, Sur un texte d'Euler relatif à une équation fonctionnelle : archaïsmes, pédagogie et style d'écriture,

mal à la suivre, alors même que peut surprendre l'écriture ici fournie. On ne peut pas la dire algébrique. Elle donne donc à réfléchir (voir par comparai-comparaison ill. 10 et la démonstration dans la première édition en grec en 1533).

La figure s'impose distinctement dans cette page qui paraît d'organisation moins claire que la figure, puisqu'apparaissent quatre colonnes assez différentes, après l'énoncé bilingue du théorème dont on apprécie le balancement rythmique, évitant le pédant vocabulaire de l'hypoténuse : l'aire du carré *BCED* est la somme des aires des carrés *BFGA* et *AHIC*. Il faut cependant un dictionnaire pour décrypter le signe égal, ici donné par 2|2, deux 2 séparés par une barre verticale qui joue à la manière du pivot d'une balance, dont la première intervention est à la ligne qui suit *Req. π. demonstr.* à la quatrième colonne. Un analogue de notre signe égal, avec deux lignes parallèles, apparaît quelques lignes plus tôt dans le même texte, mais il signifie le parallélisme justement des droites *AM, BD* et *CE*, et du coup on peut s'interroger, en lisant Hérigone aujourd'hui, sur les raisons qui ont fait plus tard adopter le signe égal des parallèles, avec un possible primat donné à la géométrie. Aucun signe égal n'était utilisé par Euclide, qui n'en avait pas moins la notion d'égalité portant sur des grandeurs, comme des aires, des longueurs, des angles, etc. Hérigone invente donc plus qu'une symbolisation : il fait intervenir un mode de pensée, avec l'égalité figurée en tant que relation ayant des propriétés que l'on reconnaît aujourd'hui sous le nom de relations d'équivalence. Pour mieux en faire voir la portée, je les donne avec la notation d'Hérigone (réflexivité *a*2|2*a*, symétrie *a*2|2*b* donne *b*2|2*a,* et transitivité, *a*2|2*b* et *b*2|2*c* donnent *a*2|2*c*). Alors que notre signe égal banalise par le sens acquis de l'algèbre. Mais justement cette banalisation n'est-elle pas le résultat lointain d'un mouvement lancé par Hérigone ? Auquel cas son 2|2 serait un signe avant-coureur !

L'écriture est de même nature que la notation avec le parallélisme qui nous paraît bien plus naturelle et riche. Hérigone entendait singulariser la relation d'égalité portant sur des mesures de grandeurs, et son choix de 2|2 signifie le maintien d'une spécificité pour les proportions que par ailleurs l'algèbre tendait à diminuer. De même, Hérigone « note » la proportion, *A* est à *B* comme *C* est à *D*, à partir du milieu de la partie de droite, avec

Sciences et Techniques en perspective, vol. 8, 1985, pp. 1-55 ; Jean Dhombres, De l'écriture des mathématiques en tant que technique de l'intellect, in Eric Guichard (dir.), *Ecritures : Sur les traces de Jack Goody*, Presses de l'ENSSIB, Lyon, 2012, pp. 157-198.

introduction de la lettre π (pour proportion sans doute) et on constate qu'il n'y a pas exhibition de l'algèbre que nous connaissons, mais un jeu qui individualise précisément le rapport dans l'agencement de la proportion, sans aller pourtant jusqu'à une égalité de rapports. Or, indépendamment de la volonté peut-on dire de Pierre Hérigone, son signe $2|2$ tendait vers une conception algébrique. A la même époque, Harriot faisait en algèbre l'assimilation avec la géométrie en donnant presque notre signe égal. Descartes écrira trois ans plus tard $\propto$, déformation de æ, pour désigner en latin une égalité qu'il situait aussi bien en algèbre, ou plutôt en fondement de sa théorie des équations. C'est ainsi constater que Descartes abrégeait seulement l'écriture dans le but de mettre en avant une indéniable nouveauté, là où Hérigone donnait une explication sous forme d'un graphème. Tous les deux, en proposant une notation alors inhabituelle, signalent un sens nouveau ; par ailleurs les deux notations ont aujourd'hui disparu dan un grand et fréquent mouvement d'élagage. En quel sens toutefois peut-on parler d'échec pour Hérigone ?

C'est qu'au delà d'une distinction entre un carré ($\square$) sans point et un carré suivi d'un point qui préfigure le sens de fonction comme $\square$.bc indiquant la puissance d'ordre 2, l'écriture d'Hérigone est dérangeante par une prétention de réduction du raisonnement mathématique. Chaque ligne du texte, au sens très matériel d'une certaine famille de mots placés horizontalement dans une même colonne, doit être une phrase complète ; elle est une étape de la démonstration, dès lors repérable en tant qu'unité, et repérable par des références à des propriétés précédentes (généralement euclidiennes). Cette spatialisation logique que l'on peut aussi décrire comme une succession ordonnée d'atomes horizontaux d'écriture, ou encore des abrégés sténographiques d'une phrase complète, est une contrainte extraordinaire. Elle implique que toutes les idées mathématiques doivent pouvoir se décomposer en unités de tailles équivalentes. La disposition en lignes séparées par des colonnes – quatre ici – est une forme de la spatialisation de l'écriture mathématique, et on a avantage à parler plus précisément d'un dispositif[18]. Il ne sera pas maintenu, ce qui ne signifie pas que leçon n'en sera pas tirée.

[18] Je ne suis pas sûr qu'on gagne autre chose qu'un effet de mode en qualifiant ce dispositif de « rhétorique ». Avec le seul mot dispositif, il me paraît utile de bénéficier de l'allusion voilée à la disposition spatiale, aussi éloignée du genre publicitaire que du bric-à-brac, ou même du style, et lui donner ainsi la signification d'un mode induit de lecture.

La naissance du calcul polynomial ou le geste des formules algébriques

La méthode des coefficients indéterminés vaut aujourd'hui comme recherche numérique et elle paraît l'exemple le plus élémentaire d'une modélisation par la détermination de paramètres à partir de données expérimentales en vue d'établir une loi de la nature[19]. Comme exemple suffisamment explicite de cette méthode, d'Alembert proposait la résolution d'une équation différentielle linéaire du premier ordre,

$$\frac{dy}{dx} + by = P ,$$

où *P* désigne un polynôme du second degré, et *b* est une constante réelle quelconque. Sans aucunement s'intéresser à l'équation homogène associée, il résolvait $Q'+bQ = P$ avec un polynôme *Q* posé *a priori* comme étant du second degré, avec des coefficients justement indéterminés, et c'est ce Q qu'il appelait « quantité ». Il conformait donc la solution. Il notait, en conservant de Descartes le fameux *xx* au lieu de la puissance d'ordre 2 :

$$Q = A+B+Cxx$$

Comme la dérivation diminue d'un degré la puissance, le système linéaire à trois équations en *A*, *B* et *C* est triangulaire non dégénéré, donc résoluble, donnant *A, B* et *C* explicitement à partir des coefficients du polynôme connu *P*. En ce sens la méthode des coefficients indéterminés est devenue le paradigme de la formulation mathématique par adaptation à l'empirisme phénoménal ou expérimental. Ce n'est pourtant pas de cette manière qu'elle fut présentée par son indéniable inventeur, Descartes, en 1637. Alors même qu'il ne s'adressait pas à un public de mathématiciens, mais voulait aider au gouvernement de chaque esprit par la seule raison. L'*Encyclopédie*, sous la plume de d'Alembert, restait bien succincte en son entrée « Méthode » en ce qui concerne du moins les mathématiques :

> La route que l'on doit suivre pour résoudre un problème. [20]

N'était pas énoncé un impératif sous la forme : « il faut suivre ». Mais l'article défini « la » induit que la « route » est unique. Chez Descartes avec cette méthode des coefficients indéterminés, ce qui est unique (mais à *n* variables) est le polynôme unitaire de degré *n*, apparaissant comme un

[19] Pour que l'on saisisse l'adaptation de la méthode à la pratique expérimentale, sinon à la réalité, il suffit de mentionner que c'est par la méthode des coefficients indéterminés que Fourier a découvert les coefficients éponymes. Voir Jean Dhombres, Jean-Bernard Robert, *Joseph Fourier, créateur de la physique mathématique*, Paris, Belin, 1992.

[20] Entrée méthode, pour les mathématiques ; repris dans *l'Encyclopédie méthodique*.

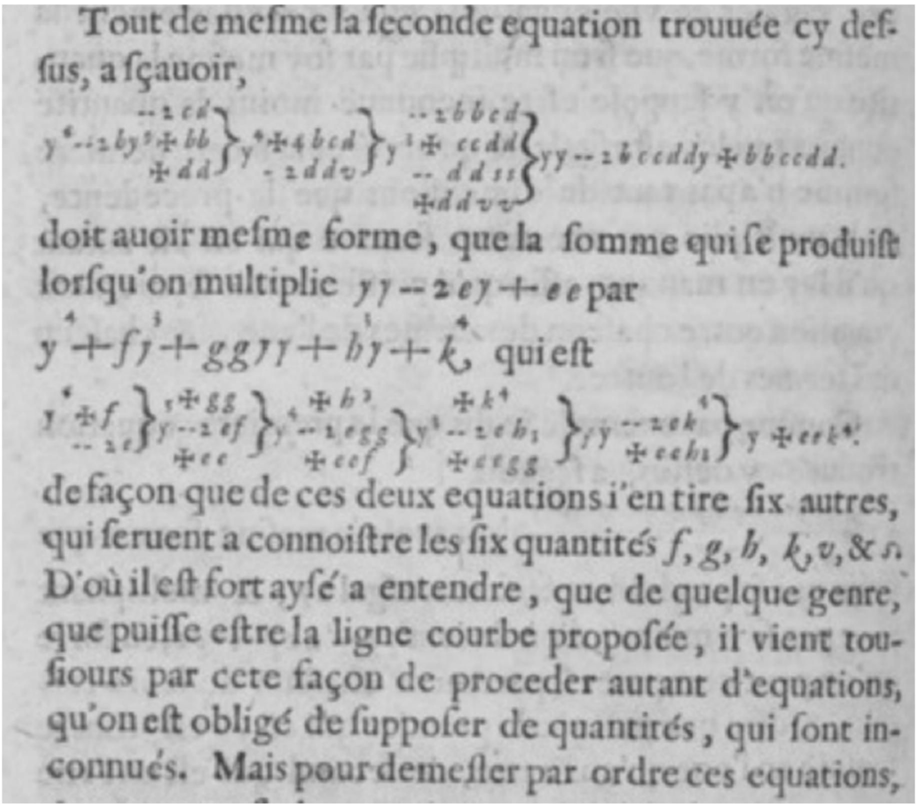
incontournable de sa postérité, alors que la méthode elle-même avait d'autres effets possibles, comme le commentateur de Descartes pour les collèges en 1730 l'indiquait suffisamment[21]. D'Alembert précisait toutefois que le mot méthode ne s'appliquait que lorsque plusieurs questions se trouvaient résolues par la même procédure. Ce qui serait une bonne lecture du commentaire « en passant » que donne Descartes de sa méthode ! Mais d'Alembert aboutissait à ce monstre épistémologique, pourtant familier, d'une qualification de « méthodes générales ». Il est soumis à la rhétorique qui les oppose aux « méthodes bornées » ! Seuls les enseignants, et non les mathématiciens chercheurs, acceptent de présenter des méthodes qu'ils savent « bornées : c'est au fond cela qui fait l'intérêt du programme d'une seule année scolaire, où le « polynôme vectoriel » peut trouver sa place.

Mais je donne directement du texte de Descartes une « équation » du sixième degré en y, la seconde équation écrit-il au livre II de la *Géométrie* à propos d'une courbe bien particulière à laquelle il s'agit de construire une tangente, et dont la nature importe peu pour mon propos[22]. Ce problème dit géométrique de la tangente à cette courbe phénoménale est réduit à celui, algébrique, de dire l'existence d'une racine double pour une équation. Descartes s'en expliqua aussi, mais là encore mon but est de suivre la démonstration en ce qu'elle porte sur des formes[23].

[21] Claude Rabuel, *Commentaires sur la Géométrie de M. Descartes*, Marcellin Duplain, Lyon, 1730.

[22] La construction d'une tangente à un cercle est un des morceaux de bravoure de toute description de la mathématique cartésienne. Voir Vincent Jullien, *La Géométrie de Descartes*, Paris, PUF, 1998, et l'édition récente de la géométrie par André Warusfel.

[23] Extrait de l'édition originale de 1637 de la *Géométrie* de Descartes, (p. 348). Se trouve dans les *Œuvres de Descartes*, t. VI, p. 420.



Tout de meſme la ſeconde equation trouuée cy deſſus, a ſçauoir,

$$y^6 - 2by^5 \left.\begin{array}{l}-2cd\\+bb\\+dd\end{array}\right\} y^4 \left.\begin{array}{l}+4bcd\\-2ddv\end{array}\right\} y^3 \left.\begin{array}{l}-2bbcd\\+ccdd\\-ddss\\+ddvv\end{array}\right\} yy - 2bccddy + bbccdd.$$

doit auoir meſme forme, que la ſomme qui ſe produiſt lorſqu'on multiplie $yy - 2ey + ee$ par $y^4 + fy^3 + ggyy + h^3y + k^4$ qui eſt

$$y^6 \left.\begin{array}{l}+f\\-2e\end{array}\right\} y^5 \left.\begin{array}{l}+gg\\-2ef\\+ee\end{array}\right\} y^4 \left.\begin{array}{l}+h^3\\-2egg\\+eef\end{array}\right\} y^3 \left.\begin{array}{l}+k^4\\-2eh^3\\+eegg\end{array}\right\} yy \left.\begin{array}{l}-2ek^4\\+eeh^3\end{array}\right\} y + eek^4.$$

de façon que de ces deux equations i'en tire ſix autres, qui ſeruent a connoiſtre les ſix quantités f, g, h, k, v, & s. D'où il eſt fort ayſé a entendre, que de quelque genre, que puiſſe eſtre la ligne courbe propoſée, il vient touſiours par cete façon de proceder autant d'equations, qu'on eſt obligé de ſuppoſer de quantités, qui ſont inconnuës. Mais pour demeſler par ordre ces equations,

Ill. 12. Extrait de la *Géométrie* de Descartes, un essai annexé au *Discours de la méthode*.

Il y a à voir la disposition spatiale d'un polynôme en *y*, à repérer des paramètres *b, c*, ou *d* et deux autres consonnes *s* et *v*, que je dirai inconnues pour rester avec le vocabulaire de Descartes et ne pas utiliser trop vite la notion de ultérieure de variable. On voit dans cet extrait entrer une autre lettre *e*, qui joue un rôle de variable réelle trop souvent passé sous silence, et quatre lettres encore : *f, g*, *h* et *k* qui sont celles mêmes de la méthode des coefficients indéterminés. Pour le vocabulaire, l'expression ici écrite « équation » de Descartes n'implique pas l'égalité à zéro d'une forme algébrique : la forme polynomiale elle-même est aussi dite une équation[24]. Descartes savait très bien mettre l'égalité à zéro lorsque ceci s'avérait nécessaire dans son exposé.

[24] En parlant de « somme qui se produit », Descartes pensait le polynôme comme objet ayant une forme, mais aussi comme quantité que je décris comme de nature vectorielle (« polynôme vectoriel »).

La dernière phrase de ce texte surprend, et inquiète : il est dit qu'il y a autant d'équations que d'inconnues, en tout cas que d'inconnues que l'on est obligé de supposer. L'inquiétude est celle de quiconque a fait un peu d'algèbre, et sait qu'une telle comptabilité, nombre d'équations et nombre d'inconnues, n'est jamais un critère de résolution, même avec des équations linéaires. J'ai recensé ci-dessous les équations en jeu. Ces équations, au nombre de six *a priori*, correspondent à l'identification des deux polynômes unitaires du sixième degré. Cette identification est le résultat du produit polynomial de $y^2 - 2ey + e^2$ par le polynôme général unitaire du quatrième degré.

$$\begin{array}{lclclcrl}
f & - & 2e & & & = & -2 & b \\
g^2 & - & 2ef & + & e^2 & = & -2 & cd + b^2 + d^2 \\
h^3 & - & 2cg^2 & + & e^2 f & = & 4 & bcd - 2d^2 v \\
k^4 & - & 2eh^3 & + & e^2 g^2 & = & -2 & b^2 cd + c^2 d^2 - d^2 s^2 + d^2 v^2 \\
 & - & 2ek^4 & + & e^2 h^3 & = & -2 & bc^2 d^2 \\
 & & & & e^2 k^4 & = & & b^2 c^2 d^2
\end{array}$$

Se pose la question de savoir quelles sont les inconnues « qu'on est obligé de supposer ». Une première réponse serait de dire que ces inconnues sont les quatre coefficients, f, g^2, h^3 et k^4, que l'on qualifiera d'*indéterminés*, ceux qui font le polynôme général du quatrième degré. Descartes prévient cette pensée, puisqu'il dit que le nombre des inconnues est 6, et non 4, correspondant aux 6 équations. Le lecteur doit donc se souvenir des quantités v et s, qui entrent dans la première équation écrite et qui ne suivent pas dans l'alphabet les b, c et d. Ces quantités v et s sont des inconnues, auxquelles viennent s'ajouter alors les quatre coefficients f, g^2, h^3 et k^4. Plane néanmoins un doute dans ce décompte sur le rôle de ce que je ne veux pas d'emblée appeler la variable e, puisque justement c'est cette qualité de variable que la méthode est aussi chargée de faire comprendre. On doit admettre que cette attitude de doute est celle souhaitée chez son lecteur par Descartes : il l'oblige à toujours être au clair des objectifs du calcul à un moment déterminé de ce calcul, précisément quant à la stratégie de l'ordre qui règle le calcul. C'est une option de didactique ! J'en souligne volontairement l'expression pour m'étonner de ne pas voir des attitudes semblables d'aujourd'hui, en vue d'un tel apprentissage. Mais n'est-ce pas parce qu'on dissocie le calcul de la méthode !

Présentement, l'ordre est dans la vision de six inconnues parce qu'il y a six équations, et on oublie provisoirement *e*, car c'est une quantité connue. J'écris ces six équations, et en passant prend conscience qu'avoir choisi des puissances comme g^2, h^3 et k^4, est une aide dans la disposition des calculs. On peut en effet vérifier qu'un terme n'est pas à sa place ! Argument *in absentia*, le visuel sert de repère du faux, dont la discrimination est jugée par Descartes comme essentielle. La nouvelle abréviation fait ressortir, au moyen d'alignements verticaux, les coefficients indéterminés que je note avec des majuscules $F = f$, $G = g^2$, $H = h^3$ et $K = k^4$, sans les membres de droite qui sont des constantes.

$$\begin{array}{llllll}
-2e + F & & & & & \\
e^2 - 2eF + & G & & & & \\
 & e^2F - 2eG + & H & & & \\
 & & e^2G - 2eH + & K & & \\
 & & & e^2H - 2eK & & \\
 & & & & e^2K &
\end{array}$$

Descartes poursuit en introduisant la prescription essentielle de sa méthode : « démêler par ordre ». Cet ordre commence par celui des seuls quatre coefficients.

> Mais, pour demesler par ordre ces equations & trouuer enfin la quantité *v*, qui est la seule dont on a besoin, & a l'occasion de laquelle on cherche les autres ; il faut, premierement, par le second terme chercher *f*, la premiere des
> quantités inconnuës de la derniere somme ; & on trouue
>
> $$f \propto 2e - 2b.$$
>
> Puis, par le dernier [terme], il faut chercher *k*, la derniere des quantités inconnuës de la mesme somme ; & on trouue
>
> $$k^4 \propto \frac{bbccdd}{ee}$$
>
> Puis, par le troisiesme terme, il faut chercher *g*, la seconde quantité, & on a
>
> $$gg \propto 3ee - 4be - 2cd + bb + dd.$$
>
> Puis, par le penultiesme, il faut chercher *h*, la penultiesme quantité, qui est
>
> $$h^3 \propto \frac{2bbccdd}{e^3} - \frac{2bccdd}{ee}.$$
>
> Et ainsi il faudroit continuer, suiuant ce mesme ordre, iusques a la dernière, s'il y en auoit dauantage en cete

somme ; car c'est chose qu'on peut tousiours faire en mesme façon.

Puis, par le terme qui suit en ce mesme ordre, qui est icy le quatriesme, il faut chercher la quantité v, & on a

$$v \infty \frac{2e^3}{dd} - \frac{3bee}{dd} + \frac{bbe}{dd} - \frac{2ce}{d} + e + \frac{2bc}{d} + \frac{bcc}{ee} - \frac{bbcc}{e^3};$$

où mettant y au lieu d'e, qui lui est esgal, on a

$$v \infty \frac{2y^2}{dl} - \frac{3byy}{dl} + \frac{bby}{dl} - \frac{2cy}{d} + y + \frac{2bc}{d} + \frac{bcc}{yy} - \frac{bbcc}{y^3}.$$

pour la ligne[25] AP.

Ill. 13. Autre extrait de la géométrie de Descartes

Cet extrait suffit pour répondre à la question de la preuve. Non pas la preuve de la méthode des tangentes toute entière, ni la preuve de la méthode des coefficients indéterminés reposant sur l'identification des coefficients des polynômes. La preuve est passée à l'ordre de résolubilité[26] des six équations, avec six inconnues. Pourquoi le faire ici ? Pourquoi scinder le discours de Descartes ? C'est que lui-même invite à cette réflexion, en venant de dire que « c'est chose qu'on peut toujours faire en même façon »[27]. La preuve est celle de Descartes avec la généralité indiquée de la résolution au-delà du sixième degré, ce que nous identifions à une résolution en cascade par alternance[28]. Bref, c'est la forme qui domine, la forme sous laquelle un problème algébrique se

25 Ce mot de « ligne » *AP* exige, pour être compris, de revenir à la signification de la première équation, et d'une courbe. Je n'en ai pas besoin pour la présente discussion. D'ailleurs Descartes lui-même se contente d'un dessin. Celle-ci clôt la discussion sur l'équation par laquelle a commencé notre entrée dans le texte de Descartes.

26 Le mot de résolubilité n'est pas admis par des dictionnaires français, solvabilité étant préféré, mais je le préfère parce que faisant paire avec le mot résolution.

27 *La Géométrie*, p.421, et ici dans la citation fournie.

28 Le lecteur moderne a tout de suite visualisé la méthode pour ces quatre coefficients, puisque ceux-ci apparaissent dans le système linéaire de six équations donné ci-dessus, et en fait deux systèmes linéaires triangulaires. L'expression cascade, comme l'expression triangulaire que j'ai utilisée, sont mal choisies, car anachroniques, l'algèbre linéaire n'étant pas faite chez Descartes. Il faudrait dire une double cascade (d'une part avec *F*, *G* et H, d'autre part en inversant l'ordre avec *K, H* et *G*), ou encore la forme croissante/décroissante, comme dans le célèbre poème Les Djinns des *Orientales* de Victor Hugo. À condition de voir que dans les équations n'interviennent ni l'inconnue *v*, ni l'inconnue *s*, à l'exception des membres de droite des équations linéaires à trois des quatre lettres *F, G, H* et *K*, ni la lettre *e* qui est pour le moment une constante parmi d'autres.

présente pour pouvoir être résolu. La forme en accordéon du système linéaire est liée à la multiplication par un polynôme de degré 2. Il y a une rhétorique de correspondance de deux ordres, de deux autres ordres, l'ordre des quantités (les coefficients indéterminés), et l'ordre des équations tel que l'a déterminé la méthode d'identification des coefficients des puissances décroissantes, ou ordre des termes. La correspondance est langagière : par la pénultième (ou avant dernière équation), il faut chercher la pénultième (ou avant dernière quantité). Mais Descartes prend soin de casser cette correspondance, puisque dès le départ il indique qu'il faut « premièrement », par le « second terme » chercher la « première » des quantités de la « dernière somme ». Il ajoutera que par le « troisième terme » il faut chercher la « seconde quantité ». L'emploi du « il faut », et non pas « il est clair », ou « on voit », est une indication que le nouvel ordre, l'écriture des équations dans un ordre autre que celui provenant des puissances décroissantes, requiert un parcours algébrique de la forme des équations.

La question de généralité soulevée est de savoir si ce nouvel ordre des équations, ou cette forme, suffit à la résolution particulière, en vue de la détermination des coefficients indéterminés, indépendamment du degré de la forme polynomiale de départ. Descartes assure : « on peut toujours faire en même façon ». Il a raison pour trouver les racines doubles. Omet-il de dire pourquoi ? On peut aisément dire cette raison avec les mots de Descartes : intervient la forme du polynôme de degré 6 obtenu par multiplication du polynôme $y^2 - 2ey + e^2$ par le polynôme « feint »[29], ou général, $y^4 + fy^3 + g^2y^2 + h^3y + k^4$. Cette multiplication, à la manière dont Descartes la conduit (commençant par le polynôme de second degré), impose que pour chaque coefficient au plus trois des lettres f, g^2, h^3 et k^4 (visualisées ici par F, G, H, K) interviendront linéairement, et qu'au début (première équation, degré 5 ou deuxième terme) comme à la fin (dernière équation, degré 0, troisième terme) une seule des lettres interviendra. Puis pour la deuxième équation (degré 4, troisième terme) comme l'avant dernière équation (degré 1, sixième terme), deux seulement des lettres seront liées. La forme des équations qui dit leur résolubilité, à condition de bien ordonner, résulte de la nature de la multiplication polynomiale, donc du « polynôme algébrique » : trois termes sont au plus en jeu parce qu'il y a trois termes dans un polynôme de degré 2.

[29] L'expression d'équation « feinte » est de tabuel, dans son *Commentaire à la Géométrie de Descartes.*

S'il y a une rhétorique fallacieuse de l'ordre dans cet extrait de la *Géométrie* de Descartes, on la trouve juste ici, avec cet ordre de la résolubilité passant de la résolubilité des coefficients indéterminés à celle de l'inconnue v. Comme s'il s'agissait du même ordre ! Par contre, la rhétorique de l'ordre avec premier/dernier, etc., par son décalage, indique bien les attentions à prendre, et la non automaticité du calcul. Il est prévisible si l'on ordonne convenablement.

C'est pourtant là que la méthode de Descartes manque sa preuve générale. Avec la possibilité même d'écrire une courbe algébrique $P(x,y) = 0$ sur la forme résolue $y = f(x)$. « On ne laisse pas de pouvoir toujours avoir une telle équation » annonçait-il quelques pages plus tôt[30]. Cette affirmation est fausse. Le calcul différentiel seul montre qu'il est possible, et seulement localement, d'exprimer $y = f(x)$, à condition d'accepter pour f, non pas un polynôme, mais une série entière. Telle est la formule de Taylor-Young[31]. Le succès de la méthode polynomiale de Descartes n'est pas garanti pour toutes les courbes algébriques, et ce n'est pas seulement une question de complication, mais tient à la nature même des courbes algébriques qu'il croyait suffisamment simple, alors qu'elle possède par certains côtés le même aspect non fini des courbes transcendantes, ou mécaniques dirait Descartes. A moins de penser que Descartes se trompe en supposant la forme fonctionnelle dans tous les cas, il ne peut raisonner que sur des exemples. Il sort au moins deux généralités de sa démarche sur le problème des tangentes. La première déjà dite est relative à l'ordre : c'est l'écriture polynomiale générale suivant les puissances décroissantes avec des coefficients généraux. La reconnaissance de la généralité des coefficients, et donc de la forme polynomiale, a été postulée par le degré 4, mais est devenue une réalité par la résolution même des coefficients en jeu, qui sont effectivement déterminés. La seconde généralité, liée à la précédente, tient à ce e qui intervient dans toutes les formules donnant les coefficients indéterminés. La signification de e est d'être un nombre réel quelconque : une abscisse. Car est ainsi fixé le point courant sur la courbe en lequel on veut calculer une tangente. C'est aussi une variable d'algèbre qui peut être mise à la place de y, précisément par la mise en facteur du carré de (y-e). On

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30 *Géométrie*, p. 416.

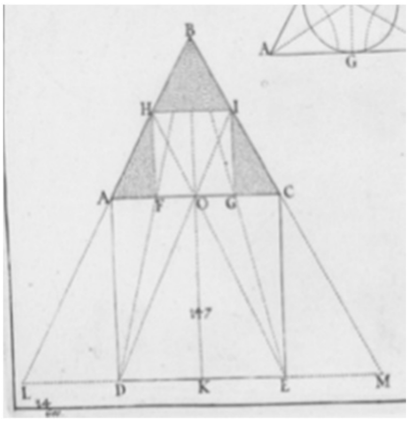
31 C'est l'écriture $f(x+a) = f(x) + af'(x) + \frac{a^2}{2!} f''(x) + \ldots + \frac{a^n}{n!} f^n(x) +$ d'une fonction suffisamment régulière f, qui a finalement détrôné la formule du binôme de sa place centrale en analyse algébrique deuxième manière.

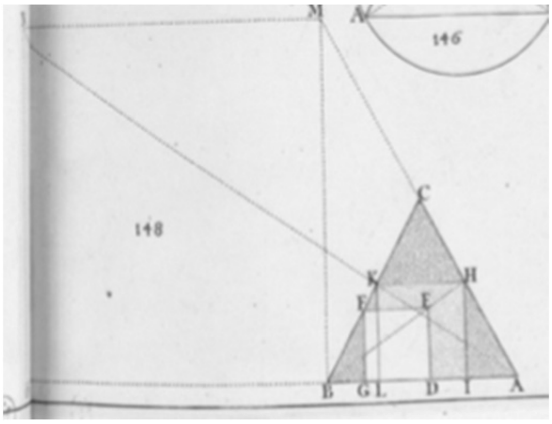
passe de la lettre algébrique muette du polynôme à la valeur réelle de la géométrie des courbes. Il y a deux constatations. D'une part, il ne fait plus de doute que les coefficients d'un polynôme ne soient des nombres réels, et pas seulement des rationnels. L'ontologie du réel est passée au polynôme par la réalité analytique d'une courbe, qui est l'œuvre de Descartes et qui tient lieu de forme phénoménale, sans seulement être une représentation ou une traduction en une autre langue. La substitution de la variable *y* en *e*, une conversion qui est partie prenante de la méthode de calcul, génère précisément le « polynôme fonctionnel », c'est-à-dire du polynôme an tant que fonction d'une variable.

Se présente ainsi un aspect épistémologiquement singulier de cette écriture dans la mesure où la pensée adopte des formes de ce qu'elle est censée donner à penser, en l'occurrence l'espace, ou l'étendue comme écrivait Descartes. Ce dernier utilisera aussi bien la spatialisation pour une forme algébrique, avec le choix d'une double écriture horizontale et verticale des polynômes ; elle sera maintenue assez longtemps. Avec cette spatialisation de l'écriture polynomiale, on ne peut pas dire qu'il s'agisse seulement d'une convention au sens où il n'y aurait aucune explication à trouver dans le moyen graphique utilisé. La spatialisation fait usage d'une « analogie », qui n'est pas entée sur le sens mathématique des proportions, mais reprend le geste du calcul d'un polynôme où l'on regarde de droite à gauche, par exemple lorsque l'on développe un produit croisé de coefficients comme $(a+b)(c+d)$, qui se déploie spatialement alors que la lecture des puissances successives de l'indéterminée est linéaire, de gauche à droite. C'est ce que j'appelle le geste de la formule.

Au contraire, le symbole de l'égalité pour Hérigone, 2|2, est simplement emblématique en ce qu'il rappelle la balance. Mais on retrouve l'une des affirmations les plus anciennes de la pédagogie de l'algèbre, avec le fait de « l'équilibre des deux membres » de l'équation, comme des poids qui, retirés car gommés d'un côté du signe, doivent être également retirés de l'autre côté, ce qui fait précisément intervenir le signe moins. Ai-je besoin d'insister sur ce qu'il y a d'invention dans cette pensée du négatif, après ce que Kant en a dit qui souhaitait son introduction dans les raisonnements philosophiques ? On rencontre ainsi une des plus belles questions de l'histoire des mathématiques en ce qu'elle peut aider à comprendre le fonctionnement de l'esprit humain dans sa réflexion collective : faut-il voir l'invention du signe moins dans l'allusion algébrique à des comptes commerciaux à équilibrer ou dans la pensée qui reste spatiale de la balance ? Je ne prétends pas trancher, mais je suis sûr, quand il y a un choix didactique pour l'une ou l'autre option, que cela répond à d'autres critères que des critères mathématiques. Je voudrais donc poursuivre sur de tels choix, à partir de problèmes particuliers et n'étant plus à la mode.

La modes de certains problèmes : l'inscription d'un carré dans un triangle





Ill. 14 et 15. Extraits de la planche XV des figures de Samuel Marolois, dans l'édition par Albert Girard en 1651 du *Traité, et practique de Géometrie, et premièrement de l'usage du compas*.

Le problème qui consiste à inscrire un carré dans un triangle est envisagé par l'auteur d'une géométrie qualifiée de pratique, affublée pourtant du nom de traité. Sans doute parce que les gestes techniques y sont justifiés. Le mathématicien professionnel qu'est Samuel Marolois - l'adjectif professionnel étant seulement mis pour dire qu'il ne s'agit pas d'un universitaire - sera « corrigé » en 1628 par un autre professionnel Albert Girard. Il fut le traducteur et commentateur en français de Simon Stevin, l'introducteur en Europe de l'écriture décimale illimitée. Les deux auteurs ne disent pas les origines du problème, ce qui ne veut pas du tout dire qu'ils ne les connaissent pas. Il est certain qu'un simple regard jeté sur les deux figures du bas de la planche (ill. 14 et 15) établit que la situation exploitée est celle de la similitude géométrique. Si elle fait fond sur la figuration la plus simple de deux triangles dont deux côtés sont parallèles, forme dite de Thalès, on voit bien plus. La construction dans la première figure numérotée 147 du carré cherché *FGIJ* dans le triangle *ABC* s'obtient par similitude du carré *ACED*, le centre de similitude étant O. Joue aussi bien dans la seconde figure le carré qui contient le numéro 148, les lettres des sommets n'étant pas toutes lisibles en raison de la pliure du livre, mais aussi bien le carré *GDLI*. La première similitude est une homothétie (de centre *O*, milieu de *AC*) suivie d'une rotation autour du même point d'un angle plat[32] (symétrie par rapport à ce point *0*), mais ce sont deux simples homothéties (de centre *B* ou *A*) pour les deux cas de la deuxième figure.

Qui pourrait, s'il n'avait déjà fait l'exercice, penser que cette preuve peut donner lieu à une formule intéressante ? En appelant *h (= BD*) la hauteur issue de *B* du triangle *ABC* (fig. 147 de l'ill. 14), et *b* la longueur delà base *AC* de ce triangle, on dispose pourtant de la longueur du côté HI du carré, noté par *x*, selon

(1) $$x = \frac{hb}{h+b}$$

Cette expression est immédiate lorsqu'on la déduit non pas de la similitude géométrique qui a servi de construction, mais de deux similitudes de centres distincts, autrement dit de deux configurations différentes de Thalès. Puisque l'on a $\frac{x}{b} = \frac{HB}{AB}$ de la similitude des triangles *BHJ* et *BAC*, et $\frac{x}{h} = \frac{AH}{AB}$ par similitude des autres triangles *ABF* et *ABD*. De sorte qu'en ajoutant, $\frac{x}{b} + \frac{x}{h} = \frac{HB}{AB} + \frac{AH}{AB} = \frac{AH+HB}{AB} = \frac{AB}{AB} = 1$, ce qui par une autre opération , donne (1) à partir de $\frac{1}{\frac{1}{h}+\frac{1}{b}} = \frac{hb}{h.+b}$. La formule (1) indique en particulier les rôles symétriques de *h* et *b*, mais suggère aussi une autre construction, à partir du constat que *2x* est en fait la moyenne harmonique de *h* et *b* : il suffit de construire une longueur au-delà de *C* et sur le prolongement de *AC* une longueur *CZ= h*, pour en traçant de *C* une parallèle à *ZB* obtenir le point *H* comme intersection avec *AB*.

[32] Je ne peux pas envisager chez ces géomètres la présence d'une homothétie qui aurait un rapport négatif.

La liberté même du carré de départ dans la deuxième figure, est un gage que la démonstration figurée est vraiment une preuve par composition, ou en synthèse comme on disait encore. Sans donc qu'il y ait besoin de supposer le problème d'abord résolu et de faire analyse. La similitude joue à plein par une propriété de conservation : un carré est transformé en un autre carré, et la conservation d'un « intérieur » ou d'une inscription du carré dans le triangle se voit aussi bien en supposant non obtus les angle à la base. La similitude comme transformation géométrique est précisément celle qui évite l'analyse, laquelle requiert de supposer le problème résolu. Dans la première figure facilitée par la propriété d'un triangle équilatéral, la similitude pourrait se réduire à celle disons plus classique des seuls triangles HFO et ECO. Dans la seconde figure, la similitude se voit au sens d'une transformation géométrique qui est l'homothétie de centre E qui produit à partir du carré $EGDI$ le quadrilatère nécessairement carré $KLIH$. La deuxième figure, et je devrais dire la deuxième démonstration figurée, est un apport de Girard à la première figure qui est due à Marolois[33] : Girard s'explique. Ce qui prouve suffisamment que le problème posé l'a été pour susciter la réflexion, et éventuellement établir une critique sur les méthodes précédemment utilisées. C'est sans aucun doute un des avantages du recours à l'histoire que de rappeler que les mathématiques ne sont pas science de recettes, mais aussi bien engagement critique sur la nature des preuves fournies.

L'aspect anachronique de la présentation que je viens de faire est-il gênant ? Dans la mesure où tant d'historiens prétendent qu'il faille attendre

[33] Le traité de Marolois paraît avec ses *Opera mathematica ou Œuvres complètes* , avec des dessins de Vredemann de Vries ("Opera mathematica", ou Oeuvres mathématiques traictans de géométrie, perspective, architecture et fortification, par Samuel Marolois, ausquels sont ajoints les fondements de la perspective et architecture, de J. Vredm. Vriese, augmentée et corrigée en divers endroicts par le mesme auteur, Hagae-Comitis : ex off. H. Hondii, In-fol. oblong, pl. et front. gr., 1615-16) ; Girard le reprend en 1628-29 (Oeuvres mathématiques de Samuel Marolois, traitant de la géométrie et fortification, réduites en meilleur ordre et corrigées d'un nombre infini de fautes écoulées aux impressions précédentes : la géométrie par Théodore Verbeeck,... et la fortification par François Van Schoten,... ,Amsterdam : G. J. Caesius, 1628, 2 parties en 1 vol. in-4° , fig. et pl.). Une seconde édition du traité a lieu en 1651 : "Opera mathematica", ou Oeuvres mathématiques traictans de géométrie, perspective, architecture et fortification, par Samuel Marolois, de nouveau reveue, augmentée et corrigée par Albert Girard, Amsterdam : J. Janssen, 1651, 2 vol. in-fol., pl. et frontisp. gr.

le XIX^e^ siècle, sinon même Felix Klein et la fin de ce même siècle, pour concevoir des transformations géométriques en tant qu'opérant dans l'espace et donc sur des figures entières en conservant précisément certaines propriétés. Ce n'est pas parce qu'il y a eu changement de conception générale sur la géométrie qu'il faut en déduire que plus tôt on ne percevait pas le rôle de la similitude, un peu comme si les mathématiciens d'autrefois ne pouvaient comprendre que le niveau du programme qu'on leur assigne ! Reste plutôt la question de savoir pourquoi cette preuve du XVII^e^ siècle, préfigurée par des démonstrations arabes du même genre, resta enfouie dans une géométrie pratique qui perdait peu à peu toute actualité à se fixer sur les seules méthodes géométriques.

Marolois ne donne d'ailleurs pas la formule (1), qui pourrait aisément s'interpréter en terme de moyenne arithmétique. Le respect de la théorie des proportions requiert d'autres formulations, par exemple celle indiquant que x est la quatrième proportionnelle de h, $h+b$, et b, ou encore celle de b, $h+b$ et h. Mais dites ainsi, quoique suggérant d'autres constructions géométriques, ce sont autant de formules de type algébrique qui sont seulement écrites avec des proportions en place des équations de type polynomial[34].Toute différente est l'attitude du mathématicien qui a donné son nom à l'algèbre. Car dans le *kitābu 'l-mukhtaṣar fī ḥisābi 'l-jabr wa'l-muqābalah* (Abrégé de calcul par la restauration et la comparaison), celui que j'écrirai simplement al-Khawarizmi s'était servi à titre méthodologique du problème du carré dans le triangle. Pour précisément faire jouer une notation algébrique, et notamment indiquer une voie qui conduise à la formule (1). Chez al-Khawarizmi, la résolution consistait à distinguer la « chose », ou côté du carré inscrit cherché, de ce carré même (*mal*). Forcément, compte tenu de la mise en algèbre, le problème devait être supposé résolu, et si l'enjeu de l'analyse était un calcul d'aires, qui contraignait donc à faire intervenir le carré de la « chose », l'objectif était l'obtention d'une équation. L'étonnant, et d'abord imprévisible résultat, est que ce carré disparaisse dans les calculs algébriques menés. Lorsqu'on écrit effectivement comme équation à résoudre l'égalité de l'aire du grand triangle à la somme des aires des trois triangles à laquelle on ajoute l'aire du carré, l'aire du triangle du haut retranche un terme en moitié du carré, et en

[34] Jean Dhombres, Sull'invenzione delle formule matematiche e delle identità delle notevoli, *Bolletino dei docenti di matematica*, 66, Mai 2013, p. 9-28.

fait de même la somme des aires des deux triangles du bas. Ce qui com-compense exactement le terme en x^2 provenant du carré géométrique[35].

Si l'on constate qu'en faisant usage du théorème de Pythagore la méthode d'al-Khawarizmi est basée sur l'application des aires, une méthode dont l'origine est jugée bien antérieure à celle d'Euclide, on ne peut que reconnaître à l'écriture algébrique l'avantage de donner à voir la disparition du terme carré, et peut-être même de la prévoir. Ce n'est pourtant pas que la présence d'un terme carré éventuel serait *a priori* une gêne pour le calcul proprement algébrique, mais en l'occurrence, sa disparition est un moyen pédagogique. Les commentateurs insistent à juste titre sur le fait qu'al-Khawarizmi joue l'algèbre au sens précis où son programme est d'atteindre une équation : il inaugure le genre algébro-géométrique, et le problème du carré inscrit le met en évidence.

Faut-il attribuer plus à la formule ? Si c'est bien Descartes qui fit apprendre à systématiquement prévoir la réduction d'une équation, et aussi bien que l'on trouve anormal de résoudre par un terme carré ce qui ne dépasse pas le premier degré pour l'inconnue choisie qui est ici *x*, il ne s'est exprimé qu'en 1637. Avec Descartes l'algèbre n'était plus la simple obtention d'une équation, mais jeu sur une structure, celle de l'algèbre polynomiale qui n'est qu'une toute petite partie du calcul. Il y a en plus la méthode des coefficients indéterminés. On peut deviner que, dans une tradition cartésienne, et pour le problème d'inscription du carré dans un triangle, on ait abandonné la méthode d'application des aires, pour en venir à une méthode épistémologiquement mieux adaptée. Serait-ce celle donnée

[35] Si l'aire du triangle du haut est évidemment $\frac{1}{2}x(h-x)$, pour la somme des aires des deux triangles du bas, il est utile de faire intervenir deux inconnues de plus, *y* et *z* , liées certes par la relation *b+y+z= b*, et de vérifier ce que j'écris lourdement, mais comme tout débutant en algèbre doit faire.

$$\frac{x}{2}(b-(x+z))+\frac{x}{2}(b-(x+y))=\frac{x}{2}(2b-(x+z)-(x+y)$$

$$=\frac{x}{2}(2b-2x-(z+y))=\frac{x}{2}(2(b-x)-(b-x))=\frac{x}{2}(b-x)$$

On voit alors bien la simplification opérée :

$$\frac{x}{2}(b-h)+\frac{x}{2}(b-x)+x^2=\tfrac{1}{2}bh\ ,$$

soit

$$hx+bh=bh.$$

par Héron d'Alexandrie ? Pourquoi pas, mais même en ce cas, rien ne per-permettrait de considérer le « retour » à un auteur antique comme une réaction. Car la formule dit au plus une dépendance fonctionnelle de type homographique en chacune des variables *b* ou *h*.

Dans les dessins arabes sur le problème, le triangle est isocèle. Mais la symétrie de *b* et *h* devient une évidence si le triangle choisi est rectangle. Cette figure du triangle rectangle intervient comme une figure de base dans le livre classique chinois des *Neuf chapitres sur les procédés mathématiques*. Mais si les exemples y sont donnés sous forme arithmétique, on s'aperçoit que la méthode utilisée est celle des aires, à la façon de al-Khawarizmi et non des proportions. Il n'y a pas d'explication algébrique, quand bien même la voie choisie consiste à supposer le problème résolu.

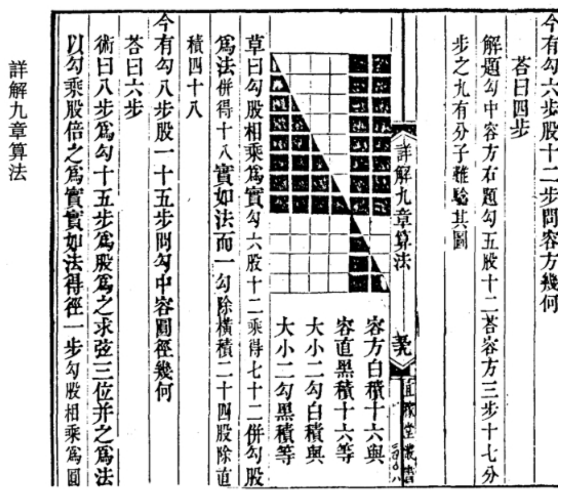

今有勾六步股十二步問容方幾何
荅曰四步
解題勾中容方右題勾五股十二荅容方三步十七分
步之九有分子難驗其圖
詳解九章算法
容方白積十六與
容直黑積十六等
大小二勾白積與
大小二勾黑積等
草曰勾股相乘爲實勾六股十二乘得七十二併勾股
爲法併得十八實如法而一勾除橫積二十四股除直
積四十八
今有勾八步股一十五步問勾中容圓徑幾何
荅曰六步
術曰八步爲勾十五步爲股爲之求弦三位并之爲法
以勾乘股倍之爲實實如法得徑一步勾股相乘爲圓
詳解九章算法

Illustration 16. Explication et figure pour le problème 9. 29 des *Neuf chapitres*, dans la version de Yang Hui en 1261, reproduit en 1993.

Ce problème se trouve dans le neuvième rouleau des *Neuf chapitres sur les procédures mathématiques*, avec les commentaires anciens de Liu Hui, un livre portant sur la base (*gou*) et la hauteur (*gu*) d'un triangle rectangle et lié à ce qui sert d'image fondamentale.

> On somme la base et la hauteur, ce qui fait le diviseur, Base et hauteur sont multipliées l'une par l'autre, ce qui fait le dividende, Et en effectuant la division du dividende par le diviseur, on obtient le côté du carré.

Par contre, le commentaire indique précisément que l'on doit faire une supposition pour entreprendre le calcul, ce qui est préciser la position d'analyse. Je donne alors la version de ce problème sous la forme due à Yang Hui en 1261 dans son *Xiangjie jiuzhang suanfa* (Explication détaillée des procédures de calcul en neuf chapitres), où l'on voit le jeu sur un décompte des aires, avec utilisation de couleurs (ici seulement noir et blanc), selon un procédé assez fréquent qui remplace les notations[36].

> Soit base 6 *bu*[37] hauteur 12 *bu*, combien fait le côté du carré inscrit ?

L'explication du problème (*jie ti*) se fait avec des petits carrés entiers (ill. 19).

> La surface blanche du carré inscrit 16 et les surfaces noires inscrites 16 sont égales. Les deux (triangles) blancs grands et petit et les deux noirs sont égaux.

Je termine ce voyage autour de l'inscription d'un carré dans un triangle avec un auteur, John Leslie, avec un exposé qu'il donna en 1821, dans un Cours de mathématiques, se voulant aussi une introduction à la philosophie naturelle. Il suffit de lire pour comprendre que la situation est celle de la similitude, à la façon dont Marolois l'abordait, et avant Klein. Mais l'auteur écossais s'acharne à défendre deux opérations distinctes, l'analyse et la synthèse, dans la mesure où, par réaction, il ne veut pas parler d'algèbre en géométrie[38]. Voilà une mathématique des classes bien dogmatique. Et comme telle sujette au ridicule. Mais, doit-on condamner ce jeu s'il n'est que destiné à apprendre une discipline du raisonnement ?

36 詳解九章算法 *xiangjie jiuzhang suanfa* (*Explication détaillée des procédés decalcul en neuf chapitres*), 楊輝 Yang Hui, 1261. Cité d'après une réimpression dans *Zhongguo kexue jishu dianji tonghui* 中國科學技術典籍通匯, Henan jiaoyu chubanshe, 1993, vol. 1, p. 981.

37 On peut traduire *bu* par un pas.

38 John Leslie, *Geometrical analysis an geometry of curves*, in *A Course of Mathematics*, W. &C. Tait, Edinburgh, 1821, vol. 2, pp. 15-16.

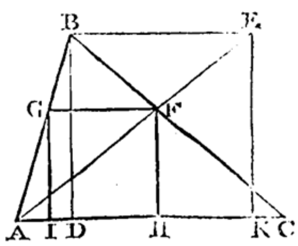
PROP. X. PROB.

To inscribe a square in a given triangle.

Let ABC be the triangle in which it is required to inscribe a square IGFH.

ANALYSIS.

Join AF, and produce it to meet a parallel to AC in E, and let fall the perpendiculars BD and EK.



Because EB is parallel to FG or AC, AF : AE : : FG : EB (VI. 2. El.); and since the perpendicular EK is parallel to FH, AF : AE : : FH : EK. Wherefore FG : EB : : FH : EK; but FG = FH, and consequently (V. 8. and 5. El.) EB = EK. Again, EK, being equal to BD, the altitude of the triangle ABC is given, and, therefore, EB is given both in position and magnitude; whence the point E is given, and the intersection of AE with BC is given, and consequently the parallel FG and the perpendicular FH are given, and thence the square IGFH.

COMPOSITION.

From B draw BD perpendicular and BE parallel to AC, make BE equal to BD, join AE, intersecting BC in F, and complete the rectangle IGFH.

Because BE and EK are parallel to GF and FH, AE : AF : : BE : GF, and AE : AF : : EK : FH ; wherefore BE : GF : : EK : FH; but BE = EK, and consequently GF = FH. It is hence evident that IGFH is a square.

Révolutionner les habitudes culturelles

Je passe justement à la question des jeux, puisque m'y introduit la remarque précédente sur le travail de Leslie. Jusqu'au début du XVIIIe siècle, le jeu mathématique, c'est-à-dire une activité ludique pour laquelle des stéréotypes mathématiques sont engagés (nombres, numérations et énumérations, équations, combinaisons de situations), se caractérise par une fixité de type algébrique sur des questions de caractère arithmétique. Fait référence le plus souvent un calcul modulo un nombre, et on pourrait peut-être dire que la preuve par neuf est typique du jeu disponible. Un exemple parmi tant d'autres consiste à demander d'écrire deux nombres en utilisant tous les chiffres entre 0 et 9 une fois seulement, d'additionner et d'effacer un des chiffres dans le résultat. Le meneur de jeu est capable de dire le nombre qui a été effacé, aussitôt qu'on lui présente ce qui reste[39]. Voici un autre jeu dit de société. On demande de choisir de tête un nombre à trois chiffres, et celui qui a fait ce choix doit, sans rien dire de ses résultats, doubler le chiffre des centaines et ajouter 5, puis au résultat ajouter le chiffre des dizaines et multiplier par 10, enfin ajouter le chiffre des unités : il prononce le résultat. Le meneur du jeu n'a qu'à soustraire 250 pour pouvoir énoncer le nombre qui fut effectivement choisi[40]. Une formule algébrique, lorsque le choix *abc* a été fait, résume le geste du meneur.

$$100a + 10b + c = [(2a+5)5+b]10+c-250.$$

Et saute aux yeux de qui sait un rien d'algèbre la convention même de l'écriture décimale, qui est de nos jours enseignée bien avant l'algèbre, et ce depuis la Révolution française. Pourquoi n'est-elle pas appelée "jeu" la parole qui consisterait à dire : "multiplie le chiffre des centaines par 100, celui des dizaines par 10 et ajoute le chiffre des unités. Tu auras le nombre cherché" ? C'est, qu'ainsi présenté, le soi-disant jeu coïncide avec l'explication du décimal que retient l'enseignement.

L'enseignant d'aujourd'hui sait que le décimal n'est en rien naturel, mais il sait que son rôle de maître est de le faire passer pour tel. Il sait que le décimal est venu à l'esprit des mathématiciens arabes au terme d'une algèbre polynomiale bien maîtrisée, alors que le décimal est désormais rangé dans l'arithmétique et dans un élémentaire du calcul [41]. Le jeu de

[39] Je suis bien incapable de faire l'histoire de ce jeu, ou plutôt de dire quand apparaît ce tour particulier. A-t-il été inventé au XVIIe siècle, alors que Pascal expliquait des questions numériques en se servant de l'analogie de l'heure, donc en faisant un calcul modulo 12 ?
[40] Ce jeu est attesté à la fin du XVI^{e} siècle, lorsque Simon Stevin signale l'avantage de la numération décimale, et ce qu'elle change aussi bien pour la conception du nombre.
[41] L'origine savante du système décimal a-t-elle jamais été occultée ?

société ci-dessus expliqué n'a donc pas un avantage dirimant sur l'enseignement usuel, et il apporte une complication algébrique. Elle est jugée mathématiquement inutile.

C'est bien le summum de la mathématique euclidienne présentée dans ces jeux, et ils ne sont pas du tout attestés dans une pratique de classe. La géométrie euclidienne est absente dans l'immense majorité des cas, alors qu'elle faisait le fond de l'enseignement des mathématiques. De tels jeux ont-ils pu manifester une innovation bien antérieure, et qui est celle de l'algèbre ? On le justifierait par les anecdotes qui font trouver l'âge d'un mathématicien au terme de la résolution d'une équation. Je ne connais guère d'histoire des mathématiques qui ne reprenne la vieille histoire grecque donnant l'âge de Diophante sous forme de rébus, ou plutôt disposant l'histoire sous forme additive de sorte que la mise en algèbre soit automatique. Le récit lui-même n'est pas jeu, mais enseignement : l'enfance de Diophante dura le sixième de sa vie, et la barbe ne lui vint qu'après un autre douzième de sa vie, alors il se maria un septième de sa vie plus tard, et un fils lui vint cinq années plus tard, dont le temps de vie fut la moitié exacte de celle du père, qui survécut quatre années à son fils ($\frac{1}{6}x+\frac{1}{12}x+\frac{1}{7}x+5+\frac{1}{2}x+4=x$). Je me demande si la résolution purement arithmétique de la devinette n'était pas au fond celle attendue, à savoir trouver un plus petit commun multiple aux dénominateurs mentionnés au cours du récit, 6, 12, 7, et 2, soit 84, et vérification qu'avec 84 on disposait de la solution. L'algèbre apporte la vérification qu'il ne peut y avoir aucun autre âge pour Diophante. Mais le seul fait de raconter avec des indications précises, signale dans un cadre d'enseignement que la réponse est possible, unique, parce que toutes les indications sont pertinentes et qu'il n'en manque aucune. Prendre la résolution de la devinette comme un jeu empêcherait de comprendre pourquoi la résolution est effectivement possible en suivant strictement toutes les données de l'énoncé. Ceci est l'opinion que je me fais pour expliquer l'absence de jeux vraiment algébriques dans les jeux dits classiques.

Que l'innovation qu'est l'algèbre puisse avoir été connue du public par les jeux, reste une question historique et elle attend une étude plus précise ; non pas un développement nouveau sur l'origine de l'algèbre, car c'est un des thèmes les plus courus de l'histoire des mathématiques, mais l'étude de la forme de sa diffusion dans l'enseignement, et la reconnaissance par la culture que l'algèbre apportait une nouvelle forme de pensée que l'on pou-pouvait jouer.

Ill. 17. Une leçon de géométrie analytique de Descartes dans un cadre mondain.

Ce qui fait vraiment histoire pour le moment, c'est qu'à partir du XVIIe siècle, toute mathématique nouvelle n'entraîna pas composition de nouveaux jeux (algèbre des polynômes, logarithmes, géométrie des courbes, enveloppes, caustiques, méthodes infinitésimales ensuite, calcul différentiel et intégral plus tard). Du point de vue de l'historien, la richesse d'un répertoire de jeux anciens tient plus à l'habillage rhétorique, et à la présentation du jeu qui renseigne sur des habitudes sociales ou des représentations, et très peu au substrat mathématique. Aujourd'hui, tout a changé en ce que les jeux mathématiques utilisent bien plus de disciplines mathématiques, et qu'une discipline mathématique même est née avec la théorie mathématique des jeux[42]. C'est cette discipline, avec sa recherche

[42] Il me semble que dans leur grande majorité les jeux mathématiques actuellement connus ne soient pas antérieurs au XIX^e siècle, et Sam Loyd paraît un fondateur. Son successeur,

des stratégies gagnantes, qui a fait entrer les jeux mathématiques dans l'enseignement ; on peut aussi penser que des jeux mathématiques indépen-indépendants de l'enseignement ont pu, au XIX^{e} siècle, conduire à cette discipline. Il me semble qu'en tout cas c'est avec le XIX^{e} siècle que peut commencer une histoire proprement mathématique des jeux mathématiques.

Puis-je alors considérer comme un jeu le problème dit des trois cercles, qui avait été splendidement reconstruit en 1600 par Viète selon une progression successive de problèmes et par une remarquable utilisation de la similitude. Dans un livre perdu, Apollonius avait expliqué comment trouver, trois cercles étant donnés dans un plan, un cercle qui leur soit tangent à tous les trois. Surprend la sincérité avec laquelle Descartes le traite, car bien loin de faire valoir le calcul algébrique avant toute autre chose, et concédant qu'il lui suffit d'avoir compris le niveau algébrique que le problème requiert, il donne d'abord à voir une fausse bonne piste.

> Mais ce chemin me semble conduire à tant de multiplications superflues que je ne voudrais pas entreprendre de les démêler en trois mois. [43]

Descartes réfléchit à haute voix à l'intention de son illustre élève, la princesse Elisabeth, fille de Frédéric V, ce roi de Bohème d'un hiver. Et manifeste l'avantage *a priori* de la géométrie repérée par des axes orthogonaux.

> J'observe toujours, en cherchant une question de géométrie, que les lignes dont je me sers pour la trouver, soient parallèles, ou s'entrecoupent à angles droits, le plus qu'il est possible ; et je ne considère point d'autres théorèmes, sinon que les côtés des triangles semblables ont semblable proportion entre eux, et que, dans les triangles rectangles, le carré de la base est égal aux carrés des côtés. [44]

Il indique que son analyse n'est pas sans fin, qu'elle permet justement de « ne plus passer outre ». Car suffit la considération du degré du problème considéré, ici deux, donc d'établir que le problème géométrique, lorsque possible, est résoluble à la règle et au compas. Du moins si on se réfère à l'ordre qu'il a su établir dans sa Géométrie.

> Enfin, retournant à l'une des trois premières équations, et au lieu d'*y* ou de *z* mettant les quantités qui leur sont égales, et les carrés de ces quantités pour *yy*

Edouard Lucas, dans ses *Récréations mathématiques* (Paris, 2e édition, Gauthier-Villars, 2 vol. 1891), présente les jeux nouveaux sous une forme d'explications par des théorèmes, et crée véritablement le genre du jeu mathématique comme enseignement.

[43] Lettre de Descartes à la princesse Elisabeth, novembre 1643, *Œuvres de Descartes*, tome IV, p. 39, orthographe modernisée. Référée par *Lettre à Elisabeth*.

[44] *Lettre à Elisabeth*, p. 38.

> et *zz*, on trouve une équation où il n'y a que *x* et *xx* inconnus ; de façon que le problème est plan, et qu'il n'est plus besoin de passer outre. [45]

Descartes succombe quasiment toujours à son esprit indépendant et altier, pour indiquer sur le champ que c'est la géométrie elle-même qui est un jeu, et non l'algèbre qui conduit la solution

> que le surplus, qui consiste à chercher la construction et la démonstration par les propositions d'Euclide, en cachant le procédé de l'Algèbre, n'est qu'un amusement pour les petits géomètres, qui ne requiert pas beaucoup d'esprit ni de science. [46]

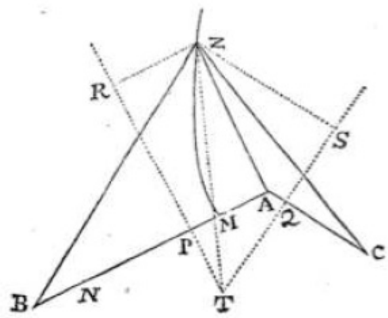
Newton répondra vingt quatre ans plus tard au premier livre de son *opus magnum*, les *Principia mathematica philosophiae naturalis*, et directement sur le problème des trois cercles, en montrant sa possible résolution à la règle et au compas, n'utilisant pas les opérations de l'algèbre et ses équations. Car Newton réduit la construction à des droites associées géométriquement à des hyperboles. Il ne se sert que des proportions, par exemple pour représenter une droite, justement là où Descartes les remplaçait par des équations polynomiales. Cet exemple est sans doute le plus beau lieu où ait pu se discuter le rôle des proportions dans la pensée. Comme Descartes, Newton dans son « Arithmétique universelle » publiée en 1707 seulement, et donnant une autre construction du problème des trois cercles, exagère radicalement sa position.

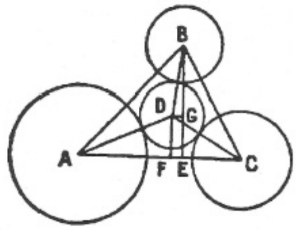
> Une équation est, en général, l'expression d'un calcul arithmétique, où l'on prononce que quelques quantités sont égales à d'autres. Une équation ne peut être géométrique qu'autant que les quantités qu'elle contient sont géométriques, telles que lignes, surfaces, solides ou proportions. C'est par une innovation des modernes qu'on y a fait entrer des multiplications, des divisions, et d'autres calculs de cette espèce ; et cette innovation n'est pas heureuse ; elle répugne aux premiers principes de la science. [47]

Ici, il m'importe d'analyser précisément la façon de travailler de Descartes avec la méthode des coordonnées pour le problème des trois cercles, dont je dois rappeler qu'elle ne nous est connue que par ce qu'il en donne à lire dans ses lettres à Elisabeth. Mais il m'a semblé qu'à l'encontre de tant de commentateurs, je ferai mieux comprendre l'originalité de

[45] *Lettre à Elisabeth*, p. 42.
[46] *Lettre à Elisabeth*, p.
[47] Isaac Newton, *Arithmétique universelle*, trad.. fr. Noël Beaudeux, Paris, 1802.

Descartes, jusque dans sa portée didactique, en donnant en premier la façon de Newton dans les *Principia.*





Ill. 18 et 19. Figure donnée par Newton au livre I des *Principia*, en justification du lemme XVI, dans une section consacrée au rôle joué par les foyers dans le travail sur les courbes coniques (éd. originale, p. 67). Figure utilisée par Descartes dans sa correspondance avec la princesse Elisabeth pour le problème avec trois cercles deux à deux tangents.

Je repère par la figure donnée par Descartes, supposant tout comme Newton le problème résolu, et exploitant la situation des trois cercles donnés, de centres respectifs *A, B* et *C* et dont il est pratique d'appeler *a*, *b*, et *c* les rayons respectif. L'objectif devant être d'exprimer à partir de ces données le centre *D* du quatrième cercle et de son rayon que Descartes note *x* à la suite de ce que lui proposait la princesse. On voit aussitôt les relations *AD= x+a*, *BD= x+b* et *CD= x+c*, du moins dans la configuration envisagée pour les tangences dites extérieures des cercles, mais pouvant être adaptées en donnant un signe aux rayons. Newton fait disparaître *x*, omet les questions de signe, en exprimant que *D* est sur l'hyperbole de foyers *A* et *B*, dont le grand axe est la différence entre le plus grand et le plus petit des rayons *a* et *b*. Il s'évite d'avoir à décider s'il s'agit de *BD-AD* ou de *AD – BD*. Naturellement, par permutation circulaire, interviennent aussi bien deux autres hyperboles. L'idée est de remplacer ces courbes par des droites, en l'occurrence leurs directrices. De façon précise, sa figure (ill. 18) qui pose *Z* en place de *D*, introduit le point *P* sur *AB*, et en pointillés la directrice de l'hyperbole (courbe dessinée selon *ZM*), venant orthogonalement en *P*, et le point *Q* sur *AC*, donnant de même la directrice posée orthogonale à *AC*. Ces deux directrices se coupent en *T*. Ce sont les proportions, en utilisant l'excentricité (donnée par le rapport de *MN* à *AB*), qui font que *P, Q* et *T* sont effectivement constructibles à la règle et au compas. Le point *Z* est effectivement repéré par orthogonalité, à partir des projections *R* et *S* sur chacune des directrices, et une fois encore les proportions sont utilisées pour établir que le rapport de *ZR* à *ZS* est connu à partir des seules données du problèmes, ce qui en quelque sorte sert d'équation à la droite issue de *T* sur laquelle se trouve *Z*. Il reste à déduire la longueur *ZT* à partir de ces seules données, ce qui dépend seulement des proportions dans les triangles. Du moins si l'on y est habitué, plus facile peut-être si l'on y met de la trigonométrie, ce que Newton ne fait pas. L'affaire est faite aux yeux de Newton, qui fait une référence à la reconstitution du problème d'Apollonius, mais pas plus que Descartes ne donne une construction explicite. Sa règle d'intelligibilité est la simplicité des proportions, explicitant aussi bien des longueurs (ainsi de la longueur *TZ*) que des directions (la droite *TZ*).

Descartes a pour règle d'intelligibilité les équations, mais précisément celles-ci ne viennent pas n'importe comment, et on doit prévoir leur intervention pour acquérir a plus grande simplicité, qui n'est pas un affichage, mais une recherche. Sans doute contraint par la princesse, qui veut travailler avec le seul rayon *x*, Descartes donne d'abord une leçon de calcul, et justement prend appui sur une formule des *Metrica* de Héron d'Alexandrie, donnant l'aire d'un cercle en fonction des trois côtés. Car l'équation dès lors la plus naturelle, provenant de la vieille méthode d'application des aires, est celle qui résulte de l'addition des aires des trois triangles *ADB, DBC, DCA*, égale à l'aire de *ABC*, ces aires étant ainsi directement exprimables à partir des données. On trouvera bien sûr une équation portant sur x, mais son degré ne signifie rien, car il faudra supprimer bien des racines carrées, et donc beaucoup multiplier. Tel est le sens de son premier avertissement.

Dès lors, il propose véritablement la méthode des coordonnées en rappelant qu'il est judicieux d'introduire d'autres inconnues que *x*, comme la position de *D*. Sitôt dit, sitôt fait, et Descartes pose *DG= z* et *DF= y* ; il a donc pris le risque d'une rupture de symétrie du rôle des trois cercles, en prenant pour axes orthogonaux la hauteur issue de *A* sur la base *AC*, fixant donc des valeurs de repérage comme *AE= d, BE = e, CE= f.* La figure (ill. 19) est remarquablement explicite. A ces trois inconnues, il faut au moins trois équations. Il paraît naturel d'utiliser le théorème de Pythagore sous sa forme algébrique, même s'il fait jouer des carrés, mais encore convient-il de choisir les bons triangles. Evidemment ils doivent contenir *DA, DB* et *DC*, de sorte que *ADF, BDC* et *DCF* s'imposent ; surtout des relations de symétrie entre les carrés de *z* et *y* doivent *a priori* permettre des simplifications dans le traitement ultérieur des équations. Tel est le jeu de la prévision cartésienne.

$$a^2 + 2ax + x^2 = d^2 - 2dz + z^2 + y^2$$

$$b^2 + 2bx + x^2 = e^2 - 2ez + y^2 + z^2$$

$$c^2 + 2cx + x^2 = f^2 + 2fz + z^2 + y^2$$

Tout est joué si l'on respecte cette fois l'ordre de résolution de Descartes, puisqu'en soustrayant la première de la troisième, on explicitera *z*, d'ailleurs sous une forme affine en *x* à laquelle Descartes a habitué son fidèle lecteur de la géométrie et en soustrayant la première et la seconde on explicitera *y*, tenant compte de l'expression précédente de *z*, donc ayant encore une forme affine en *x*. Si l'on reporte ces valeurs dans une quelconque des équations, vient évidemment une équation du second degré en *x*. C'est alors qu'il énonce non sans soulagement : « il n'est plus besoin de passer outre ». Et fustige cette fois, non les géomètres opiniâtres qui recèlent leurs méthodes, mais bien les calculateurs qui n'ont plus à en avoir

> Car le reste ne sert point pour cultiver ou recréer l'esprit, mais seulement pour exercer la patience de quelque calculateur laborieux.[48]

Notre temps en juge autrement qui témoigne d'un « théorème du cercle de Descartes »[49]. Car ce dernier n'en propose pas moins le calcul à la princesse, lui suggérant toutefois une simplification des données, en supposant tangents deux à deux les trois cercles de départ. Il conserve assurément le repérage orthogonal avec la hauteur. Le calcul moderne, qui tient mieux compte des signes selon la nature des tangences, écrit deux relations dont on ne dénigrera aujourd'hui, ni la simplicité, ni l'inefficacité, ni bien sûr l'élégance.

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{x^2}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\pm\frac{1}{x}\right)^2.$$

Newton n'a pas obtenu de résultat de cette sorte, etreprochera en quelque sorte à Descartes d'utiliser le théorème de Pythagore sous sa forme algébrique, celle issue comme nous l'avons dit d'une interprétation de l'avant dernière proposition du livre I des Eléments

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[48] *Lettre à Elisabeth*, p.

[49] C'est l'expression de D. Pedoe, On a theorem of geometry, *The American Mathematical Monthly*, 1967, p. 627 ; reprise par Coxeter dans the *American Mathematical Society* l'année suivante (p. 5).

d'Euclide, en n'allant pas à ce qui fait la « bonne » interprétation, qui est celle du livre VI, avec la théorie des proportions et les triangles semblables. Quoiqu'on pense de tels arguments, Descartes a certainement réfléchi à ce jeu sur les proportions. Il maintenait, nous l'avons lu, l'usage du théorème de Thalès, alors même que la réduction à des équations faisait disparaître les manipulations des proportions. Et restant dans l'ordre algébrique des calculs, il a trouvé une très jolie défense dans l'idée d'homogénéité avec l'écriture polynomiale, bref jouant la cohabitation entre les données et les variables, ou inconnues. Ce qui se traduit par une écriture comme *aaax* ou *bbxx* pour les termes intervenant dans un polynôme du quatrième degré : cette homogénéité devient un moyen de vérification de la qualité du calcul. Dans sa lettre à Elisabeth, Descartes insiste :

> Il est bon aussi alors d'observer que les quantités qu'on dénomme par les lettres, aient semblable rapport les unes aux autres, le plus qu'il est possible ; cela rend le théorème plus beau et plus court pour ce qui s 'énonce de l'une de ces quantités s'énonce en même façon des autres, et empêche qu'on ne puisse faillir au calcul, pour ce que les lettres qui signifient des quantités qui ont même rapport, s'y doivent trouver distribuées en même façon ; et quand cela manque, on reconnaît son erreur.[50]

Conclusion

En conclusion, je ne veux pas répéter le jeu des réactions. Partant d'un autre constat, celui d'une révolution de nature culturelle qui sous-tendait le projet des "mathématiques modernes" : c'était celle de l'éducation de masse pour laquelle on pensait que la remise en ordre des mathématiques empêcherait les seuls "héritiers" de benéficier du support familial et du pouvoir symbolique de la connaissance à l'ancienne des propriétés dans le triangle, des démonstrations par polaires réciproques en géométrie, ou des identités remarquables, etc. Il fallait donc présenter les mathématiques comme une métaphysique, presque un *a priori* de la pensée. Parce que porté par les politiques et les enseignants de mathématiques, ce mouvement français s'accompagna de la création institutionnelle des Irems, ces organismes qui heurtaient de plein fouet le monde universitaire par l'alliance toujours incongrue, quoique encore syndicalement et donc politiquement révendiquée, des diverses strates de l'enseignement de la Maternelle à l'Université, donc incluant la recherche, dérangeant aussi bien

[50] *Lettre à Elisabeth*, p.

l'Association des professeurs de mathématiques de l'enseignement public (APMEP), qui eut la force pourtant de le soutenir grâce à des hommes comme Gilbert Walusinski[51]. Il devait quand même y avoir une contradiction entre une autarcie de la discipline pour des raisons de fond, et la pratique professorale dans les classes qui fait que la mathématique n'est pas seule enseignée. Je suis sûr qu'il y aurait une histoire voisine, mais sans doute différente en Grèce. Mais quoiqu'il en soit, je sais que c'est bien par crainte de ne pas avoir suffisamment de relais auprès des enseignants de mathématiques que Stevin n'imposa pas sa réforme décimale, avec sa file de chiffres séparés par une virgule, le jeu des millièmes ou des milliers, apparue assez soudainement en Europe à la fin du XVI^e siècle dans *La disme*. Le petit livre parut en 1585 à Leyde[52], porteur aussi bien de la numération binaire illimitée dont jouent nos ordinateurs.

Rares, très rares sont les chercheurs tels Marcel Granet pour la Chine ancienne, qui ont su exprimer le jeu numérique et combinatoire, donc mathématique, dans l'expression des mythes dominants d'une civilisation[53]. Et on n'a peut-être pas fait pour l'Europe l'analyse culturelle du rôle de la fraction, s'opposant au décimal. Qu'ajoute-t-on quand, à la manière de Fernand Braudel, on dit la réforme décimale favorisée par le mouvement du très long terme de la quantification et de la pratique bancaire ? La difficulté est certes que les deux rythmes, celui de l'invention mathématique et celui de la marchandisation, ne battent pas à l'unisson : le décimal, déjà connu par exemple dans le monde arabo-musulman à partir de pratiques polynomiales, sera par ailleurs mal reçu en Europe par ceux pour qui il serait objectivement de la plus grande utilité[54], et il faudra une loi en France pour l'imposer à tous deux siècles plus tard. C'est la loi du système

[51] Son rôle a été mis en valeur par Eric Barbazo dans sa thèse soutenue à l'EHESS en 2009, L'APMEP de 1910 à 1970. Voir Eric Barbazo, Pascale Pombourcq, *Cent ans d'APMEP*, brochure de l'APMEP, n° 192, 2010. Des textes plus fouillés en sont issus aujourd'hui. Voir aussi un témoignage de l'inspection générale, Pierre Legrand, Dans la tempête des maths modernes, in Jean-Pierre Rioux, *Deux cents ans d'inspection générale*, Paris, Fayard, 2002, pp. 287-305.

[52] *La Disme*, un livre qui est inclus dans *L'Arithmétique de Simon Stevin de Bruges*, Christophe Plantin, Leyde, 1585. Le texte a souvent été réédité à l'usage des écoliers. Mais je crains qu'il soit d'emblée difficile de le trouver sur le Net.

[53] Marcel Granet, *La pensée chinoise*, Paris, 1934, réédition Albin Michel, Paris, 1968.

[54] Les rétifs au décimal, alors que les banquiers l'adoptent pour les tables d'intérêt, furent d'abord les universitaires (alors même que cela servirait à imposer l'algèbre dont ils savent les avantages), mais les astronomes aussi qui gardent le système sexagésimal.

métrique décimal, avec sa kyrielle des millimètres ou kilomètres récitée depuis l'école primaire ; on l'épinglera sous l'adjectif « républicain », preuve s'il en est qu'une réforme mathématique joue à tout le moins sur les formes de représentation d'une société. A l'époque, celle des Montagnards et des Girondins, on pensait que ce système avait quelque valeur morale en permettant à tous de juger des quantités, justement, universellement et uniformément. La réaction, il faut le rappeler, est dans l'appellation anglaise, *imperial,* pour les *measures* restées non décimales jusque dans la seconde moitié du XX^{e} siècle, et donc symboliquement jusqu'à la fin du système impérial des colonies[55]. Mais ce système de pence, de shillings et de pounds, sans parler des crowns, n'a-t-il pas un effet particulier, celui de protéger le champ marchand ? Des pédagogues ont alors prétendu qu'il incitait les gens à penser mathématiquement par la difficulté même du système, en comparaison de la facilité du décimal ! Voilà un exemple même de pensée réactionnaire, vraisemblablement nationaliste. Elle n'en mérite pas moins l'attention du didacticien et du psychologue.

Par sa volonté d'être scientifique, dans son observation d'une classe ou de l'apprentissage des notions mathématiques, la didactique se condamne, comme bien des disciplines mathématiques elles-mêmes, à une autarcie. J'ai essayé, par des exemples anciens, et tous ne dépassant pas le niveau des études secondaires, avec l'algèbre ou la similitude pour le carré inscrit dans un triangle, de montrer que les changements, dans les classes comme dans la recherche, obéissent aussi à des facteurs extérieurs, que ce soit avec la vogue du numérique à l'âge de l'exploitation des grandes découvertes, ou de l'apparition du calcul algébrique comme nouvelle forme de logique explicative. Le dire ainsi n'est pas prétendre donner la raison sociale ou idéologique de ces influences, mais en ne les éludant pas, en évitant d'affirmer que les mathématiques sont entièrement autonomes dans leurs changements, en parlant même de réactions, de permettre une compréhension du mode de fonctionnement des mathématiques et de leur enseignement.

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[55] Jean Dhombres, "Mesure pour mesure, universel contre régional : le système métrique comme action révolutionnaire", in A. Jourdan, J. Leerssen, *Remous révolutionnaires : République batave, armées françaises*, Amsterdam, 1996, pp. 159-199 ; Jean Dhombres, Résistances et adaptations du monde paysan au système métrique issu de la Révolution : les indices d'évolution d'une culture de la quantification, in A. Croix, J. Quéniart (éd.), La culture paysanne (1750-1830), *Annales de Bretagne et des Pays de la Loire*, 100, n°4, 1993, pp. 427-439.

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The Future of Mathematics: From the Pure-Applied Debate to Reality

Karl Gustafson
Department of Mathematics
University of Colorado at Boulder, USA

Abstract

I will look at Mathematics from Past to Present to Future. As each of us has a different and incomplete knowledge of each of those, I will employ the so-called case-studies approach, selecting key examples to illustrate and from which to learn. The narrative will also include some of my personal experiences. I will conclude by advancing a new mathematical ideal: *TriMathlete.* For such mathematical individuals, the future is bright indeed.

1. Introduction

This paper is an elaboration of my Keynote lecture for the Hellenic Mathematical Society at Veroia, Greece in November 7-9, 2014. I had visited Veroia previously in 2011 and 2012 to lecture in the Web Science Master Course. I was especially surprised and delighted in 2011 when two students on their own initiative organized a private visit to the nearby recently uncovered tomb of King Phillip II of Macedonia, father of Alexander the Great. It was quite spectacular. One enters the earthen tumulus to find a fantastic hidden world of undesecrated golden caskets, delicate crowns, and unique tomb wall paintings.

Greece is the cradle of mathematics and of scientific inquiry. The two are not the same, and a tension already present from the beginning continues to this day. A central tenet of this paper is that the future is bright for mathematicians who can overcome that tension.

Because the general topic of the future of mathematics is so vast, I will rely on a few key "case studies" that have caught my attention over the years to assist in making my points. To bring these to life, some of my own experiences will be woven into the narrative. I commit to making every effort in this paper to refuse to indulge in rhetoric or submerge the reader in philosophy.

As in my lecture, the following sections we take up in order are: Mathematics, the Past, the Present, the Future; The Problem of Dogma illustrated by a case study from my own institution; and Conclusions, including a proposed new ideal. In this paper I will not discuss my recommended solutions to the general institutional problems, although for those I do have some potential remedies.

2. Mathematics

Contemporary philosophers often raise the question: Was mathematics created or discovered? Of course this matter goes back at least as far as Plato, who would argue in favor of the latter. A good example is the 1995 Changeux-Connes debate [1]. I discussed that debate in [2, Section 8] from the standpoint of free will versus determinism. Going beyond that issue, I concluded, "In sum, neither of the debaters can escape their own limited experience base and so their opinions and positions are automatically prejudiced."

Nonetheless we must establish some guidelines for our words here. I shall take Mathematics to include all three of its aspects identified in my title, namely: Pure mathematics, Applied mathematics, and Reality mathematics. Rather than impose constraining definitions, I prefer to differentiate by means of descriptors. Pure mathematics for example connotes definitions, Applied mathematics connotes equations, and Reality mathematics connotes science. Another set of descriptors could be, respectively, proofs, solutions, and larger societal value. Still another set of descriptors could be, respectively, single-disciplinary, inter-disciplinary, multi-disciplinary. I will further develop this parsing of mathematics, and correspondingly mathematicians, in this paper. You can then choose your own sets of differentiating descriptors as you progress through reading. Or you can just sit back and consider your colleagues and pretty well place each of them with enforced one-bit precision into one of these three classes. Of course we could go to second level descriptors and conjunctions and endlessly finer Venn diagrams, and then even get lost into semiotics.

The point, however, is to begin a conversation about the immediate future of mathematics and, as a corollary, of mathematicians. To do so, I have looked at Mathematics in three historical periods. The Past takes us back to you, the Greeks. The Present I have chosen as encompassing the Twentieth Century, up until now. The Future takes us forward into the Twenty-first Century.

3. The Past

In the Fall of 1980, Carl Sagan presented a thirteen-part television series called *Cosmos* [3], which absolutely captivated the American public. That was also the political season that saw Ronald Reagan charm his way into the presidency of the United States of America and thereby pave the way for his Star Wars initiatives. It was also the year in which I confronted the Mathematics Department at Colorado with its need to revive our dormant Ph.D. program in Applied mathematics.

I still remember Episode 7 of *Cosmos:* The Backbone of Night. That phrase was used by the Kung Bushmen of the Kalahari desert for the Milky Way, as their explanation for what holds up the sky. However, most of Episode 7 is Sagan's tale of how scientific inquiry and mathematics originated in Greece in the 6^{th} to 4^{th} centuries B.C. You can read Episode 7 online via a number of sources.

When this episode first aired on November 7, 1980, I was struck memorably by how Sagan presented the issue of the tension that exists between abstract thought and scientific technology. He portrayed Theodorus as a master engineer, exemplifying the great science of the Ionians. In contrast, he portrayed the Pythagoreans and Plato as believers in Nature to be seen by pure thought. Sagan then took sides and accused the Pythagoreans of using the dodecahedron as a mystical device to advance their own political power. He continued further and argued that their extinguishing the light of experimental science, coupled with the later restricting policies of Christianity, inhibited scientific progress for 2,000 years, until Kepler and Galileo entered the scene.

There are many interesting observations in Sagan [3] that give great credit to many Greeks for effectively lost original scientific ideas that had to be rediscovered or recreated later. As to the politics, I checked a standard reference [4] and did not find any major discrepancies in Sagan's account.

Recalling this episode from *Cosmos* brings to mind a more recent article regarding Aristotle. Michael Rowan-Robinson [5] defends Aristotle against the attacks nearly 2,000 years later by Galileo and others. As everyone in this audience knows, Aristotle was a student of Plato and was brought to Macedonia by King Phillip II to tutor the young Alexander. In the same year, 336 B.C., that Phillip was assassinated and Alexander became King, Aristotle transferred to Athens and set up the Lyceum. Although the Lyceum would ultimately differ from Plato's Academy, which emphasized a rather elite view of knowledge, Aristotle followed the same tendencies and looked to metaphysical reasoning to discover Nature's laws.

Rowan-Robinson naturally brings his own conditioning and prejudices to his findings. He is an astrophysicist. I found his cut "…mathematicians, who love to imagine that their ideas represent some underlying reality about the universe" quite amusing. Aristotle is defended for his accounting for a real, viscous, frictional, velocity-driven world, whereas Sir Isaac Newton's later contribution was to bring in acceleration.

The noted physicist Yuval Ne'eman [6] chose to attack the thesis of [5]. He argues that the Pythagoreans had already worked out the harmonic nature of a vibrating string with fixed ends. He goes on to postulate that as the beginning of quantum mechanics. I find that quite a stretch—that classical music is quantum mechanics. No matter. Then Ne'eman goes off on an attack on the Bible and the Koran. There Sagan [3] would likely agree.

I found Ne'eman's take interesting for another reason. By chance (I am in the same volume) I was aware of his article [7] in the proceedings of the XXII Solvay Conference on Physics held at Delphi. There he attacks both irreversibility models in quantum physics and non-unitarity in quantum field theory. On the other side, he also again asserts the damage to scientific progress caused by the dogmatism of the Christian, Muslim, and Jewish religious institutions during the Dark Ages.

What are we to make of all such argumentations, all such debates? In my recent autobiography [8] I formulate a principle that, notwithstanding politicians' standard advice to address only *issues,* and not *motivations*, if you want to better understand what is really going on in any debate, it is indeed useful to discern the underlying motivations, and if necessary, bring them to the surface. Among scientists and mathematicians, beneath the rhetoric is often a compulsive seeking of recognition for themselves or for the scientific persuasion within which they reside.

Ne'eman has been generally credited with key parts of the standard model, but was not included in the Nobel Prize in Physics which went to Murray Gell-Mann in 1969. Irreversibility contributed to the Nobel Prize in Chemistry for Ilya Prigogine in 1977. Non-unitarity in quantum field theory was part of the Nobel Prize in Physics for Steven Chu in 1997. A second motivating factor is that Ne'eman's lifelong work on the standard model, with all its symmetries, preconditions his position to be against departures from the (mathematical) beauties of unitarity-based physics.

We cannot leave Greek mathematics without mentioning Archimedes. According to the noted historian E.T.Bell [9], Archimedes was one of the three greatest mathematicians of all history, along with Newton and Carl Friedrich Gauss much later. There is no need for me here to go into all of Archimedes' contributions to science, engineering, and mathematics. Nor to attempt here any detailing of the contributions to physics and mathematics of Newton. Nor all of those of Gauss, who is sometimes considered the foremost of all mathematicians.

However, as an amusing and instructive final case study for this section, I do wish to refer to an account in Bell's famous book, notably [9, pages 238-242] in his chapter on Gauss. Bell states, "Probably all mathematicians today regret that Gauss was deflected from his march through the darkness (on Fermat's Last Theorem in number theory) by 'a couple of clods of dirt which we call planets'—his own words—which shone out unexpectedly in the night sky and led him astray." Bell deplores the fact that Gauss found more intriguing the problem of computing the approximate orbits of the just-discovered dwarf planets Ceres and Pallus. Somehow Bell seems to forget his passage just above where he quotes Gauss as saying, "But I confess that Fermat's Theorem as an isolated proposition has very little interest for me, because I could easily lay down a multitude of such propositions, which one could neither prove nor dispose of." Bell ignores Gauss's stated opinion that pure mathematics was not important enough to deserve continuing support from the Duke of Brunswick. Through his results in astronomy, Gauss obtained a permanent professorship in Gottingen. Moreover, from that work Gauss gave us the method of least-squares approximation, from which came the all-important Gaussian bell-shaped probability distribution.

Bell was a very prolific research mathematician and writer and clearly a romantic, and I admire his great book with its forceful opinions which brought alive to several generations the history of mathematics. But his lifelong research interest in number theory surely conditioned and therefore influenced his judgments. Later [9, p547] he similarly disqualifies Henri Poincaré for pure mathematical greatness because of his interests in mathematical physics.

I conclude this section by forcing myself, freely acknowledging the conscious and unconscious conditioning of my experiences, to select, to one-bit precision, representative antiquity Greek mathematicians for my three classifications of mathematics. For Pure mathematician I choose Pythagoras, e.g. for his emphasis on proof; Theodorus as Applied mathematician, e.g. for his engineering inventions; and Archimedes as Reality mathematician for his broad contributions to science as well as to mathematics. You do not need to agree with these assignments and you can make your own choices as you like. We are all mixed states.

4. The Present

The dominance in the Greek era of axiomization and proof over application and rational experimentation was of course not absolute, and it actually dimmed as Christianity overwhelmed that part of the world and replaced scientific inquiry with doctrines of religious faith. The Islamic culture in its enlightened period brought mathematics along for a while, and then in the Sixteenth Century the modern Scientific Revolution could no longer be stopped. Most of classical mechanics with its accompanying mathematics developed roughly in the period 1600 to 1900. In that sense, physics, past and present, if mathematical and directed at important real tangible problems of societal need or interest, is an example of Reality mathematics. I can count at least 18 of Bell's [9] 28 men of mathematics who were seriously occupied with Reality mathematics of one kind or another, usually physics, sometimes probability. One point to be made in this paper is that the arenas for fruitful Reality mathematics are these days far more plentiful, and growing.

I will take the Present to begin about 100 years ago with David Hilbert's great address in 1900 to the Paris International Mathematics Congress, in which he outlined twenty-three important problems/tasks for mathematicians. His sixth problem postulated the axiomization of most parts of physics. This address of Hilbert constituted an important revival of the importance of axiom and proof in mathematics.

I have sometimes wondered if Hilbert was partially motivated by his frustration in trying to compete with the more intuitive physicists, such as Poincaré and Albert Einstein.

Along came Kurt Gödel and his 1931 proof that, for certain logical axiomatics, Hilbert's demand for internal consistency implies unavoidable incompleteness in the propositions that can be treated. Then came Alan Turing's 1936 unpredictability of algorithmic halting time. Gregory Chaitin[10] followed with his Omega theorem, which places randomness and even subjectiveness at the very foundations of mathematics. See my discussion in [2].

Following World War II, the Bourbaki [11] blossomed in France with their desire to build a monolithic mathematics untainted by any Applied or

Reality mathematics. One of my colleagues in South America complained to me years ago that the influence of and the young professors sent there by the Bourbaki destroyed mathematics in South America for generations.

Applied mathematics as a profession, beyond the traditional British kind of classical mechanics, arrived a little later, especially in the 1960s as a result of the advent and rapid development of electronic computers. The Cold War also helped, as I recount in [8], where you will find that I think the not-much-later evicting of Computer Science from most departments of mathematics was one of the great errors of the American mathematical community during the 1970s. As a result, mathematicians and others with significant interests in computer science had to form their own departments. A lot of Applied mathematics and Reality mathematics is being carried out in those Computer Science departments these days. At the University of Colorado, our Pure and Applied mathematics departments appear to be static or shrinking while the Computer Science department continues to grow.

It should be remembered that the great power of the method of rigorous mathematical proof was that, in antiquity and much of the Past, it was the only way to try to know the truth. We now have much better experimental facilities, and enormous computing power for simulation. For example, Monte Carlo simulations, although not rigorous proofs, enable a new "way of knowing" for many situations for which rigorous analysis is simply not feasible. And sometimes, combined with algebra or analysis, computing power has been harnessed to actually provide a rigorous proof of an otherwise unattainable proposition, by consideration of a large but finite number of cases. The solution of the four-color problem is a well-known example.

This brings to mind an incident about twenty years ago when a colleague who specialized in mathematical foundations caught me in the hall and asked me incredulously ,"Have you read that article by John Horgan?" I had not. He was referring to [12], The Death of Proof. Horgan had seized upon the completion of the 200-page proof of Fermat's Last Theorem by Andrew Wiles (with the help of others) as a sort of last-gasp of Pure mathematics, as it ran against all the other needs of modern mathematics for applications and for use with computers in solving real-world problems. I have just gone back to read [12] and Horgan interviewed many noted mathematicians and thereby brought out a lot of information and opinions. He is very good at probing until he gets the quotes he wants. Carried away with his notoriety gained from [12], he went on to write [13]

The End of Science. Perhaps he was inspired by the slightly earlier 1992 book [14] by Francis Fukuyama, who wrote about the end of everything.

It is far too easy to be a naysayer. Shortly after Fukuyama's book [14] came out, Professor Ilya Prigogine invited me to lunch one day in Brussels and asked me what I thought about it. I quote from my account in [15]. "I had read the book and told him I didn't appreciate anyone, like Fukuyama, who claimed to have such 'final answers'. To which Prigogine replied, not only that, but Fukuyama was completely wrong. Rather than society having reached some final shape, in fact human history was just beginning. Ninety-five percent of the world's population wouldn't even know what Fukuyama was talking about, as their lives still operate on a much more basic level."

It seems to me that physics is entering a new era. The new physics, with dark matter and dark energy comprising most of the universe, is just beginning. We still don't understand gravity. Quantum mechanics works but we do not understand it. The science of our minds is barely underway. Cell communication in biology, the role of proteins in regulating our DNA and RNA in genetics, the new microbiology of the bacterial component of our bodies, all are exciting new fields of science. Both with and beyond its key role in algorithmic simulations, mathematics will find many new challenges in these arenas. And also in unexpected arenas. Look at Google: Its PageRank algorithm went back to Markov processes and the old numerical power algorithm for the computation of the first eigenvector, and created an entirely new information industry of great societal and commercial impact.

Pure mathematics, because of its inherent beauty and intrinsic structure, will be with us for a long time. In a sense, it is art. The pertinent question, it seems to me, is whether it should be allowed to continue to dominate the politics and hence the make-up of most mathematics departments; and if not, how much of it should be funded?

In the last 100 years Mathematics has become a victim of its success. Much of it is now taught and practiced by non-mathematicians in areas from engineering to finance and even in the social sciences. Much like the rapid recent spreading of the English language over all continents, mathematics has become the lingua franca of the scientific world. It is more and more regarded as just a fact of life. And it is seen as a tool. There is not much interest among the general public in spending large sums pushing ahead on very technical mathematical problems for their own sake. We need English teachers and we need Math teachers. That is the public view.

Here are three selections for representative outstanding Pure, Applied, and Reality mathematicians for the Twentieth Century. I will stay with

Hilbert for Pure mathematics, especially for his 1900 formulation of his versions of it that influenced much of the first half of the century's mathematical preoccupations. Any choice of representative for Applied mathematics is more problematic since so many work in their own specialized fields. Let me just pick Andrei Kolmogorov for probability and other Applied mathematics. I nominate John von Neumann for best Reality mathematician, as he went from pure functional analysis to quantum mechanics to economics to pioneering computer science. His premature death in 1957, perhaps as a result of his exposure to radiation at Los Alamos, was certainly unfortunate. One cannot help but wonder to what else might he have turned?

5. The Future

The future for mathematicians is bright. For one thing, it is regarded as interesting and remunerative work. *Time* Magazine is among sources that have placed mathematics within the top ten best jobs. One of my colleagues told me we were placed first. I did a little checking and found [16] that we were placed third in 2011, with median earnings of $98,000 per year. To be there, we were combined with Computer Science. Above us were petroleum engineers and pharmaceutical scientists. These conclusions were pulled out of a study by the Georgetown University Center on Education. The criterion seems to have been strictly monetary.

Further browsing as I checked this propaganda I had been hearing brought me to a 2014 *Wall Street Journal* article [17]. Here, a job-search website, CareerCast.com, named mathematician as the best occupation of 2014. "Math skills unlock a world of career opportunities," the CareerCast publisher states. Statisticians were ranked #3, actuaries #4, and computer systems analysts #8. Mathematics as a field is projected to grow 23 percent in the next eight years. And the scoring method that yielded these happy outcomes was based upon four attributes: competitiveness, salaries, growth potential, and stress factors. Tenured university professors came in #2, but at $68,970, with the lowest annual salary of the top-ten rated professions.

In 1977 a colleague and I looked at the rapidly contracting academic market for our own pure mathematics Ph.D.'s and started teaching numerical analysis and optimization. In the 1980's, at the behest of our Engineering College, I moved heavily into computational fluid dynamics. For five years, 1988-1993, I was the only mathematician in a seven-department $22 million NSF engineering center at the university devoted to optoelectronic and quantum computing. I produced numerous Ph.D.'s in mathematics and the engineering sciences who have had very successful careers at Boeing, Lockheed-Martin, Silicon Graphics, U.S. Geological Science, Seagate Technologies, in ocean modeling, and in similar industries. In the last twenty years I have taught Derivatives and Risk finance and have produced two Ph.D.'s for those enterprises. Only a few of my 21 Ph.D.'s have gone into academia.

We started an Actuarial Sciences and Quantitative Finance undergraduate certificate program the University about twenty years ago. I steer promising undergraduate mathematics students into that program, which of course requires that they also take a number of courses from the Business School and from the Economics Department. Invariably these students pass one or two of the Actuarial Society exams and are offered good jobs. In addition to demonstrated mathematics and finance expertise, good people-skills, and a true interest in meeting new professional challenges, are also requisite for success in such careers.

These positive career stories bring into better focus my distinguishing of Reality mathematics from just Applied mathematics. Often Applied mathematicians remain in one specialty their whole lives. Reality mathematics requires more. In particular, you must master at least one partner field until you are as good as those who work solely in that field. You must become as competitive in Finance as those who are straight from Business schools. You must understand aerodynamics better than most engineers. You must become one of them. If it be physics, it may take you a lifetime.

One of my colleagues likes to point out that in the 1960's, when he and I obtained our Ph.D.'s in mathematics, there were 300 Ph.D.'s produced annually and 500 good tenure-track academic positions in the United States. Now there are 1500 Ph.D.'s produced annually and still 500 good tenure-track positions. Although the number of Applied Ph.D.'s has been increased, the production of Pure Ph.D.'s continues unabated. Why is that?

Part of it is just cloning. Pures produce pures. Another is cost. After all, four-year colleges are looking for mathematics teachers and not world-class interdisciplinary and multidisciplinary experts. Also, because mathematicians have been given a rather unique training in rational and critical reasoning, that niche is marketable and they all seem to be valued enough to find good jobs somewhere. And I have noticed that most of our graduate students are very computer literate.

At the undergraduate level in the United States, the government agencies are pouring money into STEM: Science, Technology, Engineering, and Mathematics. Not waiting on mathematics and education departments, engineering schools have jumped in, creating new programs to train future high school teachers who will be exposed to all four components of STEM. Here is an excerpt from a typical NSF funding solicitation that today came in to my email, for an Enriched Doctoral Training in the Mathematical Sciences. “The program will support efforts to enrich research training in the mathematical sciences at the doctoral level by preparing Ph.D. students to recognize and find solutions to mathematical challenges arising in other fields and in areas outside today's academic setting.”

David Mumford and I were both lecturers at two conferences in Vietnam in 2005, and I was surprised at learning of his abandoning his work in algebraic geometry (for which he was awarded the Field’s Medal) for a new career in patterns and vision. We had many delightful conversations on a wide range of mathematical subjects while in Vietnam. In 2011 David created a considerable stir with his op-ed [19] with Sol Garfunkel in the New York Times. Their point was that the recently imposed nationwide federal Common-Core mathematics requirements are forcing all high school students to learn lots of algebra and calculus that they will never use—and that they would be better served by the mathematics of real life, such as how mortgages are priced and what the statistics of medical trials really mean. In other words, even at the elementary and lower levels of mathematics education, Mumford and Garfunkel are advocating more Applied, and more Reality.

6. The Problem of Dogma: A Microcosm Case Study

Dogma pervades almost every cultural aspect of human society. Such was one of the messages of E.O. Wilson’s Pulitzer Prize winning book [20]. Also you may look at many of the violent conflicts currently raging or simmering in our world and see a problem of dogma at their foundation.

Here is a microcosm case study with which I have great familiarity: Mathematics at my institution, the University of Colorado at Boulder. I could write a book about it, but will not and instead will be brief and I hope not without a sense of humor.

Before turning to that tragicomedy, recall that as I recounted in Section 3 and as emphasized by Sagan [3], dogma fights were already present in ancient Greece at the very beginning. One can read Section 4 as the gradual re-emergence of the dogmas of practical mathematics and science, temporarily displaced for a very short period in the Twentieth Century by a return to a pure philosophical mathematical dogma. But Science in general ignored that counter-trend and exploded in magnificent progress. As a result, as I portrayed in Section 5, the future is bright for mathematics in its wider scope. I will say a bit more about that in the next section.

I am a local boy and stayed in Boulder to graduate from the University of Colorado in 1958, and I always wanted to get back "home" after completing compulsory military service in Washington D.C., a Ph.D. in Mathematics from the University of Maryland and postdoctoral work in Europe, plus a stint at the University of Minnesota. I finally did so in 1968. Back in Boulder, it seemed that the old mathematics dogma fights had been overcome by the administration when in 1965 they took the Applied department out of the Engineering College and the Pure department out of the Arts and Sciences College, combining them into a single Department of Mathematics—which was placed in a privileged position under the auspices of the Graduate School. Then a whole new cadre of us were hired at a salary level significantly above that of the Engineering and Law schools. We became a top-25 department and things looked rosy and promising. But it soon broke down.

A quick sketch is the following: For the fifty years before the merger, the Applied department had been larger, for it had the larger teaching mission, all of the engineering students. After the merger, the Pure mathematicians took over, requiring rigor in all courses. Ten recently hired younger members of the department were denied tenure because they had interests in applications or computer science or mathematical education or were otherwise tainted. I remember one, a champion sprinter, was even criticized for spending too much time training with the track team.

In 1970, with Stan Ulam, I and others joint with the Physics department created a new Ph.D. in mathematical physics. To oversimplify, this was simply not welcome to the Pure mathematician "powers that be", and they quickly blocked any hiring in that field. In 1980, I and others resurrected the old Ph.D. in Applied mathematics. This was also unwelcome. In 1990 the administration threw up its hands in frustration and split the single department back into two separate departments, Pure and Applied. I had to choose one, and because I did not believe in the wisdom of the split and fought those who wanted to separate the Applied department, I remained in the Pure department. You can get just a few more details in my book [8, Chapter 6]. Many amusing funny and sad anecdotes are better shared in private.

That brings us to 2014. Looking at these two relatively small and growing-weaker departments, both housed in the College of Arts and

Sciences, now the administration wants to combine them. Moreover, they wish to add a third component, of Statistics and Big Data. Really! Will the administration ever apologize to me for ignoring my foresight, or now request my help? Will the merger really happen? If so, will it work? Do I care?

7. Conclusions and Ideal

1. No matter which of Pure, Applied, or Reality mathematics one finds oneself within, there must be the component of several years of training in mathematical rigor if one is to be called a mathematician. As I state in [8, xii]:"There is a substantial, nontrivial, and not widely understood gap between the training needed to become a pure mathematician and that of just being an engineer or physicist. This gap cannot be fully appreciated by applied scientists unless and until they successfully complete the committed step of taking several course-years of algebra, topology, measure theory, geometry, and real analysis, among others."

2. For those mathematicians who can go beyond the essential training and also master a completely different field, the future will be bright indeed.

3. As an ultimate ideal, in the Greek tradition of ideals, I advance the new concept of *TriMathlete*: one who has succeeded in all three of Pure, Applied, and Reality mathematics. The analogy is with triathlons. Rather than seeking to be the world's greatest long-distance swimmer, a rather dull and monotonous prospect if you think on it, instead you get out of the water and into a new competition on your bike; and then even if you are a great cyclist, you hop off your bike and enter the final reality of running a marathon. As models, we may identify from the far past Archimedes, Gauss from the near past, and from the recent present I nominate John Von Neumann. One need not become a TriMathlete, but it is a worthy goal to hold, and in seeking toward it, both mathematicians and society-at-large will benefit deeply.

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MATHEMATICS EDUCATION FOR A NEW ERA

Marianna Tzekaki
Aristotle University of Thessaloniki
tzekaki@nured.auth

ABSTRACT

In what society do we estimate that today's students will live, what will they face in the future and what disposition will they need to meet the requirements of the new era? Can mathematics education play a role in the development of this disposition and what goals need to be set, what content and what teaching approaches to meet these requirements? Isn't it critical for researchers, programs' designers and teachers to pursue systematic answers to these questions and bring in new advances to the teaching of Mathematics?

INTRODUCTION with many questions

In his keynote speech in PME38, Luis Radford revoked Aristotle and the aims for education he set in Politics (Book Z, pp. 1337a – 1337b):

> …that it must, therefore, be adopted laws for education and that it ought to be defined equal for all, it is clear; which, then, will be the nature of education and how should it be provided, it must not escape of our attention ... indeed, we do not all share the same opinions as to how we must train young people, to virtue or to perfect life, nor is it clear whether (education) should have the aim to exercise and cultivate the mind or form a moral character…[1]

What would Aristotle feel realizing that 2350 years later the aims of general education (and mathematics included) remain controversial and ambiguous; that in Greek classrooms students typically follow programs and old fashioned teaching methods exercising them, not for mental and moral virtue, but mainly for examination procedures with questionable results (Radford, 2014)?

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[1] Translation from the ancient Greek text.

What aims would he invite us to set, what content to propose and what approach to adopt so that mathematics education would be linked to the demands of the current time, as well as of the time that follows, as suggested in his work? What factors determine these aims: the socio-historical moment, the unknown future, the needs of people, the country, the planet, of the historic- social location, of mathematics itself (Skovsmose, 2006)?

And how this future will be? In what society we estimate that today's children will live? What will be the population growth of this already globalized web dominated society, what will be the levels of health and feeding, of poverty, of increasing inequality, of conflicts and violence, already incredibly multiplied?

What situations will present students and future citizens encounter and what equipment will be need to meet the demands of this society and the creation of a better society for themselves and their descendants: to know how to learn, to effectively understand the world, to perceive and process information, to solve problems, to work systemically and creatively, to develop continuous learning, to make estimations, projections, or to adapt to changes (Yasukawa, 2010)?

Can mathematics education play a role in the development of all these skills, and if so, what should be its content, methodology and function to respond to this role? Can it stimulate the mental power of young people for a better understanding of the world and its inequalities? We usually accept Mathematics as a global, general, abstract, a- politic, neutral science, how could we envision it to serve social justice and quality of life (Gutstein, 2003)?

While wondering «*What Mathematics for a new era*», we are in front of immeasurable queries. It’s been a long time that Critical Mathematics Education is studying these issues and has produced many interesting viewpoints and research results. D’Ambrosio (2008) raises a serious concern, arguing that Mathematics is closely involved and influence today's global society. Both historically and socially, many of the recent advances in technology, industry, military equipment, economy and politics were made thanks to mathematical tools; on the other hand, respectively, the needs for these fields have supported the ongoing development of Mathematical science.

At the same time mathematics teaching, besides the positive influence that can bring in education, can also work inversely, reinforcing inequalities and exclusions, preventing understanding, leading to blind obedience, overconfidence in numbers or high authority over knowledge. Mathematics is recognized as a high level global way of thinking, but it coexists with equally recognized global problems of mankind such as injustice, war, environmental problems, survival with dignity and so on. Will these universalities be in contradiction or in collaboration, and how this result can be accomplished?

In our days in Greece, as we are looking for ways out of the deadlocks we are living in, for a better future, the answers we are seeking in order to clarify the ontology, epistemology, methodology and results of mathematical education are crucial. The essential is not about the state doings and our expectations, but about our concerns for this situation and our willingness to do something as scientists, educators and citizens of this country.

The critical question, therefore, that we set in this presentation is the following: *how can we let mathematics education remaining for decades without substantial changes in the way it works in school and is perceived by the society, while around us such imperative changes are taking place?*

WHAT GOALS for Mathematics Education?

Both in Greece and other countries, societies set very high and ambitious goals for mathematics education. Recently, responding to social demands, the program designers remove contents leading to learning without understanding and memorization of rules, and focus on new orientation on the basis of activation of students through research and active engagement in challenging and interesting mathematical situations. The curricula seek to develop high level processing of students' experiences, to enhance problem solving and modeling abilities, analytical- synthetic and reflective thinking, etc. (Cyprus, 2010; Canada, 2009; Finland, 2009; Scotland, 2007; Massachusetts, 2001). In addition to problem solving abilities, programs also encourage other mathematical processes, such as communication, reasoning, making connections, representational developments and semiotic activity, etc.

It is argued that mathematics can influence the mental development and progress of students and support their accomplishments and social interaction because of its nature linking experiences and systems of thinking and encouraging functions like self-control, dedication to a project, aesthetics, self-confidence, etc. These elements are in the aims of both Greek programs of 2003 and 2011 (DEPPS, 2003; New curriculum, 2011)

All these confirm that mathematical education shapes a high quality framework which, not only requires special adaptations for school implementation, but also makes the preparation of mathematics teaching particularly complicated. Certainly, research shows that many of these objectives can be realized by the help of appropriate mathematical instruction, but the distance between research experiments and school reality, its content, methodology and materials remains long (Clements, et als, 2013). We can continue to believe or imagine that mathematics teaching is able to assist young people's formation, but the educational practice, both globally and locally, still operates with formulations and exclusions, and, rather than enhancing students' understanding, it mainly discourages them from approaching Mathematics meaningfully. Thus, it is generally accepted that aims leading to these high level results are indispensable for the preparation of young students for a New Era, but the accomplishment of these results requires careful and systematic design.

What NEW ERA for our students

A United Nations committee working on a quality education by 2050 (UN Oprn Working Group, 2014), records a future where the world's population will be 9 billion, many planetary boundaries will be diversified, global warming will be intensified, the food crisis will also increase, there will be more poverty and fragility, rising inequality, conflicts, violence health problems in many countries. This admittedly depressing picture, presented by the Commission as a framework to help the design of an education that could possibly improve it, is visible even from today. What will be the new power balance in the world and what will be the position of our small country in this uncertain environment?

The following eras, besides the shocking speed of changes, will include (not so new anymore) globalized and complex systems such as web network, multimedia, advanced communication and calculations tools, virtual realities, online education environments etc.: Sriraman, Roscoe, and English, (2010) summarize them in three key axes: systems with applications in everyday life, conceptual understanding systems, and systems modeling and designing real or conceptual systems.

They argue that our students will need in the future knowledge concerning complex understandings of both actual and conceptual systems, as well as of the models they structure them. They will also need skills to create new models for situations and phenomena that will occur. At the same time, collaboration skills as well as communication and technology abilities will become increasingly necessary. Few would dispute these necessities, but how many of us keep wondering (theoretically or practically) whether these elements are encountered by the education we offer today?

We all see that the situations and phenomena that surround us are gradually becoming more and more complex; we miss a lot of what is happening worldwide. Still current citizens require considerable knowledge and cognitive structures to understand, for example, how the economy works, what is meant by debt, what model of development will reverse this situation? Or to investigate thoroughly what is the role of the internet, how it works, what will be the investment for its development and what will be the future of technology? Respectively, what is the Cern experiment and its consequences for the evolution of humanity, or what is happening in the environment (natural and anthropogenic), how it operates, and what will be the consequences in this field? These are only few of the phenomena that form the present and the future of our world.

In the past, talking about situation problems and models, we referred to something rather simplistic, as for example to make an algebraic equation to solve a real problem. In the current and future era, all understanding, description and interpretation require capturing of models that rely heavily (as is the case now) in mathematics, technology and often in physical or other related disciplines. These models (just like all mathematical models) are complicated, complex, with levels of emerging meanings and concepts and increasingly evolving (Lesh, & Sriraman, 2010).

Let us ask ourselves, though: are all these cognitive structures indispensable for an individual to understand the complex phenomena surrounding him? Do they mean something for his survival, his health, his professional, social and personal life? A few years ago we might still be able to answer differently these questions, but the shocking events that took place recently make these answers obvious. The way we perceive our world leads to decisions and actions for maintaining or changing it. In this direction, mathematics play an important role, certainly if its teaching highlights the specific nature of mathematical activity and enables young people to understand and process situations and phenomena (Smokovse, 2006). How close is our education to this approach?

What MATHEMATICS EDUCATION for the future?

It is generally acknowledged that Mathematics is a high-level way of processing, development of concepts and procedures, symbolization, instrumentalization, modeling etc. Therefore, it's argued that mathematics education should not seek to the teaching of the products of this process, but of the way that produces these products. The accessibility to this way of thinking becomes critical for the cognitive development of young people, because, historically and epistemologically, it worked and is still working in all mechanisms of understanding, designing and changing the world around us.

Attempting to justify the importance of mathematics education and the extent to which we should be concerned about the content and teaching of mathematics, let's recall the lecture of Marshall Stone[2] in the seminar, organized in France in 1959 by the Organization of European Economic Co-operation (and late OECD) provoked by 'Sputnic shock' (OEEC, 1961, p.17, in Skovsmose, 2006):

> … we are literally compelled by this destiny to reform our mathematical instruction as to adapt and strengthen it for its utilitarian role carrying the ever heavier burden of the scientific and technological super- structure which rest upon it…

We highlight in these words the special interest attributed to the appropriate teaching of mathematics for the economic, political and social future of an era. Relevant attempts to change mathematics education positively for the future have been historically recorded in critical times: in Germany after the war people sought for an education after Auschwitz and in N. Africa after Apartheid etc. (Skovsmose, 2006). Thus, what mathematics education do we need to prepare our country for a time after the crisis?

As already indicated in the objectives, presented earlier, mathematics education turns to the active participation of students in mathematical experiences that allow the emergence of processing, problem solving and reasoning abilities. How many of them are currently promoted in the classroom? How can we create, through education - as Aristotle told us - a moral active citizen, when the educational process developed in the classroom make him passive receiver reproducing - often without understanding - readymade and rather incomprehensible ideas?

The connection of the students' experiences with high mathematical systems of thought requires the development of a genuine mathematical processing, i.e. an activity relevant to mathematical activity within the

2 O Marshall Harvey Stone was an important American mathematician, with rich contribution in mathematics, but also participation in the research activity of the American Ministry of War during the second world war. Later he continued as professor in the University of Chicago.

science. Thus, deepening our reflection about a substantial mathematical education, you need to first analyze what is *problem solving ability* and then what is *authentic mathematical activity*?

MATHEMATICAL ACTICITY

Many researchers, attempting to explain the elements that describe an authentic mathematical processing, study the way in which mathematicians work. Schoenfled since 1992 argued that '*thinking mathematically'* is closely related, not only to problem solving, but also to metacognitive processing of problem's solutions (i.e awareness concerning solving process and self- regulation). He approached the problem solving process by analyzing its stages: reading the problem, analyzing it, exploring for solutions, finding solution plans, implementing a plan and verifying the result. Research has shown that, while young and inexperienced solvers read very quickly the problem they encounter and use all the time searching for a solution, which abandon quickly if it doesn't work, an experienced solver focuses on the systematic study and analysis of the problem, the design of a plan solution, implement this plan with feedbacks and finally always validate the correctness of the solution. These findings helped significantly the development of teaching proposals exercising students, from early ages, in problem solving abilities, concerning not only arithmetic or mathematical but also more general problems (Stacey, 2005).

Respectively, the study of the elements of a *genuine mathematical activity* could support the design of mathematics education aiming at developing this special activity. So, what do we understand as mathematical processing?

In a previous study we collected a number of similar or complementary views on the issue (Tzekaki, 2014; Tzekaki, 2011). Some of these approaches consider the mathematical activity as a way of *modeling* to address and deal with real situations (Freudenthal, 1983), other as finding *appropriate solutions* for situation-problems (Brousseau, 1997), and others as transferring these solutions to a *more general* framework (Radford, 2006, 2014). Researchers argue that mathematical meanings derive from *mathematical connections* or the *process of symbolization* (Noss, et als, 1997; Ernest, 2006) or more generally, from a dynamic link of *situations, with signs and concepts* (Steinbring, 2005).

Even different approaches for the same issue, they all converge to the position that the students need to reach a way of a high-level thinking and processing that involves habits and mental routines. This special processing comprises research for properties and relationships, identification of patterns and common structures, analysis and synthesis in parts and unit parts, connections, links to language, representations, signs and symbols, explanations / justifications, reflections and generalizations, and so on.

In addition to all above aspects, the study of gifted students revealed dimensions of the mathematical way of working that illuminates further the study of the phenomenon, adding to *mathematical abilities* related to what was reported previously (finding relations analysis - synthesis, representations and explanations), *cognitive processes* involving connections between generalizations, flexibility and creativity, and finally *hyper-cognitive processes* associated with self-dedication to work, perseverance and confidence, etc. (Leikin, 2007; Kattou et als., 2012; Kontoyianni, 2014). The study of competent persons in Mathematics gives us very important information about the skills we seek to develop in mathematics education.

The obvious question coming out of such an analysis is how we can develop all this knowledge, abilities and attitudes to our students? And in the broader perspective of critical mathematics education, what mathematical knowledge is needed by our students in order to keep reflecting upon the ways the reality is presented, upon justice, equality and exclusion, in an attempt to understand or change it?

WHAT SCHOOL, WHAT CLASSROOR, WHAT Mathematical classroom?

From all the elements presented earlier, it becomes apparent that the teaching and learning of mathematics, aiming at supporting a substantial students' development, seeks to cultivate in young people this particular human activity with the aforementioned elements, by creating appropriate conditions and environments, and not limited to the reproduction of concepts and procedures (Frade, et als, 2013). The formation of such a framework is certainly not simple endeavor, but it is important to start thinking about whether the activity we encouraged in today's classroom has the characteristics of an authentic mathematical activity. Or also to start searching for content, problems, tasks, situations and environments that will allow such a development? And moreover, inversely, to start wondering if the content of today's mathematics teaching and the way we follow lead to the exactly opposite consequences?

For more than at least 50 years researchers agree that substantial mathematical students' development requires their activation upon real questions, problems, unfamiliar situations, games, technology environments etc., that can encourage them to make assumptions, to find solutions, to seek modeling, to use sources or tools to justify, elaborate and formulate (e.g. Freudenthal, 1983; Brousseau, 1997; Radford, 2006; Perry & Dockett, 2008). Such an education presupposes major changes both in collective and individual level; only the strong belief (especially of teachers) that it is really helpful for the students and the country can support them.

European Commissions' proposals about the quality in education (European Commission, 2013) are limited to *structural* (class size, student/teacher ratio, teacher quality and working conditions) and *procedural changes* (teacher - student interaction, appropriate classroom practices that encourage students' involvement, enhance identity, and sense of 'belonging and learning'), while supporting a unitary structure that prevents artificial division (elementary, high school, etc.) in front of the continuum of human development.

Getting into the heart of the problem, in a recent speech Radford (2014) argues that the way in which the mathematics class works derive from prevailing conceptions (and policy pursuits) about the forms of *(re) production of knowledge*, but also about the ways the teacher and students should interact (human cooperation). From this approach it becomes clear that if our aim is not the reproduction or the individual acquisition of knowledge, but the development of the youth, oriented to pedagogical and cognitive disposition, operating collectively to address the characteristics of the new era, then our conceptions and relevant practices require major revisions.

The traditional transmissive educational model, Radford analyzes, perceive knowledge as a *product* (and wealth) that someone holds and someone else should acquire. Even in the progressive educational programs, like constructivism, knowledge is still treated as an individual construction and personal property, following the logic of private production, i.e. a 'private property'. Instead he argues that (2014, p. 18):

> … Mathematical knowledge is not something possessible. It is not yours or mine. Mathematical knowledge appears as pure potentiality—virtual possibilities for mathematical understandings, meanings, and course of action. To be materialized, knowledge has to be set into motion through teachers' and students' labour.

In his view (and historical- cultural approach) knowledge is a synthesis of what people do and process about this doing, which is a dynamic and evolving way with action, thought and interaction in the world. In this sense, it cannot be seen as something that one gets and can record or transmit it (which gives a completely wrong perception even of the mathematical science itself), but as something that is produced and accessed by teachers and students in cooperation within appropriate actions and contents, giving to Mathematics its precise meaning (Radford, 2006).

CONCLUDING REMARKS

Summarizing the importance of mathematics education in the new era, it is clear that we need to move away from the simplistic and conventional approaches, for example, that mathematics is useful, is everywhere and necessary for everyday life. We also need to move away from the idea that working with mathematical objects (in simple or more advanced form, like numbers shapes, equations, functions, etc..) teaches students mathematical thinking or finally that mathematics education alone supports the critical thinking of the citizens of the future.

As Skovsmose (2006) presents in details that depending on the forms, mathematics education can encourage good but also bad practices, as already mentioned, like blind obedience, exclusion, lack of cooperation, an over-reliance on numbers or a development of high power through knowledge. Similarly, some elements resembling 'democracy', as unique approaches to a solution, or an unique book, a simplification and standardization to make some aspect of mathematics understandable could not help but rather exclude and lead to opacity, special distribution knowledge and consequently of the power (Yasukawa, 2010).

The substantial benefit of mathematical education, both in individual and in collective level, presupposes an engagement that meets the young students' needs (present and future) and the emergence of the important dimensions of mathematics, like multiple approaches, alternative assumptions and conclusions, choices and consequences from those choices. We should not give to students and society the image that mathematics is a set of eternal and undeniable truths, but a science that is continuously formed, constructed, could be falsified, be corrected, in an attempt to better understand the world (Davis & Hersh, 1991).

These aspects of mathematics can give the dimensions of critical mathematics education and provide new disposition for the unknown future that follows. On this issue there has been substantial research and experiments (see. Gutstein, 2003), but the design and implementation of these elements in all education require radical changes in mentality and re-training of teachers. In our country these changes are critical because of the particular impasse we face, but it is not known when they would be mature in the field of mathematics education. However, we can hope that together with the revisions that are gradually taking place in Greek society, a substantial mathematical education will arise.

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Can preschool children collaborate in mathematical tasks productively?

Sonia Kafoussi
Department of Sciences of Preschool Education and Educational Design
University of the Aegean, Greece
kafoussi@aegean.gr

Abstract

The purpose of this study was to investigate how pairs of preschool children collaborated productively during their efforts to solve mathematical tasks. Fifteen pairs of preschool children (five years old) participated in this research. The results revealed three types of collaborative actions that allowed the preschool children to arrive at the solution of a mathematical task: a) using materials simultaneously, b) evaluating their different ways of solving a mathematical problem jointly and c) inspecting the different answers to a mathematical problem jointly. All the collaborative actions that the preschool children used gave them learning opportunities, as they critically examined their proposed solutions by repeating, through their actions, each other's ideas or by confirming their answers using alternative methods. The development of collaboration from the preschool age could help the children to succeed productive collaborations during their future mathematical education.

Keywords
preschool education; collaboration; small groups; mathematics

Introduction

Recently the study of children's collaboration in mathematics has been at the heart of research, as the social dimension of the construction of mathematical knowledge plays an essential role in the contemporary perspectives of mathematical learning and teaching (e.g. Chronaki & Christiansen, 2005; Cobb & Bauersfeld, 1995; Dekker & Elshout-Mohr, 1998; Lerman 1998; Sfard 2001). Collaborative learning is not just an instructional arrangement that can be used to foster active children's learning, but it is in the essence of the learning processes. According to Burton (2002), during collaboration children work together to solve a problem as they share "the same disciplinary approaches" (p. 162).

In mathematics education, many researchers have mentioned that collaborative learning allows children to feel responsible both for their own learning and that of the other members of the group, the acquisition of collaborative skills and the development of problem-solving ability at a higher level (e.g. Chaviaris, Kafoussi & Kalavassis, 2007; Cobb, 1995; Francisco, 2013; Good, Mulryan & McCaslin, 1992; Martin, Towers & Pirie, 2006; McCrone, 2005; Pijls, Dekker & Van Hout-Wolters 2007; Stacey & Gooding 1998; Voigt 1995; Weber, Maher, Powell & Lee, 2008). Children's collaboration allows them to check their ideas, listen and refine their classmates' ideas, formulate their thoughts through arguments in order to understand better the mathematical concepts and/or procedures.

Moreover, many researchers investigating children's collaboration have focused on learning contexts concerning small groups of children (usually two children). The research has revealed various factors that influence children's mathematical learning, including group composition, their beliefs about collaboration in mathematics, their achievement in mathematics or the quality of mathematical tasks (e.g. Good et al., 1992; Kafoussi, Chaviaris, & Dekker, 2010; Webb, 1989). These findings suggest that productive children's collaboration in small groups for a meaningful learning is hardly an easy goal to achieve in a mathematics classroom, as a productive collaboration means to succeed a shared goal mutually through a continual negotiation of collaborative actions and mathematical meanings. Furthermore, little research has been done concerning preschoolers' collaborative learning in mathematics (Tarim, 2009). The aim of this study was to investigate how pairs (small groups of two) of preschool children collaborated productively during their efforts to solve mathematical tasks.

Theoretical background

Investigating the circumstances that facilitate children's collaboration in pairs in mathematics in primary education, Cobb (1995) found that when two children are working together to find a solution in a mathematical problem, they may collaborate directly or indirectly. In the first case, the children explicitly coordinate their efforts to solve the problem and they make "taken-as-shared" interpretations of it. In the second case, one or both children think aloud while they seem to solve the problem independently. By this way, they can capitalize on each other's comments. According to the same researcher, indirect collaboration is frequently more productive than direct collaboration which doesn't usually allow for learning opportunities.

Martin, Towers and Pirie (2006) have used the notions of interactions and coactions in order to describe the ways of children's collaboration in groups. Interactions involve "the process of acting on the ideas of another in a reciprocal or complementary way", whereas coactions refer to the process "of acting with the ideas and actions of others in a mutual, joint way" (p. 156). Both interactions and coactions can be crucial in order to promote mathematical understanding (Francisco 2013).

Moreover, in an effort to gain a deeper understanding of the constituting elements of a productive collaboration among children, researchers have given emphasis on the notion and the elements of argumentation that provide learning opportunities in mathematics classrooms (e.g. Francisco, 2013; Krummheuer, 1995; McCrone, 2005; Weber et al., 2008). They usually use Toulmin's model (1969) about argumentation, according to which an argument needs: a) a claim, that is an assertion of which an individual is trying to convince others, b) data, that is some statements on which the conclusion can be based and c) a warrant, that is an explanation as a means of legitimizing why the claim supports the conclusion. According to Krummheuer (1995), especially in the primary education, participants usually try to demonstrate the rationality of their methods when they solve a problem through their actions during their collaboration. Hence, argumentation is manifested through ways that help the participants to demonstrate, implicitly or explicitly, the rationality of the action while they are acting. He mentioned that, in the primary education, the children's mathematical statements "carry the significance of acting on experientially real mathematical objects" (e.g. counting on fingers) (Krummheuer, 1995, p. 236) and these actions convey their process of argumentation.

According to the above, our hypothesis was that the investigation of a productive collaboration at the preschool age means to identify the types of collaborative actions in which the preschool children engaged in order to accept or assess the rationality of their solutions and arrive at the answer of a mathematical problem.

Methodology

Fifteen preschool children's pairs (30 children, 17 girls and 13 boys) participated in the research that took place in November and December 2011. The children were five years old and they attended public kindergarten schools of the city of Rhodes, in Greece. We observed three pairs from five different schools. They volunteered to participate in this research and they were asked to solve mathematical problems in pairs. Each pair had to solve one task. The tasks were not the same for all the pairs. They were designed to cover basic mathematical topics for this age, including numbers and operations, measurement and probabilities (Clements & Sarama, 2004). The small-group activity lasted about 15 minutes for each pair and it took place outside the classroom (in the library of the kindergarten schools' buildings).

All the dialogues during the collaborations of the pairs were tape-recorded. Data analysis was conducted with the transcripts of tape-recordings of the children's small-group activity, as well as with the researcher's hand-scripted observations. Drawing upon Krummheuer's analysis about argumentation (1995), we first described the data and the possible warrants that the preschool children accepted as "taken as shared" during their collaboration. These elements allowed us to identify the collaborative actions they supported in order to arrive at an acceptable solution of the mathematical task.

The researcher presented the mathematical tasks to the children and participated in their discussion by asking for clarifications when needed. Concerning fostering children's collaboration, the only advice that she gave to them was that they should solve the proposed mathematical problem jointly.

Results

The analysis of our data revealed three types of collaborative actions that helped the preschool children to work together productively and to arrive at the solution of a mathematical problem: a) using materials simultaneously, b) evaluating their different ways of solving a mathematical problem jointly and c) inspecting the different answers to a mathematical problem jointly. In the following we are analyzing examples of the preschool children's collaboration in mathematics identifying the above mentioned actions.

a)using materials simultaneously

This action is related to the use of the given materials of the task by both children. They used the materials simultaneously while they were acting in order to proceed to a conclusion about the solution of the mathematical problem. In the following examples, we are analyzing the collaborations of two pairs of children.

Example 1

The children (two girls, Chara (C) and Ioanna (I)) had to measure the side of a square table to help a squirrel to buy a tablecloth for its party. They had big and small straws, yellow and green respectively at their disposal. They should measure the side of the table by using these two different kinds of straws in order to take a decision for the squirrel. The following dialogue took place between the children.

	Utterances	*Actions*
1	C: Let's start with the yellow straws.	
2a	I: Yes. Yes, with the yellow firstly.	[2] They are putting the
2b	I: Let's put them. Do we have to count	yellow straws together
3a	them?	one by one along the
3b	C: Yes.	side of the table. Ioanna
4	C: 1,2,3,4… Come on, we will do it	is putting one straw and
5	together.	then Chara is putting
6	I: 1,2,3,4,5,6.	the other one and this
7	C: They are six. Can I put the green ones?	process is continuing.
8	I: Take some of them and I'll take the rest	[6] Ioanna is giving
9	of them.	some straws to Chara
	C: 1, 2, 3, 4, 5, 6, 7, 8. You have to count	and they are putting
10	too.	them together in the
11	I: 1, 2, 3, 4, 5, 6, 7, 8. They are eight.	same way that they did
12	R: So, how many yellow straws do you	with the yellow ones.
	need and how many green?	
	C: Six yellow and eight green.	
13	R: Why do you need less yellow than green	
14	straws?	
	C: Because this is smaller *(she is looking at*	
15	*a green straw*) and it is taking up less room.	
	Eh; *(She is looking at Ioanna)*	
16	I: Yes.	
17	R: What are you going to tell the squirrel in	
18	order to buy the tablecloth?	
19	C: It is the same, but this one is smaller *(she*	
20	*is looking at a green straw).*	[19] Ioanna is writing
	R: Do you want to write your answer?	the symbols of the two

I: Yes. C: I don't remember how we write the eight. I: It is two small circles. I'll show you. C: We finished.	numbers.

The children shared the given materials for the measuring of the side of the table and they put them together along the side of the table in order to find the result. Moreover, as they tried to find the answer to the problem, both children used the counting process. The actions of putting the straws together and the repetition of the counting served both as their data and as their warrant for their answer. During this process they accepted this collaborative action as a legitimate one for their conclusions.

Example 2

At the second example another pair (two girls, Stella (S) and Evi (E)) had to find the distances that the turtles of a forest had to cover in a speed contest and to compare them. They had small markers at their disposal. Following we are presenting the dialogue that took place between the children.

	Utterances
1	S: Have you understood what we are
2	going to do?
3a	E: No! You?
	S: Yes. We have to help the turtles to
3b	measure the distances.
4	S: 1,2,…
5	E: OK.
6	S: 1, 2, 3, 4, 5.
7	E: 1, 2, 3.
8	S: 1, 2.
9a	E: 1, 2, 3, 4, 5.
9b	S: Be careful, the markers are slipping.
10	S: 3, 4.
	R: So, which of the turtles have to run the
11	longest distance?
12	S: The first one.
13a	E: This one *(She showed the last one.)*
13b	S: No! The first one.

Actions

[3b] Stella is putting the markers along the first road and she is counting simultaneously.
[6] Evi is putting the markers along the second road and she is counting simultaneously.
[7] Stella is repeating the same action for the third road.
[8] Evi is repeating the same action for the fourth road.
[13b] She is counting the markers again along the last road.

14	S: You measured it in a wrong way. 1, 2, 3, 4.	
15	R: And which turtle has to run the smallest distance?	
	S-E *(together)*: This one *(They showed the third road)*.	

Stella proposed a concrete way of solution, by counting the markers as she was putting them along the length of the road. Evi accepted this process as their data. Subsequently, the children measured the different distances together, successively. That is, Stella counted the first distance with the markers and after that Evi made the same action for the second distance and so on. When a child put the markers and counted, the other controlled the whole process. This control functioned as a warrant for their measurement and helped them to correct the wrong solutions (cf. utterances 8-9, 13b). The collaborative action of realizing the measuring of the distances using both the markers helped them arrive at their conclusions.

In both examples the collaborative action of using the materials simultaneously seemed to be a fruitful one for the children's learning. At the first example, the children used the materials by putting the straws together for the measuring of the length of the table each time. At the second example, they divided the task in subtasks and they made the measurements with the materials alternatively by checking each other. In both cases, this action gave them the opportunity to understand better how to put the informal measure units along a distance correctly in order to measure it as well as to compare different lengths.

b) evaluating their different ways of solving a mathematical problem jointly

This collaborative action emerged when the children were engaged in a mathematical task and they investigated different ways in order to find the answer of the problem, as we can observe in the following example.

Example 3

Similarly to the first example, the children (two boys, Ilias and John) had to measure the side of a square table in order to help a squirrel buy a tablecloth for its party. Firstly, they were asked to use their hands and secondly cubes in order to make this measurement. The following dialogue took place between Ilias (I) and John (J).

Utterances *Actions*

1 J- I *(together)*: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
2 J: They are ten.
3 I.: There is another way. 1, 2, 3, 4, 5, 6, 7.
4 R: Oh! Why did John find ten and you
5 seven?
I: But John counted in this way *(he is*
6 *repeating his own action and then John's*
7 *action).*
8 J: Yes! One way or another.
9 R: So, how long is the table?
10 I: We will do it in this way.
11 J: 1, 2, 3, 4, 5, 6, 7. Seven.
12 I: Yes, seven.
13 J: 1, 2, 3, 4, 5, 6, 7, 8. No, they are eight!
14a I: 1, 2, 3, 4, 5, 6, 7, 8.
14b R: What happens now? Seven or eight?
15 J: My hand is smaller…
16 J: Do you want to count the cubes?
17 I: Yes.
18 J: 1, 2, 3, 4, 5, 6. Six.
I: 2, 4, 6. Six. I counted them two by two.
19 R: So, what are you going to tell the
20a squirrel, seven, eight or six?
20b J: Maybe we are confused…
21 I: No…We counted them! *(He is angry.)*
I: The cubes are big. The hands are small.
22 R: So, what is more helpful for you? The
23 hands or the cubes?
24 I-J: The cubes.
25 R: Why?
26 I: It is easier.
27 J: I was confused with the hands. The
28 cube is better.
29 I: We put the cubes more quickly.
R: So, what are you going to tell the
30 squirrel?
I: Six.
J: Yes. We will give him the cubes and
we will say six.

[1] John is putting his hands vertically along the side of the table and the boys are counting together.

[3] Ilias is putting his hands horizontally along the side of the table. John is making the same action in the air.

[8] He is putting again his hands horizontally along the side of the table.

[9] John is counting Ilia's hands.

[11] John is counting his own hands by putting them horizontally.

[12] Ilias is counting John's hands.

[15] They are putting the cubes together one by one along the side of the table. John is putting one cube and then Ilias is putting the other one and this process is continuing.

[29] He is putting again the cubes along the side of the table

I: Yes.

During their collaboration, John and Ilias discussed different ways of solving the mathematical problem. Although they both accepted the process of counting as their data, when they used their hands to measure the side of the table, the placing of their hands vertically or horizontally challenged their finding of an acceptable answer. However, this challenge emerged when John tried to repeat the process of the counting that his partner proposed by making the same action (cf. utterance 11). The action of the evaluation of the different ways for the solution of the problem by both allowed them to reach at their conclusions about the suitability of the measurement units that they used (cf. utterances 21-26). They had the opportunity to make comparisons between their hands and the cubes and to formulate an explanation about the choice of the cubes for the measurement process. Moreover, we should mention that they also acted collaboratively in using the materials simultaneously, when measuring the side of the table with the cubes (cf. utterance 15).

c) inspecting the different answers to a mathematical problem jointly

This collaborative action was emerged when the type of the problem that the children had to deal with required more than one answers. In that case, the children inspected their different answers jointly while working together.

Example 4

The children (a boy and a girl, George (G) and Katerina (K)) were asked all the different combinations in order to hide six whistles in Bob's house. Bob would organize a party and he would like to play this game with his friends. Bob had a two-storey house (a model of the house was given to the children) and they had to think that Bob could hide some whistles on the first floor and the rest on the second. The question was how many whistles Bob could hide on the first and how many on the second floor. The children had six whistles and a sheet of paper to record their answers at their disposal.

	Utterances	*Actions*
1a	G: This is the first floor and this is the	
1b	second.	[1b] George is writing
	G: He can put three whistles on the first	the answer (3, 3).
2a	and three whistles on the second.	
2b	K: Yes! Three on the first and three on	[2b] She is counting four

the second.
3 K: I think that he can put four here *(she*
4 *showed the first floor)* and two here *(she*
5 *showed the second floor)*.
6 G: No, four on the first and one on the
7 second.
8 K: No, because four plus two makes six!
G: Ah! Yes!
9 K: If he put one…
G: They remain five.
10 K: Maybe he can put five on the first and
one on the second floor.
11 G: Yes, one here *(he showed the first*
12 *floor)* and five here *(he showed the*
13a *second floor)*.
13b K: No, I said five here *(she showed the*
14 *first floor)* and one here *(she showed the*
15 *second floor*).
16 G: Ok, five on the first and one on the
17 second, eh?
18a K: Yes, I think so…
18b G: Yes. Can I write it?
19 G: 1, 2, 3, 4, 5, 6.
K: Did you put them all together?
20 G: Yes.
21 K: Will we put all of them on one floor?
G: Yes.
22a K: And nothing on the other?
22b K: That is zero?
22c G: No, he will put six here *(he showed*
23 *the first floor)* and seven here *(he showed*
24 *the second floor)*.
25 K: Six and seven?
26 G: Yes, six *(he showed again the first*
floor) and seven here *(he showed again*
the second floor).
K: But six plus seven is...
K: Does he happen to put six and zero?
K: I think so…

fingers in her hand and she is putting two more fingers in the other hand.

[5] He is writing the answer (4, 2).

[6] She is holding one whistle in her hand.

[7] He is counting the rest whistles.

[13a] He is writing the answer (5, 1).

[13b] He is putting all the whistles on the first floor.

[18b] She is counting six fingers.

[22a] She is trying to count her fingers, but she is troubled.

[23] He wrote the answer (6,0)

G: Mm…Ok.
R.: Can you put the whistles in another way?
G: No.
K: Ok.

In the above example, the children found some of the different combinations of this problem jointly. When a child told an answer, the other one inspected it and this action led them to find the right one (cf. utterances 4-5, 22-24). This checking also gave them the opportunity to think of more answers. For example, George's mistake that 4 plus 1 could be an acceptable answer gave the opportunity to Katerina to propose 1 and 5 as an alternative answer. Furthermore, in contrast to the previous examples, the children used different ways in order to support their answer. Katerina used the counting of her fingers as a warrant for the right solution, whereas George counted the whistles. These two different kinds of perceptual counting unit items (Steffe et al., 1983) seemed to be acceptable by both of them implicitly, as it was not a topic of discussion during their collaboration.

Discussion

The purpose of this study was to identify the types of collaborative actions that the preschool children realized as they worked together when they had to solve a mathematical problem in pairs and to be led to a conclusion. Our results showed that the preschool children can collaborate productively when dealing with mathematical tasks, through coactions, as they worked together on the same idea (Martin et al., 2006). Therefore, we could argue that the preschool children interpreted the word "together", according to the researcher's advice about the way they were expected to deal with the tasks, as a realization of coactions for every step of the solution of the problem. In all cases, the children explicitly coordinated their efforts as they solved the mathematical problems.

In our research we identified three types of collaborative actions of the preschool children: a) using materials simultaneously, b) evaluating their different ways of solving a mathematical problem jointly and c) inspecting the different answers to a mathematical problem jointly. All the collaborative actions that the preschool children used gave them learning opportunities, as they critically examined their proposed solutions by repeating, through their actions, each other's ideas or by confirming their answers using alternative methods. An interesting question for further investigation is the relation of these collaborative actions to the zone of proximal development for different groups of children.

We should mention that in many studies the role of a teacher is significant for the construction of a productive collaborative learning environment (Cobb et al., 1992; Dekker & Elshout-Mohr, 2004; Edwards, 2002). The main habits that students have to adopt with the help of the teacher during their efforts to collaborate are usually: listening, explaining, justifying, reconstructing. In our study the preschool children hadn't been taught any kind of collaborative skills, as our purpose was to investigate possible collaborative actions that they might use on their own. Thus, another research question concerns how a kindergarten teacher could encourage children to use these collaborative actions, as they seemed to be effective for their mathematical learning.

Finally, according to our findings, the preschool children's spontaneous collaborative actions allow us to assert that they do not experience essential difficulties when they try to work together, which is in contrast with the performance of older children who have already shaped more stable beliefs about the learning and teaching of mathematics as well as about the individuality that characterizes traditional mathematics classroom (Cobb et al., 1992). Unlike the preschoolers, the older children have to change their beliefs in order to succeed a productive collaboration (Kafoussi, Chaviaris & Dekker, 2010). This fact could lead to the hypothesis that the development of collaboration from the preschool age could help the children to succeed productive collaborations during their future mathematical education.

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The APOS/ACE Instructional Treatment for Mathematics: A Fuzzy Approach

Michael Gr. Voskoglou
Professor Emeritus of Mathematical Sciences
School of Technological Applications
Graduate Technological Educational Institute of Western Greece
E-mail: mvosk@hol.gr , voskoglou@teipat.gr
Home page URL: http://eclass.teipat.gr/eclass/courses/523102

Abstract

The APOS/ACE theory for learning and teaching mathematics was developed during the 1990's in the USA by a team of mathematicians and mathematics educators led by Ed Dubinsky and one of its central ideas is the use of computers as a teaching tool. In this paper we introduce principles of fuzzy logic on comparing the performance of two student groups concerning the comprehension of the real numbers in general and of the irrational numbers in particular. The first group was taught the subject in the traditional way (control group), while the APOS/ACE instructional treatment was applied for the second group (experimental group). The two groups are represented as fuzzy subsets of the set of the grades (from A to F) achieved by the students in a pre-instructional and a post-instructional test and the çentroid defuzzification technique is applied on comparing their performance. The results of our classroom experiments showed that the application of the APOS/ACE approach can effectively help students to enlist the real numbers in a powerful cognitive schema including all the basic sets of numbers.

Keywords: Fuzzy sets, centroid defuzzification technique, teaching and learning the real numbers, APOS/ACE theory.

1. Introduction

The *fuzzy sets theory*, introduced by Zadeh[24] in 1965, gave genesis to fuzzy logic, a rich and meaningful addition to standard logic. The applications which may be generated from or adapted to fuzzy logic are wide-ranging and provide the opportunity for modelling under conditions which are inherently imprecisely defined, despite the concerns of classical logicians. Many systems may be modelled, simulated and even replicated with the help of fuzzy logic, not the least of which are human cognitive systems (e.g. [3], [5], [7], [8], [10-12], [16-19], etc)

Fuzzy logic offers a much higher problem solving capability than the standard probability theory opening the door to the construction of mathematical solutions of computational problems which are stated in a natural language.

The methods of assessing the individuals' performance usually applied in practice are based on principles of the bivalent logic (yes-no). However these methods are not probably the most suitable ones. On the contrary, fuzzy logic, due to its nature of including multiple values, offers a wider and richer field of resources for this purpose. This gave us several times in the past the impulsion to introduce principles of fuzzy logic in assessing the performance of student groups in learning mathematics and in problem solving (e.g. see [8], [11-12], [16], [19], etc).

In this paper we shall apply such kind of principles in comparing the results of performance of two groups of students of the Graduate Technological Educational Institute of Western Greece concerning their comprehension of real numbers in general and irrational numbers in particular. The first group was taught the subject in the traditional way (control group), while the APOS/ACE [acronyms constituted by the words **A**ctions, **P**rocesses, **O**bjects, **S**chemas for the former and **A**ctivities (on the computer), **C**lassroom (discussion), **E**xercises (done outside the class) for the latter]. instructional treatment was applied for the second group (experimental group). The APOS/ACE theory for learning and teaching mathematics was developed during the 1990's in the USA by a team of mathematicians and mathematics educators led by Ed Dubinsky and one of its central ideas is the use of computers as a teaching tool (see [1-2]).

The rest of the present paper is formulated as follows: In the next (second) section we present the headlines of the APOS/ACE theory and we provide some simple examples for its better understanding. In the third section we describe our classroom experiment, while in the fourth section we apply a simple fuzzy model on our experiment's process and we use the centroid defuzzification technique for assessing the performances of the two groups. Finally, in the last (fifth) section we state our conclusions and we discuss the future perspectives of this research.

2. The APOS/ACE theory for learning/teaching mathematics

APOS is a theory based on Piaget's principle that an individual learns (e.g. mathematics) by applying certain mental mechanisms to build specific mental structures and uses these structures to deal with problems connected to the corresponding situations [6]. Thus, according to the APOS analysis, an individual deals with a mathematical situation by using certain mental mechanisms to build cognitive structures that are applied to the situation. The main mechanisms are called *interiorization* and *encapsulation* and the related structures are *actions, processes, objects* and *schemas*.

The theory postulates that a mathematical concept begins to be formed as one applies transformations on certain entities to obtain other entities. A transformation is first conceived as an action. For example, if an individual can think of a function only through an explicit expression and can do little more than substitute for the variable in the expression and manipulate it, he/she is considered to have an action understanding on functions.

As an individual repeats and reflects on an action it may be interiorized to a mental process. A process performs the same operation as the action, but wholly in the mind of the individual enabling her/him to imagine performing the transformation without having to execute each step explicitly. For example, an individual with a process understanding of a function thinks about it in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs.

If one becomes aware of a mental process as a totality and can construct transformations acting on this totality, then we say that the individual has encapsulated the process into a cognitive object. In case of functions encapsulation allows one to form sets of functions, to define operations on such sets, to equip them with a topology, etc. Although a process is transformed into an object by encapsulation, this is often neither easy not immediate. This happens because encapsulation entails a radical sift in the nature of one's conceptualization, since it signifies the ability to think of the same concept as a mathematical entity to which new, higher-level transformations can be applied. On the other hand, the mental process that led to a mental object through encapsulation remains still available and many mathematical situations require one to *de-encapsulate* an object back to the process that led to it. This cycle may be repeated one or more times.

For example, in defining the sum f + g of two functions possessing a common domain, say A, it is necessary to reconsider again f and g at a process level and thinking of all x in A to obtain a new process mapping each x in A to the sum f (x) + g(x). Then this new process must be encapsulated, in order to obtain the function f + g at an object level.

A mathematical topic often involves many actions, processes and objects that need to be organized into a coherent framework that enables the individual to decide which mental processes to use in dealing with a mathematical situation. Such a framework is called a schema. In the case of functions it is the schema structure that is used to see a function in a given mathematical or real-world situation.

The APOS theory has important consequences for education. Simply put, it says that the teaching of mathematics should consist in helping students to use the mental structures that they already have to develop an understanding of as much mathematics as those available structures can handle. For students to move further, teaching should help them to build new, more powerful structures for handling more and more advanced mathematics. Dubinsky and his collaborators realized that for each mental construction that comes out of an APOS analysis, one can find a computer task of writing a program or code, such that, if a student engages in that task, he (she) is fairly likely to build the mental construction that leads to learning the mathematics. In other words, performing the task is an experience that leads to one or more mental constructions. As a consequence of the above finding, the pedagogical approach based on the APOS analysis, known as the *ACE teaching cycle*, is a repeated cycle of three components: *Activities on the computer, classroom discussion* and *exercises done outside the class.* The target of the activities on the computer is to help students in building the proper mental constructions for the better understanding and learning of the corresponding mathematical topic. The students discuss later in the classroom their experiences from the computer tasks performed in the laboratory, they repeat the same tasks without the help of computer this and they reach, under their instructor's guidance and help, to the proper conclusions. Finally, the purpose of the exercises, which are given by the tutor as a home work, is to check and to embed better the new mathematical knowledge (for more details see [1-2], [17], etc).

3. The classroom experiment

The implementation of the ACE cycle and its effectiveness in helping students making mental constructions and learn mathematics has been reported in several research studies of the Dubinsky's team. A summary of earlier work can be found in [21]. More recently this approach was applied in studying the pre-service teachers understanding of the relation between a fraction or an integer and its decimal expansion [22-23].

In developing and applying in practice the ACE design for teaching the real numbers in general and the irrational numbers in particular we performed during the winter semester of the academic year 2012-13 (October 2012) a classroom experiment with two groups of students of the Graduate Technological Educational Institute (T. E. I.) of Western Greece (ex Patras) being at their first term of studies. The subjects of the experimental group were 90 students of the School of Technological Applications (prospective engineers) attending the course "Higher Mathematics I"[1]. The students of this group were taught the real numbers in the computers' laboratory and in the classroom according to our ACE design that we shall present below. The subjects of the control group were 100 students of the School of Management and Economics of the same T. E. I. attending a similar mathematical course (the instructor was the same person). In this group the lectures were performed in the classical way on the board, followed by a number of exercises and examples. The students participated in solving these exercises. Notice that the students of both groups had more or less the same mathematical background from secondary education, since they had finished the same type of Lyceum (the upper level of secondary education in Greece). Further, the grades that they obtained in the Panhellenic exams for entrance in the higher education were of about the same level. Also, since they were in the first term of their studies, they had attended no previous mathematical courses at the T. E. I. of Western Greece.

On the first day in class the students of both groups completed individually a five-item pre-instructional written questionnaire (see Appendix I). The instrument served to establish the similarity of the two groups and to guide the development of the teaching process. The results of

[1] This course involves an introductory chapter repeating and extending the students' knowledge from secondary education about the basic sets of numbers, Complex Numbers, Differential and Integral Calculus in one variable, Elementary Differential Equations and Linear Algebra.

the pre-instructional test for the two student groups are presented in the following table:

Table 1: Results of the pre-instructional test

Experimental group (G_1)

% Scale	Grade	Amount of students	% of students
89-100	A	0	0
77-88	B	17	18.9
65-76	C	18	20
53-64	D	25	27.8
Less than 53	F	30	33.3
Total		90	

Control group (G_2)

% Scale	Grade	Amount of students	% of students
89-10	A	0	0
77-88	B	18	18
65-76	C	20	20
53-64	D	30	30
Less than 53	F	32	32
Total		100	

The interpretation of the data of Table 1 will be presented in section 4 in terms of our fuzzy assessment method.

Our APOS/ACE approach for teaching the real numbers to the students of the experimental group involved three iterations of the ACE cycle. Each cycle consisted of two class sessions, one for computer activities and one for classroom discussions. Homework exercises were assigned and collected. Notice that, since the proper understanding of rational numbers is an essential pre-assumption for the comprehension of irrational numbers, our design involved frequent repetitions of the corresponding situations for rational numbers. Some of these repetitions were adapted from [22].

In an action level the concept of an infinite decimal (rational or irrational number) is understood by considering its finite decimal approximations. The target of the *first iteration* of the ACE cycle was to facilitate the interiorization of this action to a process. The students completed in the computer laboratory activities with a preloaded decimal expansion package. They developed general descriptions of what was stored and answered various questions about an infinite digit string such as: What is a repeating decimal? Which of the strings are repeating decimals? What are the digits in the first 20 places after the decimal point and what would appear in the 1005th place? Further, students were asked to calculate the successive finite decimal approximations of several square roots with gradually increasing accuracy.

In the classroom discussion the students reported their group responses and the class negotiated agreements. A notational system for infinite decimals was devised. For example, since $1<\sqrt{2}<2$, $1.4<\sqrt{2}<1.5$, $1.41<\sqrt{2}<1.42$, $1.414<\sqrt{2}<1.415$, $1.4142<\sqrt{2}<1.4143$, $1.41421<\sqrt{2}<1.41422$, etc, $\sqrt{2}$ can be written as $\sqrt{2}= 1.41421.....$. The dots at the end indicate that the sequence of the decimal digits is continued to infinity. Therefore, by accepting this symbolic representation of an infinite decimal we can not see written all its decimal digits. We can only see the digits of its given decimal approximation each time. The instructor recalled at this point that a repeating decimal (rational number) can be written in the form $a,b\overline{c}$. Here *a, b, c* are natural numbers, where *a* denotes the integer part of the rational number, *b* is its decimal portion that possibly appears before the repeating cycle (in case of mixed periodic numbers) and *c* is the repeating cycle (period) of the number. A finite decimal can be written as a repeating decimal with period *0* or *9*; e.g. $2.5 = 2.5\overline{0} = 2.4\overline{9}$ The exercises included problems where certain information about an infinite digit string was provided that was sufficient to specify the string.

The target of the *second iteration* of the ACE cycle was to facilitate the encapsulation of the concept of a real number to a mental object. During the computer activities students were asked to work out examples with *transparent* and *opaque* decimal representations of real numbers like the following:

The rational numbers $\frac{3}{5}=0.6$, $\frac{1}{3}=0.33...$, $\frac{281849}{99900}=2.82113113113...$, have transparent decimal representations, since we can foresee their decimal digits in all places; but the same is not happening with $\frac{1}{1861}= 0.0005373...$, which, possessing a period of 1860 digits, has an opaque decimal representation. Notice that decimal representations of certain irrational numbers, despite of their complex structure in general, are also transparent. For example, this happens with the numbers *2.0013131131113111131111113...* where *1*, following *13*, is repeated one more time at each time, and *0.282288222888222288882...*where *2* and *8*, following *28*, are repeated one more time at each time. Taking this opportunity the instructor clarified to the class that an infinite decimal is an incommensurable (non periodic) decimal <u>not</u> because its decimal digits are not repeated in a concrete process (this in fact <u>can</u> happen according to the above two examples), but because <u>it has not a period</u>, i.e. its decimal digits are not repeated in the same concrete series. Some standard cases of decimal expansions of transcendental numbers like π and *e* were also added to the above examples. Students were also asked to convert fractions and roots of second or higher order to decimals and vice versa. Further, the computer activities included arithmetic operations among irrational and rational numbers by using their finite decimal approximations.

In the classroom the students performed the same mathematical activities without using the computers. In this way they realized that in converting a fraction to a decimal, if the quotient obtained is an infinite decimal having a long period, a long and laborious division is reached in general, which is not possible to be determined soon. At this point the instructor emphasized that given a fraction $\frac{\mu}{\nu}$, $\mu,\nu \in Z,\ \nu \neq 0$, the quotient of the division $\mu : \nu$ is <u>always</u> a periodic decimal. The probability to be a finite decimal is small enough, since a fraction, whose denominator is not a product of powers of *2* and/or *5*, cannot be written as a finite decimal. In case of an infinite decimal, since the remainder of the division $\mu : \nu$ is smaller that ν, performing the division and after a finite number of steps (at most *ν-1*) the same remainder will reappear at some step. This means that the resulting decimal is a periodic one, having a period of at most *ν-1* digits. Conversely, a standard method for converting periodic numbers to fractions (although other methods could be used as well) is by subtracting both members of proper equations containing multiples of a power of *10* of the given number. For example, given *x=2.75323232...*, we write *10000x=27532.3232...* and *100x=275.3232....*, wherefrom we get that *9900x=27532-275*, or $x=\frac{27257}{9900}$. Reflecting on the above examples the students reached to the conclusion that periodic decimals and fractions are the same numbers written in a different way. Students' contact at school with the definition of irrational numbers as *incommensurable decimals* is usually rather slim, while emphasis is given on defining them as *non rational numbers* (i.e. they cannot be written as fractions $\frac{\mu}{\nu}$, with μ, ν integers and $\nu \neq 0$). However, students must clearly understand the equivalence between the above two definitions: Similarly to the fact that rational numbers and periodic decimals are the same numbers written in a different way, the same holds for non rational numbers and incommensurable decimals. Thus, the set of real numbers ***R*** can be defined as the set of all commensurable and incommensurable decimals and their opposites. In closing the classroom discussion the instructor presented empirically the concept of a sequence of finite decimals and of its limit (i.e. what it means to "tend" to a number) and explained it to students by using the appropriate examples, like this with $\sqrt{2}$ mentioned above. In no case is it necessary for the teacher to give the analytic definition of the limit of a sequence. The above empiric approach is enough for helping students to encapsulate the concept of a real number to a mental object. The homework exercises were standard problems related to the topics mentioned above aiming to consolidate the students' knowledge and understanding of these topics.

The target of the *third* iteration of the ACE cycle was to help students to enlist real numbers in general and irrational numbers in particular in their cognitive schema related to the already known basic sets of numbers (natural numbers, integers and rational numbers). A prerequisite for this is that they must be able to transfer in comfort among the several representations of real numbers. Therefore, the computer activities in this cycle involved among others examples of constructions of line segments with incommensurable lengths; either classical geometrical constructions by using the Pythagorean theorem, like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc, or cases where the construction of the graph of a function is necessary, like $\sqrt[3]{2}$ with the function $f(x)=\sqrt[3]{x}$ (or $f(x)=\sqrt[3]{x}-2$) etc. They involved also examples of writing real numbers in the form of a series $x=\sum_{n=0}^{\infty}\frac{\kappa_n}{10^n}$, where κ_0 is an integer and $\kappa_1, \kappa_2, \ldots, \kappa_n, \ldots$ are natural numbers less than 10. Finally, the computer activities involved also examples of interpolation of rational and irrational numbers between two given integers, or between two rational (irrational) numbers aiming to promote the later discussion in classroom about the density of the sets of rational and real numbers.

In the classroom discussion the instructor recalled first that in defining the set **Q** of rational numbers as the set of all fractions and in order to count each fraction only once, we considered only the fractions of the form $\pm\frac{\mu}{\nu}$, where μ and ν are non negative integers ($\nu \neq 0$), with greatest common divisor equal to *1*. In an analogous way, since for all integers κ and α with $1 \leq \alpha \leq 9$ we have $\kappa.\alpha = \kappa.(\alpha\text{-}1)\overline{9}$ and $\kappa.\overline{9} = \kappa+1$ [2], in defining ***R*** as the set of all decimals and in order to count each real number only once , we must exclude all infinite decimal expansions of the form $\kappa.\kappa_1\kappa_2$…….. , in which there exists a natural number ν such that $\kappa_\mu=9$ for all $\mu \geq \nu$.

Activities on geometric constructions of irrational numbers were also organized in classroom combining history of mathematics with Euclidean Geometry. For example, as we have already mentioned above, these activities included the construction of the line segments of length $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc with the ruler and compass only in terms of the Pythagorean Theorem and the proper placing of them on the real axis, followed by proofs of the fact that the above lengths do not correspond to rational numbers (since they cannot be written in the form $\frac{\mu}{\nu}$, with μ and ν natural numbers, $\nu \neq 0$). Within the culture of ancient Greek mathematics the geometric figure was the basis for unfolding mathematical thought, since it helped in obtaining conjectures, fertile mathematical ideas and justifications (proofs). In fact, convincing arguments are built by drawing auxiliary lines, optical reformations and new modified figures, and therefore mathematical thinking becomes more completed in this way. Therefore the geometric representations of real numbers enrich their teaching, connecting it historically with the discovery of incommensurable magnitudes and the relevant theory of Eudoxus. Following these historical steps of the human thought is therefore a good way for helping students to accept the existence of incommensurable magnitudes.

[2] In this case, if we denote by *[x]* the integral part of *x*, we have that $[x] = \kappa_0$ and at the same time that $[x] = \kappa_0+1$, which is absurd! Therefore there is a debate in the literature whether or not decimal expansions of the form $\kappa_0.\overline{9}$ are representing real numbers; e.g. see [15]. Fortunately the results obtained when using these representations are conventionally correct because the corresponding operations could be performed in an analogous way among the sequences of the partial sums of the corresponding series. This allows us at elementary level to pass through this sensitive matter without touching it at all.

Another crucial matter for the instructor is to find the proper way to explain to students the continuum of ***R*** with respect to the density of ***Q***. In other words to persuade them that in a given interval of numbers it is possible to have an infinite number of elements of a certain type (rational numbers) and at the same time to be able to add an infinity of elements of another type (irrational numbers), when this is not compatible with the usual logic and intuition. It seems that the use of the geometric representations of real numbers is a proper way to deal with this problem (an interval of points on the real axis cannot be "filled" with rational points only). The difficulty in this case is that most of the irrational numbers, like $\sqrt[3]{2}$, *π, e,* etc, are associated with lengths of line segments that cannot be constructed geometrically. Therefore, we correspond to all these numbers points of the real axis in an approximate way by using their finite decimal approximations and our fantasy[3].

However, things became more complicated when we arrived to the natural, but crucial, question asked by students: *"Which numbers can be written in the form of an incommensurable decimal number?"* At the lower high-school level (Gymnasium) students learnt that this happens with the square roots of positive rational numbers that they cannot be exactly determined (i.e. they have not an exact value). Later, at the upper high-school level (Lyceum), they learnt that this also happens with the n-th roots, $n \in \mathbf{N}$, $n \geq 2$. However, as the instructor emphasized at this point, the converse is not true, since they are incommensurable decimal numbers that cannot be written as roots, or in a more general expression they are not roots of an algebraic equation with rational coefficients. Thus we arrive to the concept of the *transcendental numbers*. This new kind of numbers usually activates students' imagination and increases their interest by creating a pedagogical atmosphere of mystery and surprise. The instructor informed students (without giving any proofs) that the set of algebraic numbers is a denumerable set, while, as Cantor has proved, the set of transcendental numbers has the power of continuum (i.e. it is equivalent with the whole set ***R*** of real numbers, "filling" all the points of the real axis).This practically means that transcendental numbers are much more than algebraic numbers,

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[3] Mathematically speaking the above correspondence is based on the *principle of the nested intervals* connected to the method of *Dedekind cuts* for defining the real numbers (e.g. see section 2 of [13]), an approach not compatible with an elementary presentation of real numbers to students.

but, apart from some characteristic examples, like π and e, the information that we have about them is very small relative to their multitude. That is why one can characterize them as a “black hole” (in analogy with the astronomical meaning of term) in the “universe” of real numbers [14].

In concluding, our general didactic approach included: A fertile utilization of already existing informal knowledge and beliefs about numbers, active learning through rediscovery of concepts and conclusions, construction of knowledge by students individually or as a team in the computer laboratory and in classroom. Construction of knowledge followed in general student’s perspective, while teacher’s role was limited to the discussion in the whole class of wrong arguments and misinterpretations observed. The teaching process was based on multiple representations of real numbers (rational numbers written as fractions and periodic decimals, irrational numbers considered as non rational ones and as incommensurable decimals which are limits of sequences of rational numbers, geometric representations, etc) and on flexible transformations among them. It was hoped that this approach could help students in building a powerful schema for real numbers.

At the end of the instructional unit students of both groups completed a new ten-item post-instructional written questionnaire (see Appendix II). Students were instructed to work on the questionnaire individually and to answer each question thoroughly. The instrument counted as a progress grade added to the course’s final exam results. The results of the post-instructional test for the two student groups are presented in the following table:

Table 2: **Results of the post-instructional test**

Experimental group (G_1)

% Scale	Grade	Number of Students	% of students
89-100	A	3	3.3
77-88	B	21	23.3
65-76	C	28	31.1
53-64	D	22	24.4
Less than 53	F	16	17.8
Total		90	

Control group (G_2)

% Scale	Grade	Amount of students	% of students
89-100	A	1	1
77-88	B	10	10
65-76	C	37	37
53-64	D	31	31
Less than 53	F	21	21
Total		100	

4. Application of fuzzy logic in assessing the student's performance

As we have already stated in our introduction, we have applied principles of fuzzy logic in comparing the results of performance of the two groups of students of the previous experiment by implementing on the experiment's process the following fuzzy model and the defuzzification technique known as the *centroid method*. According to this method, the centre of gravity of the graph of the membership function involved provides an alternative measure of the system's performance. The application of the centroid method in practice is simple and evident and, in contrast to the measures of uncertainty which can be also used as alternative defuzzification techniques (for example see [11] and its references), needs no complicated calculations in its final step. The techniques that we shall apply here have been also used earlier in [7], [16-18], etc.

For a better understanding of the present section, the readers who are not familiar with fuzzy logic are advised to study first the introduction of [18], where the basic principles of this multi-valued logic are briefly presented. For a detailed description of the fuzzy sets theory, the ideas and principles of fuzzy logic and the uncertainty connected to it we refer to the book [4].

Here, for reasons of continuity, let us start with the definition of fuzzy sets [24]: Let U be the universal set of the discourse, then a *fuzzy subset A of U*, or equivalently a *fuzzy set in U* is a set of ordered pairs of the form $A=\{(x, m(x)) : x \in U\}$, where m: $U \rightarrow [0, 1]$ is the associated to *A membership function*.

The value $m(x)$ of each x in U is called the *membership degree* of x in A. The closer is $m(x)$ to 1, the better x satisfies the property characterizing A. For example, if A is the set of the tall students of a class and m(x) = 0.9, then the student *x* is rather tall. On the contrary, if *m(x)* = 0.2, then *x* is rather short, while if *m(x)* = 0.5, then *x* is of middle height.

Obviously, any crisp subset *A* of *U* can be considered a fuzzy subset of *U* with membership function *m* defined by *m(x)* = 1, if *x* belongs to *A* and *m(x)* = 0, if *x* is not in *A*. Most of the classical notions for crisp sets (e.g. subset, intersection, union, complement, etc) can be generalized in terms of the above definition to corresponding notions of fuzzy sets.

Now, given a fuzzy subset $A = \{(x, m(x)): x \in U\}$ of *U* with membership function *m*, we associate to each $x \in U$ an interval of values from a prefixed numerical distribution, which actually means that we replace *U* with a set of real intervals. Then, we construct the graph of the membership function *y=m(x)*. There is a commonly used approach in fuzzy logic to measure performance with the pair of numbers (x_c, y_c) as the coordinates of the *centre of gravity (centoid)*, say F_c, of the level's section *F* contained between the graph of *m* and the OX axis, which we can calculate using the following well-known from Mechanics formulas:

$$x_c = \frac{\iint_F x\,dxdy}{\iint_F dxdy},\; y_c = \frac{\iint_F y\,dxdy}{\iint_F dxdy} \qquad (1)$$

Concerning the described experiment, we characterize a student's performance as very low (F) if $x \in [0, 1)$, as low (D) if $x \in [1, 2)$, as intermediate (C) if $x \in [2, 3)$, as high (B) if $x \in [3, 4)$ and as very high (A) if $x \in [4, 5]$ respectively. Denote by G_1 the experimental group and by G_2 the control group and set *U* = {A, B, C, D, F}.

We are going to represent the G_i's, i=1, 2, as fuzzy subsets of *U*. For this, if n_{iF}, n_{iD} n_{iC}, n_{iB} and n_{iA} denote the number of students of group G_i who achieved very low, low, intermediate, high and very high success respectively, we define the membership function m_{Gi} in terms of the percentages of the students who achieved the corresponding performance. More explicitly for each x in *U*. we define [4]

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[4] We recall that the methods of choosing the suitable membership function are usually empiric, based either on the common logic (as it happens in our case) or on the data of experiments made on a representative sample of the population that we study. For general facts on fuzzy sets we refer freely to the book [4].

$$y = m_{G_i}(x) = \begin{cases} 1, & if \quad 80\%\ n < n_{ix} \le n \\ 0,75, & if \quad 50\%\ n < n_{ix} \le 80\%\ n \\ 0,5, & if \quad 20\%\ n < n_{ix} \le 50\%\ n \\ 0,25, & if \quad 1\%\ n < n_{ix} \le 20\%\ n \\ 0, & if \quad 0 \le n_{ix} \le 1\%\ n \end{cases}$$

Then G_i can be represented as a fuzzy subset of *U* by $G_i = \{(x,\ m_{G_i}(x)) : x \in U\}$, i=1,2. Therefore in this case the level's section *F* defined by the graph of the membership function of the corresponding fuzzy subset of *U* is the bar graph of Figure 1 consisting of five rectangles, say F_i, i=1,2,3, 4, 5 , whose sides lying on the x axis have length 1. It is straightforward then to check (e.g. see section 3 of [18]) that in this case formulas (1) give:

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right),$$
$$y_c = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right) \qquad (2)$$

with $y_i = \dfrac{m(x_i)}{\sum_{x \in U} m(x)}$, where x_1 = F, x_2 = D, x_3 = C, x_4 = B, x_5 = A respectively and $y_1+y_2+y_3+y_4+y_5=1$.

Further, using elementary algebraic inequalities, it is easy to check that $y_1^2+y_2^2+y_3^2+y_4^2+y_5^2 \ge \frac{1}{5}$, with the equality holding if and only if $y_1=y_2=y_3=y_4=y_5=\frac{1}{5}$ (e.g. see section 3 of [18]). Then a combination of formulas (2) shows that the unique minimum value for y_c ($y_c = \frac{1}{10}$) corresponds to the centre of gravity $F_m\left(\frac{5}{2}, \frac{1}{10}\right)$.

The ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from formulas (2) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the centre of gravity in this case is the point $F_i\left(\frac{9}{2}, \frac{1}{2}\right)$. On the other hand, the worst case is when $y_1=1$

and $y_2=y_3=y_4= y_5=0$. Then from formulas (2), we find that the centre of gravity is the point $F_w(\frac{1}{2}, \frac{1}{2})$.

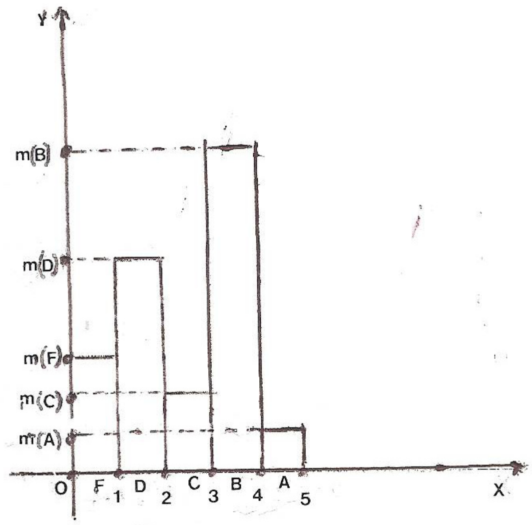

Figure 1: Bar graphical data representation

Therefore the "area" where the centre of gravity F_c lies is represented by the triangle $F_w F_m F_i$ of Figure 2.

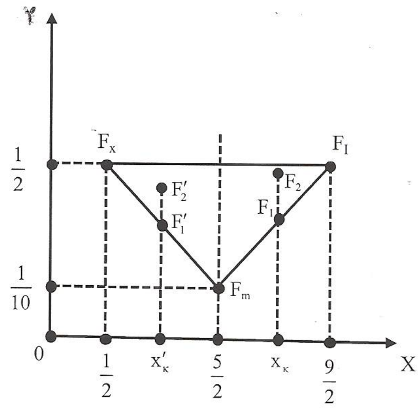

Figure 2: Graphical representation of the "area" where the centre of gravity lies

Then from elementary geometric considerations one obtains the following criterion for comparing the groups' performances:

- *Among two or more groups the group with the biggest x_c performs better.*
- *If two or more groups have the same $x_c \geq 2.5$, then the group with the higher y_c performs better.*
- *If two or more groups have the same $x_c < 2.5$, then the group with the lower y_c performs better.*

For a more detailed description of the centroid defuzzification technique see section 3 of [18].

We apply now this model to the data of Table 1 (pre-instructional test, see section 3). The two student groups can be represented as fizzy subsets of *U* by G_1 = {(A, 0), (B, 0.25), (C, 0.25), (D, 0.5) (F, 0.5)} = G_2

Therefore the students of the two groups responded similarly to the questionnaire's items.

In the same way, on applying the model to the data of Table 2 (post-instructional test, see section 3) we represent the two groups as fizzy subsets of *U* by G_1 = {(A, 0.25), (B, 0.5), (C, 0.5), (D, 0.5) (F, 0.25)} and
G_2 = {(A, 0), (B, 0.25), (C, 0.5), (D, 0.5) (F, 0.5)}.

Then, from the first of formulas (2), we find for the pre-instructional test that x_c= 0.5*(0.5+3*0.5+5*0.25+7*0.25+9*0) = 2.5 for both groups, while for the post-instructional test we find that
x_c= 0.5*(0.25+3*0.5+5*0.5+7*0.5+9*0.25) = 5 for the experimental group and x_c= 0.5*(0.5+3*0.5+5*0.5+7*0.25+0*9) = 3.125 for the control group.

So, according to the above stated criterion, both groups demonstrated a better performance in the post-instructional with respect to the pre-instructional test (as it was logically expected), but the experimental group demonstrated a significantly better performance than the control group in the post-instructional test. Notice also that two analogous experiments were performed during the academic year 2011-12. In this case the students' performance was assessed by applying the classical *GPA method* [5] and the results obtained were similar to the above experiment [20].

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[5] The **G**reat **P**oint **A**verage (GPA) is a weighted average of the students' performance. For this, if *n* is the total number of students and n_A, n_B, n_C, n_D, n_F denote the numbers of students getting the marks A, B, C, D, F respectively, GPA = $\frac{0.n_F + 1.n_D + 2.n_C + 3n_B + 4.n_A}{n}$.
Obviously we always have $0 \leq$ GPA ≤ 4.

In concluding, it seems that the application of the APOS/ACE teaching style enhances significantly the students' understanding of the real numbers in general and of the irrational numbers in particular.

5. Discussion and conclusions

Fuzzy logic, due to its nature of including multiple values, offers a wider and richer field of resources in assessing the students' performance than the practice usually applied in bivalent logic does. This gave us the impulsion in this paper to introduce principles of fuzzy logic on comparing the performance of two student groups concerning the comprehension of real numbers in general and of irrational numbers in particular. The first group was taught the subject in the traditional way (control group), while the APOS/ACE instructional treatment was applied for the second group (experimental group). The APOS/ACE theory for learning and teaching mathematics, developed during the 1990's by Ed Dubinsky and his collaborators, is based on principles of Piaget for the learning process and one of its central characteristics is the use of computers as a teaching tool.

The two groups were represented as fuzzy subsets of the set of the grades (from A to F) achieved by the students in a pre-instructional and a post-instructional test and the çentroid defuzzification technique was applied on comparing their performance.

The results of our classroom experiments, performed during the academic years 2011-12 and 2012-13 at the graduate T.E.I of Western Greece showed that the application of the APOS/ACE approach for teaching the real numbers in general and the irrational numbers in particular can effectively help students for building a powerful cognitive schema for the basic sets of numbers.

However, all those discussed in this article must not be considered as an intention to impose a model of teaching (at an elementary level) the real numbers. On the contrary, our general belief is that the instructor should be able to make a small "local research", readapting methods and plans of the teaching process according to the teaching environment and the special conditions of each class [9]. Therefore this is simply an effort to introduce an alternative approach that could help the instructor to reconsider and organize better his/her teaching plans for this subject.

Among our future research plans is the performance of more classroom experiments with different groups of students (high school students as well!) in order to obtain statistically stronger results and conclusions. It will be interesting also to apply to the same data, on the one hand assessment techniques of the students' performance based on fuzzy logic, and on the other hand usual assessment techniques based on classical logic (like the GPS method mentioned above), and then to compare and analyze their results, advantages and disadvantages. Furthermore, since the centroid deffuzzification technique is part of a general fuzzy model introduced in earlier works ([12], [17]) for dealing with situations in a system's operation characterized by a degree of fuzziness and/or uncertainty, another direction of our future research concerns the effort of representing in terms of this model even more such situations related to several sectors of the human activity (e.g. education, management, artificial intelligence, everyday life, etc).

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Appendix I: Pre-instructional questionnaire

1. Which of the following numbers are natural, integers, rational, irrational and real numbers?

-2 , $-\frac{5}{3}$, 0 , 9.08 , 5 , $7.333...$, $\pi = 3.14159...$, $\sqrt{3}$, $-\sqrt{4}$,

$\frac{22}{11}$, $5\sqrt{3}$, $-\frac{\sqrt{5}}{\sqrt{20}}$, $\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)$, $-\frac{\sqrt{5}}{2}$, $\sqrt{7}-2$, $\sqrt{\left(\frac{5}{3}\right)^2}$

2. Are the following inequalities correct, or wrong? Justify your answers.

$\frac{2}{3} < \frac{14}{21}$, $\frac{2002}{1001} > 2$

3. Convert the fraction $\frac{7}{3}$ to a decimal number. What kind of decimal number is this and why we call it so? .

4. Find the integers and the decimals with one decimal digit between which lies $\sqrt{2}$. Justify your answers.

5. Find two rational and two irrational numbers between 10 and 20. How many irrational numbers are there between these two integers?

Appendix II: Post-instructional questionnaire

1. Which is the exact quotient of the division *5:7*?

2. Are *2.8254131131131…* and *2.0013131131113111…* periodic decimal numbers? In positive case, find the period and convert the corresponding number to a fraction.

3. Find the square roots of *9, 100* and *169* and describe your method of calculation.

4. Characterize the following expressions by C if they are correct and by W if they are wrong: $\sqrt{2}=1.41$, $\sqrt{2}=1.414444...$, $\sqrt{2}\approx 1.41$, there is no exact price for $\sqrt{2}$.

5. Find two rational and two irrational numbers between $\sqrt{10}$ and $\sqrt{20}$. How many rational numbers are there between these two square roots?

6. Are there any rational numbers between $\frac{1}{11}$ and $\frac{1}{10}$? In positive case, write down one of them. How many rational numbers are between the above two fractions?

7. Are there any rational numbers between *10.21* and *10.22*? In positive case, write down one of them. How many rational numbers are in total between the above two decimals?

8. Characterize the following expressions as correct or wrong. In case of wrong ones write the corresponding correct answer.

$\sqrt{3+5}=\sqrt{3}+\sqrt{5}$, $\sqrt{3\cdot 7}=\sqrt{3}\cdot\sqrt{7}$, $\sqrt{\frac{2}{9}}=\frac{\sqrt{2}}{3}$, the unique solution of the equation $x^2=3$ *is* $x=\sqrt{3}$, $\sqrt{\left(1-\sqrt{17}\right)^2}=1-\sqrt{17}$

9. Construct, by making use of ruler and compass only, the line segments of length $\sqrt{5}$ and find the points of the real axis corresponding to the real numbers $\sqrt{5}$ and $-\sqrt{5}$. Consider a length of your choice as the unit of lengths.

10. Is it possible for the sum of two irrational numbers to be a rational number? In positive case give an example.