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HELLENIC MATHEMATICAL SOCIETY

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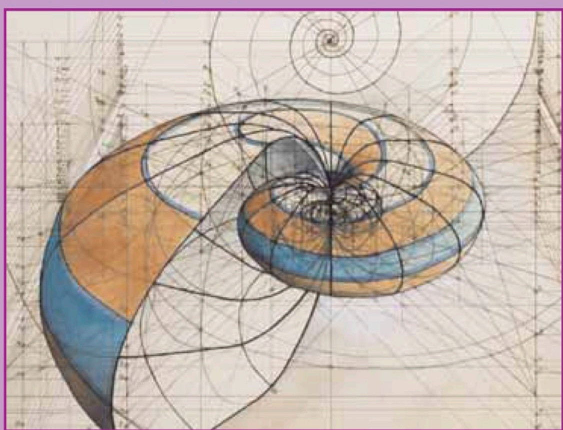
HMS i JME

VOLUME 7, 2015-2016

including special issue
"Investigating Complex Systems in Mathematics Education"



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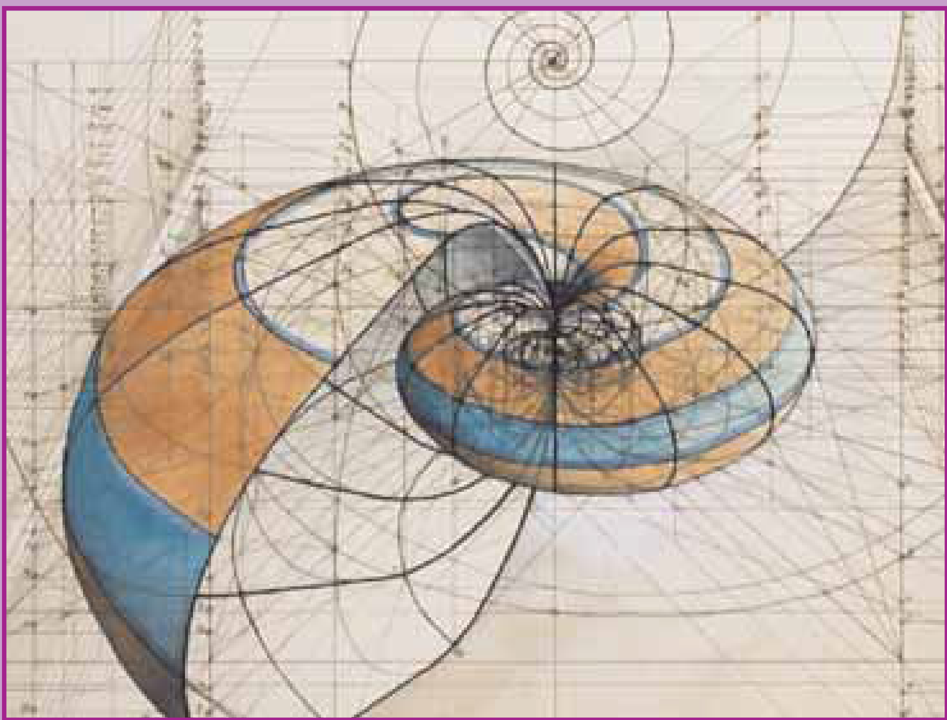

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International Journal for Mathematics in Education (*HMS i JME*)

The Hellenic Mathematical Society (HMS) decided to add this Journal, the seventh one, in the quite long list of its publications, covering all aspects of the mathematical experience. The primary mission of the HMS International Journal for Mathematics Education **(*HMS i JME*)** is to provide a forum for communicating novel ideas and research results in all areas of Mathematics Education with reference to all educational levels.

The proposals must be written almost exclusively in English but may be admitted, if necessary, also in French, in German or perhaps in Spanish.

The proposals could concern: research in didactics of mathematics, reports of new developments in mathematics curricula, integration of new technologies into mathematics education, network environments and focused learning populations, description of innovative experimental teaching approaches illustrating new ideas of general interest, trends in teachers' education, design of mathematical activities and educational materials, research results and new approaches for the learning of mathematics.

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Introduction

In this volume of HMS-iJME eight contributions are included, organised in two parts.

In the first part we present a special issue with five invited papers concerning a special topic: Investigating Complex Systems in Mathematics Education. The purpose of this collection is to identify diverse systemic perspectives with respect to the complexity in mathematics education.

The contemporary reality spans across the perceptual and the virtual world, across the teaching practices and the international comparative studies with proximity being topologically redefined to transcend its geographical meaning. The re-conceptualisation (seemingly reversal) between the local and the global objectives, as well as between the didactical research and the teaching practices are actually becoming more evident and it is very difficult to describe the connections and to conceptualize the emerging situations for the mathematics education.

Important time-spaces for these exchanges are created throughout the mathematics teaching and learning communities ranging from relatively informal structures (such as personal blogs), to official communities' communication spaces (such as official fora of research or of teachers' associations), or even to international committees or organisations (such as ICMI, CIEAEM, PME, MES or the TIMSS and PISA, the UNESCO and OECD reports for mathematics education).

Furthermore, the academic output -with a continuous revalorisation of the dimensions that affect the phenomenology of the teaching and learning of mathematics- is now more than ever transparent and accessible to the mathematics education community and the broader society, due to technological and sociocultural convergences (including the internet, the file formats, the communication language).

In this complex framework, it seems reasonable to investigate mathematics education through the conceptualisations of complexity and of the systemic approach, in order to facilitate our deeper understanding the interactive characteristics of mathematics in education.

Mathematics educators have discussed issues about complexity and systems in the last decades. Notably, in March 2003 in the same issue of Journal for Research in Mathematics Education Brent Davis and Elaine Simmt drew our attention to complexity and learning systems, while Helen Doerr and Lyn English discussed basic complex systems through mathematical modelling. Less than year later, in 2004, Complicity: An International Journal of Complexity and Education was launched (with Brent Davis as the co-editor), investigating related issues in education. In Greece, researchers in Didactic of Mathematics have recently developed a variety of publications, round tables and conferences on complexity and on systemic ideas.

It is an honour for HMS-iJME that Brent Davis contributes to the present collection of papers, with his paper entitled “Complexity as a prompt to rethink school mathematics: From computation to modelling”. Davis discusses the diverse conceptualisations of complexity adopted by mathematics educators. Subsequently, he presents an empirical study about a teaching experiment concerning exponentiation. Davis questions what constitutes the “basics” within school mathematics, arguing at the same time for mathematics education towards modelling, rather than computation.

The collection continues with Dimitris Chassapis’ theoretical discussion about “Conceiving mathematics classrooms as activity systems”. Chassapis employs activity theory to gain deeper understanding about the complex phenomena that occur in mathematics classrooms. He argues that mathematics education research and practice may benefit from conceptualizing mathematics classrooms as complex systems of activities, taking also into account their interactions within the educational system and the broader sociocultural context.

Konstantinos Nikolantonakis’ contribution is an empirical investigation entitled “A Mathematical activity for the training of In-Service Primary school teachers using a Systemic Approach”. Nikolantonakis synthesises systemic and cybernetic ideas with radical constructivism ideas to discuss in service primary school teachers' dealing with "The target number", utilised as a field for repositioning the concept of division within the tool-object dialectic. The systemic concepts of framing, interaction and co-construction are at the crux of his framework of analysis, emphasising the importance of the construction of a multi-levelled dialogue amongst teacher, student - trainee, class.

Fessakis and Kirodimou's paper concerns an empirical investigation about the teachers' professional development entitled "Improving the teachers' understanding of complex systems through dynamic systems modelling and problem solving". Through a research by design methodology (combining systems dynamics, authentic problem solving and Digital Games Based Learning), they argue that the employment of dynamic systems modelling, as applied in sustainability problems on the field of ecosystems, may facilitate the teachers' understanding of complex systems.

The collection concludes with a paper authored by Andreas Moutsios-Rentzos and Francois Kalavasis entitled "Systemic approaches to the complexity in mathematics education research". By conceptualizing learning as linking links and the school unit as a learning organization, they introduce a framework to empirically reveal implicit inter-/intra- systemic links in the system of scientific disciplines and the system of school unit as experienced by the educational protagonists. They argue that this approach helps the didactical planification towards mathematics learning as linking links by identifying a communication space amongst the seemingly incongruent experience spaces of the educational protagonists.

The second part of the volume includes three papers investigating diverse mathematics education topics.

Katerina Kasimatis, Tasos Barkatsas and Vasilis Gialamas in their paper entitled "Values about mathematics learning: focusing on Greek high school students" discuss the structure of mathematics values of Greek students. The conducted analysis revealed nine value factors, containing both inter-cultural and specific to Greek population aspects, with the identified gender and grade level comparisons providing deeper understanding of the development of the values the Greek students hold about mathematics and mathematics learning.

Ann Luppi von Mehren in her paper entitled "Inspiration for Elementary Mathematics Descriptions from a "Heritage" Reading of On the Nonexistent by Gorgias" adopts Ivor Grattan-Guinness' distinction between history and heritage to present a heritage reading of Gorgias' *On the Nonexistent*. She argues that Gorgias' text can be helpful for elementary mathematics teachers in their designing and communicating challenging mathematical ideas.

Michael Voskoglou and Igor Subbotin with their contribution entitled "An Application of Fuzzy Logic for Learning Mathematics according to the Bloom's Taxonomy" discuss an improved version of the Trapezoidal Fuzzy Assessment Model to assess the students' learning with respect to "Real numbers" in line with Bloom's Taxonomy. They employ the Center of Gravity defuzzification technique to more efficiently (in comparison with traditional assessment methods based on bivalent logic) treat the student scores that are at the boundary between two grades.

The editors

First part

Special issue

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Complexity as a prompt to rethink school mathematics: From computation to modeling

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Abstract

This writing begins with a brief survey of some of the disparate ways that complexity thinking has been taken up by mathematics educators. That review is used to frame a report on a brief teaching experiment that was developed around the topic of exponentiation, which is used in turn to introduce two main arguments that are rooted in complexity thinking. Firstly, it is asserted that the notion of "basics" within school mathematics must be interrogated and elaborated. Secondly, a case is made for shifting the core of school mathematics away from computation and toward modeling. This proposed shift toward modeling is situated in the literature, specifically among mathematics education researchers with an interest in complex modeling.

Keywords
complexity thinking; school mathematics; mathematics curriculum

Introduction

I am currently involved in a longitudinal investigation of "changing the culture of mathematics teaching at the school level." Through this design-based inquiry (Hoadley, 2004), a group of university researchers has teamed with the staff of a school in a 7-year commitment to work together in transforming how mathematics is seen and engaged.

The project's key defining features are, firstly, that "learning" is understood to happen simultaneously across multiple levels of organization and, secondly, that learning at the institutional level may be the most important consideration for those interested in substantial and sustainable educational change. While we are of course also attentive to adaptive and transformative processes at individual and social levels, our principal focus is on the culture of mathematics teaching and learning in the school. How is mathematics talked about? How do people perceive themselves in relation to mathematics? How does the discipline figure into student choices in courses and career trajectories? How are beliefs held in place or interrupted?

At the start of the project, three years ago, researchers and teachers met to brainstorm concerns, followed by a collective distillation of the more-than-50 topics that were identified. The result comprised the following three clusters of questions:

- Mathematics curriculum – e.g., What mathematics is important to teach? Is that the same was what is in the curriculum? Where did that curriculum come from?
- Individual understanding – e.g., How does understanding of a concept develop? Is there a "best" way to structure/sequence teaching to support robust conceptual development? Are individuals' understandings necessarily unique, or is there a way of nudging learners to "true" interpretations of concepts?
- Social process – e.g., How do groups support/frustrate the development of individual understanding? How does individual understanding support/frustrate the work of groups?

While there was some comfort in being about to organize the questions into just three categories, the research team was at first somewhat taken aback with the full range of concerns. The breadth of topics and the challenge of framing their questions in manners that are simultaneously provocative and pragmatic seemed to be beyond our reach. However, with some time for examination and reflection, we soon changed our minds.

While these three clusters of questions might seem on the surface to be focused of disparate matters, “inside” them we perceived a uniting theme: complexity.

More precisely, we elected to regard these clusters of issues as emergent phenomena – that is, as forms and agents that obey an evolutionary dynamic and that arise in and transform through the interactions of other forms and agents. That realization shifted our principal focus from the three clusters of questions to a single unifying theme. In the process, as is reported below, a space was opened both to move toward productive and pragmatic responses to the questions posed and to make meaningful strides toward the grander intention of the project.

What is “complexity” … within mathematics education?

Before getting into some of the specifics of those developments, it is important to situate the intended meaning of *complexity*.

Unfortunately, there is no unified or straightforward definition of the word. Indeed, most commentaries on complexity research begin with the observation that there is no singular meaning of complexity, principally because researchers tend to define it in terms of their particular research foci. One thus finds quite focused-and-technical definitions in such fields as mathematics and software engineering, more-indistinct-but-operational meanings in chemistry and biology, and quite flexible interpretations in the social sciences (cf. Mitchell, 2009).

Within mathematics education, the range of interpretations of complexity is almost as divergent as it is across all academic discourses. This variety of interpretation can in part be attributed to the way that mathematics education straddles two very different domains. On one side, mathematics offers precise definitions and strategies. On the other side, education cannot afford such precision, as it sits at the nexus of disciplinary knowledge, social engineering, and other cultural enterprises. Conceptions of complexity among mathematics education researchers thus vary from the precise to the vague, depending on how and where the notion is taken up.

However, diverse interpretations do collect around a few key qualities. In particular, *complex* systems adapt and are thus distinguishable from *complicated* (i.e., mechanical) systems, which may consist of many interacting components and which can be described and predicted using laws of classical physics. A complex system comprises many interacting agents – and those agents, in turn, may comprise many interacting subagents – presenting the possibility of global behaviors that are rooted in but not reducible to the actions or qualities of the constituting agents. In other words, a complex system is better described by using Darwinian dynamics than Newtonian mechanics.

Complexity research only cohered as a discernible movement in the physical and information sciences in the middle of that 20th century, with the social sciences and humanities joining in its development in more recent decades. To a much lesser (but noticeably accelerating) extent, complex systems research has been embraced by educationists whose interests extend across such levels of phenomena as genomics, neurological process, subjective understanding, interpersonal dynamics, mathematical modeling, cultural evolution, and global ecology. As discussed elsewhere (Davis & Simmt, 2014; 2016), these topics can be seen across three strands of interest among mathematics education researchers – namely,

- Complexity as a disciplinary discourse – i.e., as a digitally enabled, modeling-based branch of mathematics.
- Complexity as a theoretical discourse – i.e., as the study of learning systems, affording insight into the structures of knowledge domains, the social dynamics of knowledge production, and the intricacies of individual sense-making.
- Complexity as a pragmatic discourse – i.e., as a means to nurture emergent possibility, with advice on how to design tasks, structure interactions, etc.

Unfortunately, however, it is difficult to consider all of these topics at the same time, and any attempt to do so would result in an unwieldy paper that would likely be of little use. Re-emphasizing that the theme of complexity reaches across all aspects of our collaborative work, in this article I focus on mathematics curriculum and how complexity thinking, as a disciplinary discourse, might inform curriculum development.

A teaching experiment on exponentiation

For centuries, the basics of school mathematics have been identified as addition, subtraction, multiplication, and division. Notably, these operations are "basic" not because they are foundational to mathematics knowledge, but because they were vital to a newly industrialized and market-driven economy a few hundred years ago.

It is easy to see why computational competence would be useful to a citizen of that era, and to ours as well. If anything, the need has been amplified in our number-dense world. However, it is not clear that these four operations are a sufficient set of basics today, given that some of the most pressing issues – such as population growth, the rise of greenhouse gases, ocean acidification, decline in species diversity, cultural change, increases in debt, and so on – have strongly exponential characters. More descriptively, these sorts of pressing issues are instances of complexity, evidenced in part by their potentials for rapid change and unpredictability.

Understandings and appreciations of the volatility of prediction have become rather commonplace, no doubt in part because of the way the "Butterfly Effect" has captured the collective imagination. However, while awareness of this popular trope might suggest that complexivist sensibilities have gained traction, it might also indicate limited understanding of the actual mechanisms at work inside complex dynamical systems. The Butterfly Effect is most often stated in terms of a system's sensitivity to initial conditions, but what really matters is the power of iteration to amplify or dampen. That is, the Butterfly Effect only makes sense within a frame of exponentiation.

Prompted by this thought, I wondered how I might structure a study of exponentiation that treated the concept as a useful interpretive tool rather than a site for symbolic manipulations, an emphasis that represents a significant departure from our program of studies. As it is currently represented in local curriculum documents and textbooks, exponentiation is an exemplar of questionable practice. Guides and resources tend to frame the topic almost entirely in symbolic terms and to focus on a single interpretation of the concept – namely, as "repeated multiplication." For example, 9th-grade learning outcomes in my home province of Alberta are stated as follows:

1. Demonstrate an understanding of powers with integral bases (excluding base 0) and whole number exponents by:
   - representing repeated multiplication, using powers
   - using patterns to show that a power with an exponent of zero is equal to one
   - solving problems involving powers

2. Demonstrate an understanding of operations on powers with integral bases (excluding base 0) and whole number exponents:
   - $(a^m)(a^n) = a^{m+n}$
   - $a^m \div a^n = a^{m-n}, m > n$
   - $(a^m)^n = a^{mn}$
   - $(\frac{a}{b})^n = \frac{a^n}{b^n}$, $b \neq 0$ (Alberta Education, 2007, p. 17)

In other words, in the approved classroom resources, the pedagogical sequence is the reverse of conventional wisdom. Texts start with abstract and symbolic manipulations, move to applications (e.g., the Rice-Doubling question, the Richter scale, half lives, compound interest), and do not touch the ground of immediate personal experience.

In order to avoid the curriculum emphasis on symbolic manipulations, the decision was made to work at the 8th-grade rather than the 9th-grade level. The host teacher generously offered a week of class time, and the outline of lesson topics for that week is presented in Table 1.

Table 1. *An overview of a weeklong unit on exponentiation.*

Day	**Focus**	**Activities**
Monday	Images of Exponentiation	- drawing pictures of exponential change - web searches (“exponentiation,” “exponential growth,” “powers of two,” and related terms)
Tuesday	Exponentiation Lattice	- collectively assembling a lattice - looking for patterns - contrasts to addition and multiplication lattices
Wednesday	Analogies to Other Binary Operations	- symbolism and vocabulary - noting similarities between addition and multiplication, and extending these to exponentiation
Thursday	Exploring the Validity of those Analogies	- justifying and questioning - thinking about the structure of mathematics and mathematical ideas
Friday	Consolidation and Examples	– other illustrations of exponentiation – instances of exponentiation in the world we inhabit

A more detailed, general overview of the classroom activities has been presented elsewhere (Davis, 2015), and so only summary descriptions are offered here.

Monday. The unit’s opening task was an invitation to create images of exponential change. Students were instructed on drawing grid-based images of sequential doubling – starting by outlining a single square, then doubling the figure to enclose two squares, and so on to the limits of their sheets of paper. T-tables were incorporated into the activity to record quantities and make number patterns more apparent, and students were then tasked with creating similar images and tables for bases of 3 to 9. They were encouraged to do web searches, and together generated a rich range of associated figures that included images of exponential growth/decay and exponential curves.

Tuesday. On the second day, students were asked to compare exponentiation to addition and multiplication. Earlier in the school year, the class had created poster-sized lattices for addition, subtraction, multiplication, and division on *xy*-coordinate grids. On these charts, values on the *x*-axis served, respectively, as augend, subtrahend, multiplier, and dividend; values on the on the *y*-axis as addend, minuend, multiplicand, and divisor; and corresponding positions on the grid as locations for sums, differences, products, and quotients. Figure 1 presents small portions of these lattices.

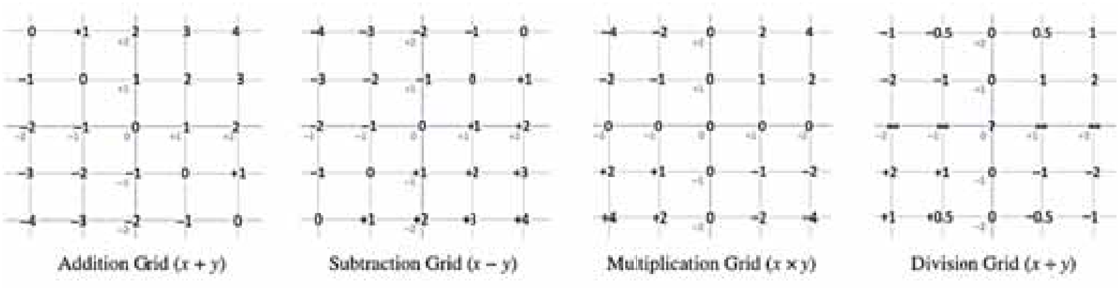


Figure 1. Core portions of the addition, subtraction, multiplication, and division lattices generated earlier in the school year.

In the earlier unit, these devices proved to be powerful tools for noticing patterns and, in the process, interpreting identity elements, commutativity, and other concepts and properties. We imagined a chart for exponentiation might serve similar purposes and began the second class with the construction of an exponentiation lattice spanning values of –10 to +10 on both axes – that is, covering the range of -10^{-10} to 10^{10}. A core portion of the exponentiation lattice is presented in Figure 2.

$x \uparrow y$

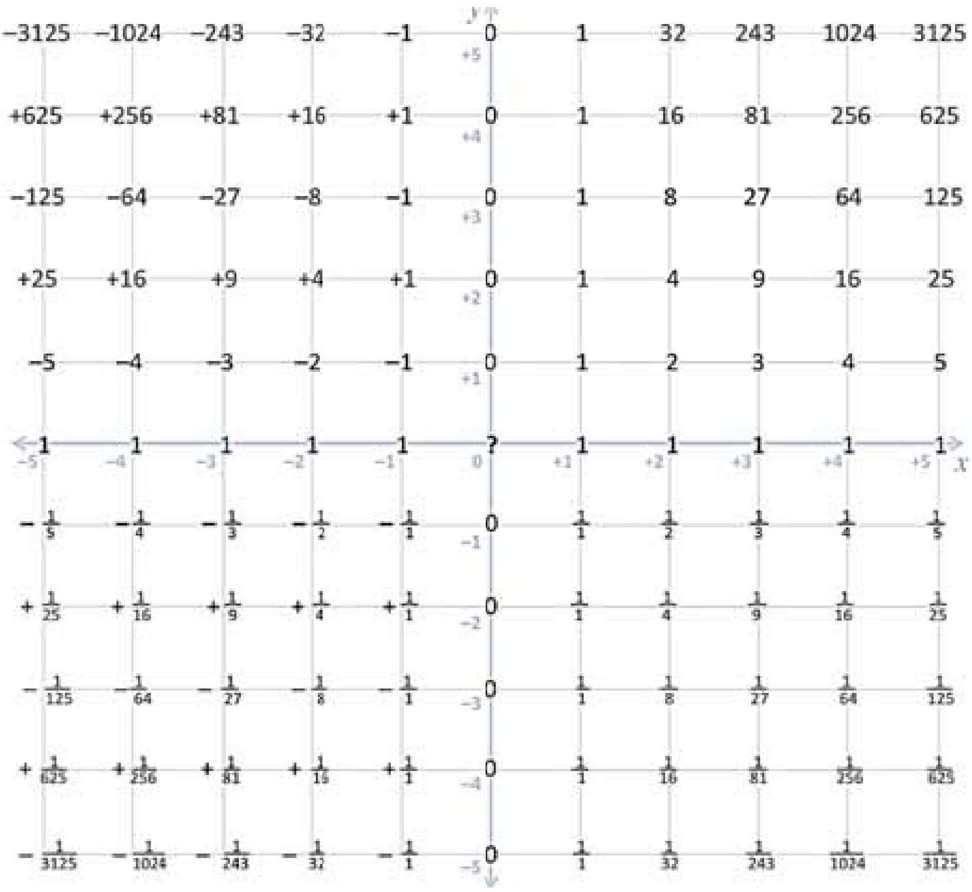


Figure 2. A piece of the exponentiation lattice.

The collective analysis of the result began by examining the first quadrant. Students compared its patterns to those in the addition and multiplication lattices, posted nearby. Three observations were immediately noted. First, students remarked on the "steeper and crazy-steeper" increases in values as one moves away from the origin, contrasted with the "flattening" feel of the addition lattice and the "gentler rising" of the multiplication lattice. Second, it was noted that the exponentiation chart "doesn't fold over like adding and multiplying" – that is, whereas the addition and multiplication lattices are symmetric about the line $y = x$, the exponentiation lattice is not. Third, "the diagonal of one table is the 2-row of the next." That is, just as the values along the $y = x$ diagonal of the addition lattice correspond to those of the $y = 2$ row of the multiplication lattice, so the values along the $y = x$ diagonal of the multiplication lattice correspond to those of the $y = 2$ row of the exponentiation lattice.

Those observations set up preliminary discussions of the lack of commutativity of exponentiation and nature of the values on the $y = x$ diagonals of the lattices. Students were asked to settle the first issue themselves, quickly agreeing that steepnesses and flattenings had to do with the fact that multiplication "can change things faster than addition, so multiplying over and over will *really* change things." As for the second observation, after analyzing several examples, students agreed there was good reason for the lack of symmetry in the exponentiation lattice. As one student explained, "Any number to the 1 [first power] is just itself, but 1 to any number is 1. So it all breaks down right away … and it gets worse as the numbers get bigger." The third observation, however, proved to be more of a sticking point. Students were able to recognize the relationship between the values along the $y = x$ diagonal in one chart and the $y = 2$ row of the next, but were clearly struggling with how to represent the values appearing along the $y = x$ diagonal of the exponentiation chart. The break came when a cluster of students noticed a pattern in the different ways relationships can be expressed when moving from addition through multiplication to exponentiation. That is, on the addition lattice, the values along the $x = y$ diagonal are doubles, and so can be written as either "$x + x$" or "$x \times 2$". And so, on the multiplication lattice, the values along the $x = y$ diagonal are squares, and so can be written as either "$x \times x$" or "x^2". That set up the realization that "x^x" should describe the values along $y = x$ diagonal on the exponentiation lattice – that is, 1^1, 2^2, 3^3, 4^4, and so on. However, that insight was accompanied by a question: what is the second way of writing "x^x"? The query sparked a flurry of discussion, and my immediate sense was that the issue revolved around the notation for exponentiation.

Moving on to the fourth quadrant, students quickly noted vertical patterns of decrease that reflected patterns of increase in the first quadrant. Examination of the left side of the lattice was not so quick, however. A number of quandaries arose: Predictably, the oscillation between positive and negative values was confusing for many, but satisfactory explanations based on even and odd numbers of multiplications were quickly offered. The more confounding question for most was, "What happens between the rows?” on the left side. Their calculators indicated “ERROR” when negative bases (e.g., $(-4)^{2.6}$) were entered. We deflected the queries, advising that there were online tools (e.g., a calculator available at www.mathisfun.com) available to dig into the emergent issues. We also flagged that a new number system is needed to talk about some of those values. While we elected not to delve into imaginary and complex numbers, we drew an analogy to other operation lattices and other number systems. In particular, the need for signed numbers arose in creating addition and subtraction lattices, and for fractional numbers when creating multiplication and division lattices. It makes sense that another operation might present the need for another set of numbers.

Wednesday. The third session dealt with analogies between exponentiation and the operations of addition and multiplication. Prompted by the problems encountered with x^x the previous day, we began by noting that the symbolism for exponentiation might obscure the relationship to other operations. To highlight similarities to “2 + 3” and “2 × 3,” then, we proposed “2 ^ 3,” which is one of several accepted notations (Cajori, 2007). The resulting set of pairs:

$$x + x = 2x$$
$$x \times x = x^2$$
$$x \wedge x = x^x$$

seemed to satisfy the desire for parallel representations that had emerged the day before.

We set up the day’s task with a version of Table 2 (below), which was an extension of a chart they had done earlier in the year comparing properties of addition to properties of multiplication. We reminded them of that detail to get things started, and then invited suggestions for completing the row labeled “Commutative Property.”

Table 2. *The blank speculation table*

Topic/ Property	**How it looks for addition ($x + y$)**	**How it looks for multiplication ($x \times y$)**	**Speculation for exponentiation (x ^ y)**	**T/F**
Commutative Property				
Reverse operation				
Identity element				
Inverse values				

The main point of this activity was to deepen understandings of exponentiation. A second purpose was to support understandings of the relationship among concepts, based on a vital difference between topics studied at elementary and secondary levels. Whereas almost all the concepts encountered at the elementary level can be interpreted in terms of (i.e., are analogical to) objects and actions in the physical world, the analogies for concepts at the secondary level are mostly mathematical objects (see Hofstadter & Sander, 2013). Making analogies, then, is both a mechanism for extending mathematical insight and a window into the structure of mathematics knowledge.

Before setting the students to work on their own, we indicated that they should not worry about the last column, as we had already planned that for the focus of the fourth session. The rest of the class was devoted to filling in blank cells, an effort that began in small groups and that ended in whole-group negotiations of acceptable, parallel phrasings for each entry (see the second row in Table 3). Notably, the final three rows of the chart were additions proposed by the students themselves.

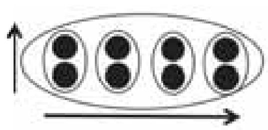
Table 3. *Conjectures for exponentiation based on analogies to addition and multiplication*

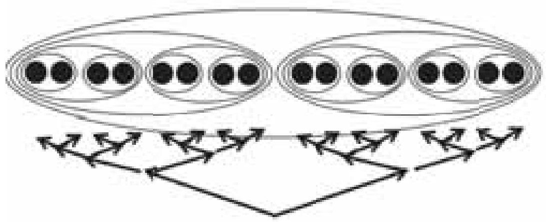
Topic/ Property	**How it looks for addition ($x + y$)**	**How it looks for multiplication ($x \times y$)**	**Speculation for exponentiation (x ^ y)**	**T/F**
Commutative Property	$a + b = b +$	$a \times b = b \times a$	$a \wedge b = b \wedge a$	False $2^3 \neq 3^2$
Reverse operation	Subtraction (–)	Division (÷)	De-exponentiation (∨)	
Identity element	0 … as in $a + 0 = 0 + a = a$	1 … as in $a \times 1 = 1 \times a = a$	1? … since $a \wedge 1 = a$ … although $1 \wedge a = 1$	
Inverse values	Additive inverse of a is $0 - a$, or $-a$; $a + (-a) = 0$	Multiplicative inverse of a is $1 \div a$, or $\frac{1}{a}$; $a \times \frac{1}{a} = 1$	Exponentative inverse of a is $1 \vee a$, or $\vee a$; $a \wedge (\vee a) = 1$	
Operating on the opposite	Subtraction can be done by adding the [additive] inverse: $a - b = a + (-b)$	Division can be done by multiplying the [multiplicative] inverse: $a \div b = a \times \frac{1}{b}$	De-exponentiation must be doable by exponentiating the [exponentative] inverse: $a \vee b = a \wedge (\vee b)$	
"Next" operation	A repeated addition is a multiplication.	A repeated multiplication is an exponentiation.	A repeated exponentiation must be a … something.	
"Next" set of numbers	When you allow subtraction, you need signed numbers.	When you allow division, you need rational numbers.	When you allow de-exponentiation, you need another set of numbers.	

Thursday. The fourth session was devoted to exploring the truth or falsity of the speculations from the day before. Students worked in small groups and focused on speculations of their choosing. They also made free use of the Internet to help them in their deliberations. Topics in the follow-up discussion included a problem with the speculation on inverse values (i.e., that for every a there is a $\downarrow a$ such that $a \wedge (\vee a) = 1$), because the exponentiation grid suggested $a \wedge 0 = 1$ (for all $a \neq 0$). If the speculation were true, it would mean that the exponentative inverse of every number would be 0, which most felt to be nonsensical – in addition to rendering the speculation on “operating on the opposite” similarly troublesome. We elected to leave these discussions unsettled, suggesting that our simple analogies might be misleading. We also suggested that further studies in high school would shed some light on a few of the details – a point that was supported by topics that came up in students’ web searches, including logarithms, imaginary and complex number systems, and titration (i.e., the next hyperoperation after exponentiation).

Friday. The final session was devoted to review and consolidation. We framed the session by developing the table presented in Table 4, through which we suggested that the geometric image best fitted to addition is the line, to multiplication is a rectangle, and to exponentiation is a fractal. That thought was tied in to a “fractal cards” activity (Simmt & Davis, 1998) that the students had undertaken earlier in the school year.

Table 4. *Some geometric analogies to arithmetic operations*

Operation	Principal Visual Metaphors	Common Applications/Interpretations (using whole number values)
2 + 4		• combining of sets or lengths along 1 dimension • can be consistently represented in linear form
2 × 4		• sets of sets or array/area generated by crossing dimensions • can often be represented as a rectangle

2 ^ 4		• sets of sets of sets (etc.) or multi-dimensional form • representable in a fractal-esque, recursively generated and/or branching image

The balance of the lesson was given to conducting searches and looking across instances of exponential growth and decay (e.g., creating fractal cards, population growth, species decline, greenhouse gas increase, technology evolution), framed by Charles and Ray Eames' (1977) film, *Powers of Ten* and Cary and Michael Huang's (2012) interactive Prezi presentation, *The Scale of the Universe*. Exponential growth curves emerged to be a uniting image across these explorations, and also proved useful as a recap on the week as they linked back to the images and grid developed on Monday and Tuesday.

Complexity as a disciplinary discourse: moving from computation to modeling

While the explicit topic of instruction in the above teaching episodes was not complexity, it was oriented toward and by a deeper awareness of complex modeling – evident in three specific aspects:

- the examination of the mathematics of rapid change – The contrast between the exponentiation lattice and the addition and multiplication lattices afforded insight into a mathematics of sudden growth and decay, which is vital for appreciating the dynamics involved in complex modeling.
- the encounter with the emergence of mathematical constructs – Looking at an instance of complex emergence through the explicit discussion of how new number systems have arisen iteratively as operations on established systems presented the need for new number systems.
- the treatment of mathematics a means to model experiences and phenomena – Exponentiation was used to interpret a range of phenomena, especially those that have an element of self-similarity.

As a researcher with interests in the possible contributions of complexity thinking to school mathematics, I see these points as illustrative of a powerful opportunity to transform school mathematics. Specifically, they present the possibility of rethinking "doing math" as more an exercise in *modeling* than one in *computation*. That is, being mathematically competent also is also about being able to interpret and simulate real-life situations with mathematical constructs. It was in this spirit that exponentiation was studied in the reported classroom episode. While some calculations were involved, computation was always a means to an end. It was a tool within the modeling activity.

To elaborate, a "model" is a representation – a description, an image, a copy – which is intended to highlight vital, defining attributes of some phenomenon. Most often, a model is a simplification, one that is useful as a tool for understanding. A "mathematical model" is thus a description of a phenomenon using mathematical constructs. Examples abound, and range from the mundane to the enormously complex. On the more familiar end of the spectrum, every act of counting or measuring is an act of mathematical modeling – that is, of representing a situation in terms of an appropriate number system. At the more complex end of the spectrum, mathematical models are used in the natural sciences (e.g., physics, chemistry, biology, geology, meteorology, astronomy), engineering, and the social sciences (e.g., economics, psychology, political science, sociology) to interpret, explain, and predict phenomena that arise in the interactions of many, many interacting agents.

In this sense, the discipline of mathematics has always been about modeling – although this core emphasis has often been obscured by the computational demands of some models. In particular, prior to rapid and inexpensive computing, the modeling of systems was largely focused on those dynamics that could be studied through differential linear equations. Poincaré was notable among those who examined non-linear dynamical systems, doing so from a theoretical perspective (Bell, 1937). The computational power of digital technologies in the second half of the 20th century was necessary for the investigation of dynamical systems began to flourish. Computing power brought about possibility of doing “experimental mathematics” (Borwein & Devlin, 2008) and numerical analysis, triggering a rebirth of the modeling of non-linear dynamical systems. Importantly, digital computing provided not only a means of computing extremely large data sets and iterating functions through hundreds of thousands of repetitions, it also provided means for converting numerical data to visual representations, enabling the generation of new insights and, consequently, new forms of mathematics (Mitchell, 2009).

It might be tempting to characterize the ever-growing gap between the research mathematics and school mathematics in terms of the contrast between the emphasis on modeling in the former and the emphasis on computation in the latter. That distinction would be unfair, however. Every topic in school mathematics was originally selected for its power to model, and this detail helps to explain the traditional pedagogical emphasis has been on rote application. In the first public schools, learners were being trained not to model, but to apply established mathematical models, and to do so efficiently and effectively. Routinized, repetitive instruction that does not allow for much divergent thinking is arguably the best way to do that.

In other words, schooling’s emphasis on computation was once fitting. However, circumstances and sensibilities have changed, along with the needs of a mathematically literate citizen. But so too have the affordances of the world in which we live, such as access to data, computational speed, and spatio-visual interfaces. Such evolutions were behind Lesh’s (2010) assertion that complexity has emerged as “an important topic to be included in any mathematics curriculum that claims to be preparing students for full participation in a technology-based age of information” (p. 563). To be clear on the point of this article, the suggestion is *not* that study of complex systems is new, but that the mathematics of complexity could represent a significant shift from traditional emphases on computation to a new emphasis on modeling – and, in that shift, possibly nudge school mathematics closer to its parent discipline. As Stewart (1989) has reported, mathematicians have long seen their work in terms of modeling. Just as significantly, they were perfectly aware when they were using linear approximations and other reductions in order to avoid computational intractability. Lecturers and texts followed suit in omitting nonlinear accounts; hence generations of students were exposed to over-simplified, linearized versions of natural phenomena. In other words, non-complex mathematics prevailed in public schools not because it was ideal but because it lent itself to calculations that could be done by hand. The power of digital technologies has not just opened up new vistas of calculation, they have triggered epistemic shifts as they contribute to redefinitions of what counts as possible and what is expressible, and this insight has been engaged by many mathematics education researchers (e.g., English, 2011; Hoyles & Noss, 2008; Moreno-Armella, Hegedus, & Kaput, 2008).

Notable in the movement toward recasting school mathematics in terms of modeling is the seminal work of Papert (e.g., 1980), particularly his development of the Logo programming language in the late 1970s. The language was designed to be usable by young novices and advanced experts alike. It enabled users to solve problems using a mobile robot, the "Logo turtle," and eventually a simulated turtle on the computer screen. While not intended explicitly for the study of complexity, Logo lent itself to recursive programming and was thus easily used to generate fractal-like images and to explore applications dynamically – opening the door to more complexity-specific topics. To that end, different developers have since offered Logo-based platforms that are explicitly intended to explore complex systems (and other) applications. For example, StarLogo (lead designer, Mitchell Resnick; http://education.mit.edu/starlogo/) and NetLogo (lead designer, UriWilensky; http://ccl.northwestern.edu/netlogo/). Both platforms were developed in the 1990s and extended Papert's original Logo program by presenting the possibility of multiple, interacting agents (turtles). This feature renders the applications useful for simulating ranges of complex phenomena. Both StarLogo and NetLogo include extensive online libraries of already-programmed simulations of familiar phenomena (e.g., flocking birds, traffic jams, disease spread, and population dynamics) and less-familiar applications in a variety of domains such as economics, biology, physics, chemistry, neurology, and psychology. At the same time, the platforms preserve the simplicity of programming that distinguished the original Logo (e.g., utilizing switches, sliders, choosers, inputs, and other interface elements), making them accessible for even young learners. Other visual programming languages have been developed that are particularly appropriate to students (e.g., Scratch, scratch.mit.edu, and ToonTalk, www.toontalk.com).

Over the past few decades, hundreds of speculative essays and research reports (see, e.g., http://ccl.northwestern.edu/netlogo/references.shtml) have been published on these and other multi-turtle programs. Regarding matters of potential innovations for school mathematics, in addition to well-developed resources, there have been extensive discussions and there exists a substantial empirical basis for moving forward on the selection and development of curriculum content that is fitted to themes of complexity. Not surprisingly, then, with the ready access to computational and imaging technologies in most school classrooms, some (e.g., Jacobson & Wilensky 2006) have advocated for the inclusion of such topics as computer-based modeling and simulation languages, including networked collaborative simulations (see Kaput Centre for Research and Innovation in STEM Education, http://www.kaputcenter.umassd.edu). In this vein, complexity is understood as a digitally enabled, modeling-based branch of mathematics that opens spaces (particularly in secondary and tertiary education) for new themes such as recursive functions, fractal geometry and modeling of complex phenomena with mathematical tools such as iteration, cobwebbing, and phase diagrams.

The shift in sensibility from linearity to complexity is more important than the development of the computational competencies necessary for modeling. The very role of mathematics in one's life is transformed through this shift in curriculum emphasis. As Lesh (2010) described, "whereas the entire traditional K–14 mathematics curriculum can be characterized as a step-by-step line of march toward the study of single, solvable, differentiable functions, the world beyond schools contains scarcely a few situations of single actor–single outcome variety" (p. 564). Extending this thought, Lesh highlighted that questions and topics in complexity and data management are not only made more accessible in K–14 settings through digital technologies, current tools have made it possible to render some key principles comprehensible to young learners in manners that complement traditional curriculum emphases.

Despite the growing research base and the compelling arguments, however, few contemporary programs of study in school mathematics have heeded such admonitions for change. It is perhaps for this reason that many mathematics education researchers have focused on familiar topic areas (such as those just mentioned; see Davis & Simmt, 2016, for other examples) as means to incorporate studies of complexity into school mathematics. Discussions of and research into possible sites of integration have spanned all grade levels and several content areas, and proponents have tended to advocate for complexity-content, but in a less calculation-dependent format.

Closing Remarks

For many mathematics educators, complexity thinking might seem like a Pandora's box. If the field were to open it and take up the topic seriously, an array to world-changing possibilities would impose themselves. Complexity thinking challenges many of the deeply engrained, commonsensical assumptions on how humans think and learn. It interrupts much of the orthodoxy on group process and collective knowledge. And, in particular, as a curriculum topic, there is no straightforward way to fit complex modeling into the mold of contemporary school mathematics. It transcends procedures with its invitation to experiment; it demands precision, but in the service of playful possibility; it is rooted in computation, but offloads most of that work onto digital technologies; it requires facility with symbol manipulation, but that manipulation is more for description than deriving solutions. In other words, merely considering complex modeling as a possible topic for today's classrooms forces a rethinking of not just *what* is being taught, but *why* some topics maintain such prominence and *how* topics might be formatted to engage learners meaningfully and effectively.

Indeed, as the example of exponentiation might be used to illustrate, if complexity were to be seriously considered as a curriculum topic, it would compel reexamination of the very foundations of school mathematics. Not only must the "basics" be available for interrogation and revision, emphases of computation-heavy and symbol-based processes would have to be complemented with modeling-rich and spatial-based possibilities. Importantly, this is not an either–or situation. Taking up modeling as a focus of school mathematics does not negate computation and symbolic manipulation, but such a shift does reposition them as means rather than ends.

It will be interesting to see if and when the culture of school mathematics is able to move in the direction of complexity thinking. The discourse itself suggests that, while a sudden and dramatic shift could happen at any time, it is more likely that the grander system will find ways to maintain its current emphases for some time longer. Caught in a tangle of popular expectation, deep-rooted practice, entrenched curricula, uninterrogated beliefs, and lucrative publishing and testing industries, school mathematics is an exemplar of a complex unity. This insight, more than any other, is the one that sustains my interest. Sooner or later, a well-situated wing flapping will trigger that moment of exponential change through a cascade of transformations that pull school mathematics into a new era.

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Conceiving mathematics classrooms as activity systems

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Abstarct
This paper contributes to the efforts of mathematics education community to understand the complex phenomena which occur in mathematics classrooms by highlighting the usefulness of activity theory. Activity theory and the associated concepts attempt to overcome the separation of human action from the relevant elements embedded in its context by means of a functional structuring of the key elements involved in social and institutional situations. Conceptualizing mathematics classrooms as complex systems of activities and considering their interactions with other activities within educational system as well as outside it in their broader societal and cultural context may offer a new look at the challenges of mathematics education.
Keywords: Activity theory; Complex systems; mathematics classrooms;

Introduction: New perspectives on learning and teaching

During the past few decades, a spectrum of socio-cognitive perspectives has emerged which consider knowledge as an emerging characteristic of activities taking place among persons in specific contexts and view learning as a developmental process appearing twice: first, on an *inter*personal, socio-cognitive, level positioned between people and second, on an *intra*personal, cognitive, level situated inside the individual person. On this ground, learning is regarded as a constructive activity that over and over again requires active and extensive reorganization of existing conceptual structures. As a consequence, an everyday growing amount of research and developmental studies propose ways of applying, and in many cases put in practice, these socio-constructivist approaches in re-conceptualising school curricula, teaching practices, and learning activities, however, not effecting, in my view, the promised or anticipated learning gains in mathematics education. Although more research into refining and extending these socio-constructivist approaches in applied contexts of education is needed, I consider that their relative inefficiency in transforming their theoretical assumptions to practice originates from a not balanced approach between the interpersonal and intrapersonal aspects of cognition and learning when implemented in school contexts. This fact results, after all, either in distorted applications of socio-constructivist theoretical assumptions or in school projects having not actual theoretical relation to the socio-constructivist approach they claim. A balanced approach transcending the distinction between the interpersonal and intrapersonal level of cognition and learning offering promising options both in studying and reforming school pedagogies, curricular materials, and educational tools is offered by conceptualizing the teaching and learning in school classrooms as complex systems of activities.

A complex system, as is aptly clarified by Davis & Simmt (2014),

> "*comprises many interacting agents – and those agents, in turn, may comprise many interacting subagents – presenting the possibility of global behaviours that are rooted in but that cannot be reduced to the actions or qualities of the constituting agents. In other words, a complex system is better described by using Darwinian principles than Newtonian ones. It is thus that each complex phenomenon must be studied in its own right. For each complex unity, new laws emerge that cannot be anticipated or explained strictly by reference to prior, subsequent, or similar systems" (p. 88).*

In the following, the activity systems of school learning and teaching, as modelled by Engeström (1987), is outlined with specific references to mathematics education where appropriate.

The activity system of learning-teaching

According to Engeström (1987), who expanded Leontiev's theory of activity as originated from the ideas of Vygotsky, the system of human activity combines a set of primary components and their reciprocal relationships (Fig. 1).

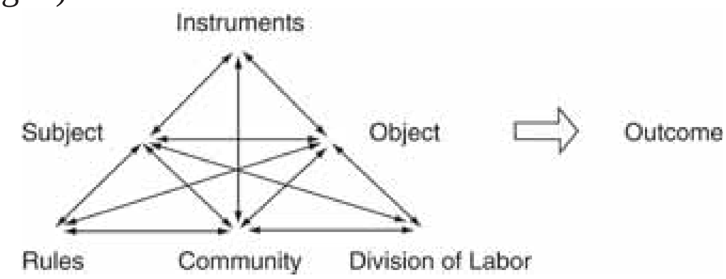

Fig. 1. The structure of a human activity system (Engeström, 1987, p.78).

Every human activity is animated by a *subject*, usually a collective one, who is working towards some *object*, tangible or intangible, in order to transform it into some *outcome*. In order to achieve the expected outcomes, the subject uses specific *instruments* which, broadly defined, include the mediating, material or symbolic tools, which they shape the activity (and, in turn, they are shaped by the activity). Persons, either as individuals or in groups, who have the same object of activity, constitute a *community*.The community binds individuals together through *rules* and *division of labour*. Rules signify norms, conventions, or social traditions that are established by the community to govern its members and as a result guide the system's actions and interactions and the division of labour defines the allocation of works among community members both horizontally and vertically with respect to expertise, power and status (Engeström, 1996, 1998) .

In mathematics education, the *subject* refers to a student or a group of students ranging from primary to higher education who follow a mathematics course, to a teacher or teachers of mathematics, to mathematics education policy makers or curricula developers or even to parents and other groups of social agents that are involved and to some extent effect schooling. The *object* refers to mathematical knowledge, conceptual tools, problem solving techniques, mathematical practices, ways of thinking or even assessment and examinations at which the activity is directed or more broadly, as put by Kaptelinin (2005, p 5), to “sense-making” or “ultimate reasons” for the behaviours of the subjects. On this account, the objects of activity of mathematics teachers may be described as a productive teaching attaining the goals set by curricula or developing mathematical skills or motivating their students to learn, while for students the objects of mathematics classroom activity may be defined as responding to the teacher demands or solving assigned mathematical problems or preparing for successful examinations or appropriating concepts and techniques of mathematics. The *instruments* are both material tools (e.g. paper and pencil, chalkboards, manipulatives, calculators, computers, etc.) and symbolic tools (e.g. language, symbols, diagrams, charts, pictures, etc.), which mediate acting and thinking of the subjects in any activity. The instruments by their use transform the activity itself and at the same time alter the subjects’ behaviour (Chassapis, 1999). The *community* in mathematics classrooms is usually constituted by the teacher and students but also, according to the case, it may include teachers, family members, friends, educational officials, policy makers and other actors. The *rules* of mathematics teaching and learning activity are implicit and explicit. Implicit are the rules which set permissible and non-permissible behaviors and acts aiming to regulate the community’s discussion and argumentation in the mathematical classroom, e.g. raising the hand before responding a teacher’s question or not making noise when the teachers address the class, etc. On the other hand, explicit are the rules set by the school regulations or by the administrative decisions of the school authorities, e.g. assessments procedures and norms or absence from lessons rules, etc. The *division of labor* refers to the sharing of tasks and responsibilities between the members of the community in a vertical dimension according to their roles as defined by power relations and hierarchies of authority (e.g., students, teachers, school head, regional director, etc.) and in a horizontal dimension as stated by the requirements of the teaching method (e.g., student or teacher centred pedagogy etc).

There are studies of mathematics education which focus on a particular component of the activity of mathematics teaching and learning or in the interrelation of two or more components in a mathematics classroom (e.g., Chassapis, 1999, Zevenbergen & Lerman, 2008, Groves & Dale, 2004 are focused on mediating instruments; Jaworski, 2005 on communities; Hardman, 2007 on the object of a teaching/learning activity, etc.)

It must be pointed out that an activity system is an intertwined system whose elements are tied together by “a collective object and motive [that] is realized in goal-oriented individual and group actions” (Hasu & Engeström, 2000, p. 63) and therefore whenever a component changes the system becomes unstable and must develop or change to obtain a renewed stability. That is, an activity system in not a static but a dynamic structure and all components of the system reciprocally and dynamically influence each other so that the system is continually adjusting, adapting, and changing.

At the same time, every activity system is not an isolated entity but interacts with other activity systems and is crucially influenced by changes in its environment (Kuutti, 1996). This interaction takes place concurrently across two dimensions. On a vertical, let say on an inter-level dimension, an activity system is nested in other superposed activity systems and is interacting with one or many of their components (Nunez, 2009). In educational contexts of learning and teaching mathematics, the activity system of a mathematics classroom or a computer laboratory is nested in the primary or secondary schooling activity system and this, in its turn, is nested in the broader institutional activity system which is structured by the educational system of a country and ultimately in the particular society considered as a culturally and historically framed system of activities (Fig.2).

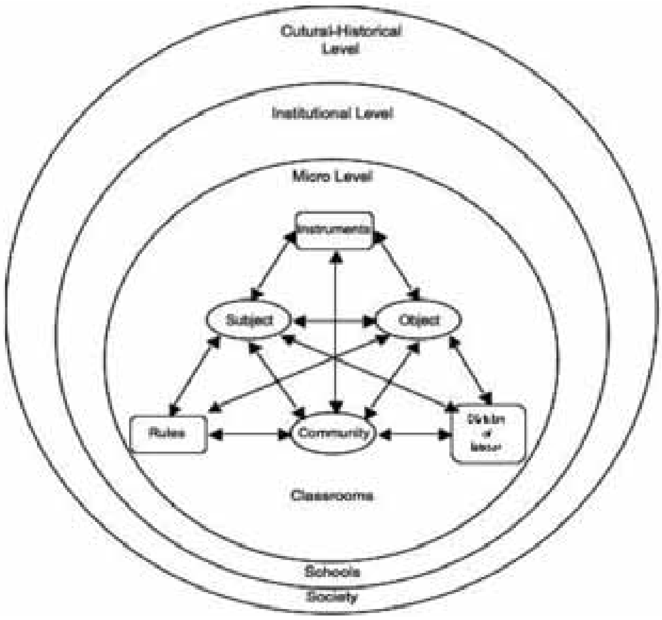

Fig. 2: Nested activity systems within Educational Context Levels (Nunez, 2009, p. 10)

Mathematics education literature includes research papers selecting as their minimum unit of analysis various activity systems at any one of the mentioned levels or at their mutual interaction. Citing just a few quite indicatively, Zevenbergen & Lerman (2008) focused on a mathematics classroom using interactive whiteboard, Jaworski & Potari (2009) related a mathematics classroom to its educational and social context, Venkat & Adler (2008) studied the implementation of a mathematics education reform on the school level, Jurdak (2006) considered activity systems of problem solving in the socially situated real world and school contexts and Kanes (2002) approaches numeracy as a cultural-historical activity system in a society.

On a horizontal, let say on an intra-level dimension, an activity system interacts with other systems functioning on the same level of analysis. In a formal mathematics classroom, for instance, there are at least two different but interacting activity systems: that of the teacher and that of the students. These two systems may, and usually, differ in their objects, as well as, in their intended outcomes. The object for students may be meeting examination requirements, so their participation in the mathematics lessons aims at acquiring knowledge and skills required to success in examinations typically transforming mathematics texts into grades and test scores. On the other hand, the object for the teacher of mathematics may be to teach students the proper subject matter transforming mathematics texts into teaching tasks in order to accomplish the mathematics curriculum requirements to the maximum feasible extent. Therefore, even though temporary exceptions may exist, teachers and students in a mathematics classroom do not usually engage in the same activity and thus, at least two (or maybe in some cases multiple) coexisting in a mathematics classroom activities influence the pedagogy chosen teachers and the learning climate formed. In a corresponding way, the communities of students and teachers and their division of labour are different, while the rules, which can be both explicit and implicit, are normally the same for both students and teachers imposed by the school regulations and classroom rituals (Fig. 3).

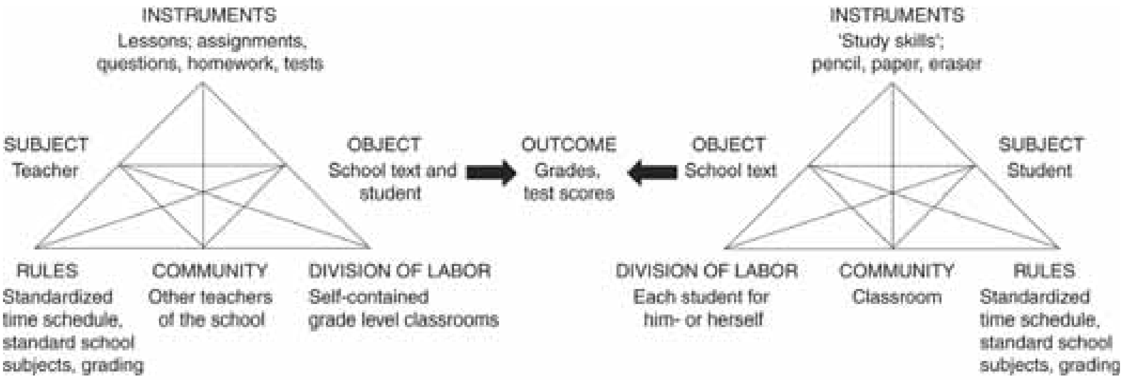

Fig. 3: Traditional teaching and school-going as interconnected activity systems (Engestrom, 1998, p.80)

Basic functioning principles of an activity system

As epitomised by Kaptelinin et. al. (1999), the functioning of any activity system is structured by the following basic principles:

- Any activity is oriented or directed towards an object, which is related to the subject (a person or a group engaged in the activity). This "object (in the sense of 'objective') motivates activity, giving it a specific direction" (Nardi, 1996, p. 73). In contrast to a "thing" which may be regarded as any element of reality characterised physical specificity, by

"object … we understand a form embodying the socio-historical

experience of mankind. An object is the vehicle of this experience and embodies a specific aspect of human social practice; it is the form in which a physically defined thing functions as people go about their life activities in society. The essence of an object functioning in the social process is constituted not by its physical properties, but by the specific connections and relations that become known in the process of collective activity" (Stetsenko, 1995, p. 59).

As Jonassen & Rohrer-Murphy (1999, p. 8) further exemplified it , "the object of an activity can be anything, so long as it can be transformed by subjects of the activity system".

- Activities are long-term formations, whose objects are transformed into outcomes through a process consisted of several steps. Activities are realised by certain actions or chains of actions, which in turn are consisted of operations, both being short-term processes (Leontiev, 1978). These actions are directed towards a conscious goal, and are related to one another by the same overall objective. Actions, in turn, are composed of operations, which are automatic processes unconsciously performed by the individual subject(s) of the activity. Operations, are not directed toward a goal as are actions, but they provide an adjustment of actions to the existing situation and the prevailing conditions. In a few words, at the level of actions, the questions posed are about "what": What must be done to be the activity accomplished? At the level of operations the questions to be answered are about the "how": How has each action to be executed? How things are made to happen? Operations are bound to conditions under which their respective actions are being carried out. These, actually hierarchical, levels of an activity system are not fixed but subject to developments; operations can become actions through internalization and actions in one context may be transformed to activities in another situation.
- In any activity system they are involved internal, mental, processes and external, physical, processes, which although differentiated are in a constant reciprocal relationship. The subject is transforming

the object of an activity, while the properties of the object penetrate into the subject and transform him or her (Kuutti, 1996). Thus, the internal “mental” processes of the subject cannot be understood without reference to external processes manifested in his or her behaviours. The internal processes are formed at a social level in collaboration and interaction with others or by the use of external tools. The other way around, “mental processes manifest themselves in external actions performed by a person, so they can be verified and corrected, if necessary” (Kaptelinin, 1996, p. 109).

- As Vygotsky (1978) has analysed, all human activities are mediated by tools which are carriers of cultural knowledge and social experience and alter by their use the nature of the activities. This knowledge and experience is manifested in both the structural properties of the tool, and in the way the tool should be used. A tool comes fully into being when it is actually used and knowing how to use it is an essential part of the tool (Kaptelinin et. al. 1999).
- Activity systems are not static, but dynamic processes under continuous change and development (Kuutti, 1996). Therefore, knowing the history and development of an activity is a requirement for understanding its essential aspects.

The before mentioned principles which rule the functioning of an activity system should be conceived in a holistic way since each one is associated with all others and is in many ways related to the various aspects of the whole activity.

Developmental changes in activity systems

The development and change of every human activity is an outcome of contradictions generated in the activity itself, and at the same time is a result of transformations imposed by new needs which are produced by one or more of the components of the activity (Engeström, 2001). Contradictions should not be understood as single conflicts or even complex but solvable problems but as fundamental tensions and misalignments in the structure of the activity system that typically manifest themselves as problems, ruptures, and breakdowns in its functioning (Virkkunen and Kuutti, 2000, p 302).). It follows that overcoming contradictions is key to, and at the same time the explanatory principle of, changes and developments in activity systems.

Engeström (1987) has defined four types of contradictions:

- Primary inner contradictions, which occur within each constituent component of an activity system, e.g. using a paper and a pencil as an instrument to carry on number operations versus using a calculator for the same task.
- Secondary contradictions, which occur between two or more components of the activity system, e.g. between students and division of labour in a mathematics classroom decided to work in groups for solving a problem.
- In cases that the object and motive of a culturally more advanced activity is introduced in an activity system, tertiary contradictions may occur between the new and the previous object of the activity, e.g. between the study of number operations in a primary school classroom introduced by the teacher in replacement of playing with numbers. Furthermore, in cases that two or more activity systems are interacting, as for instance are the systems of learning and teaching in a mathematics classroom or the system of a classroom activity and the broader system of schooling, tertiary contradictions may occur between the objects of the interrelated activities.
- A quaternary type of contradictions may occur between the rest of the components (except the object) of an activity system, i.e. subjects, instruments, rules, community or division of labor, and its adjoining or overlying linked activity systems, e.g. between learning activity in a mathematics classroom and learning mathematics in an everyday real world contexts as is for instance game playing or between an activity of teaching mathematics and a rule producing activity of educational administration.

Contradictions in activity systems take place through, and are driven by, the reciprocal and integrated processes of internalization or literally appropriation and externalization. Internalization is the process by which external activities are transformed into or modify internal ones, e.g. when is learnt to mentally perform number operations. It is a process which allows potential interactions with reality without performing actual manipulation with real objects (ideas, imaginations, mental simulations, etc.). By the way, it must be underlined that the terms "appropriation" and "internalization" do not assume a transmission method of knowledge that neglects the active participation of the learner. Externalization, correspondingly, is the process by which internal, mental, activities become external or are manifested by the creation of new tools and social practices e.g. drawing a diagram illustrating a relationship.

In this account, development and change of an activity system means resolution or transformation of contradictions and tensions occurred between individuals and socio-cultural influences, between two or more elements of an activity system, and between interconnected activity systems resulting in the construction of a new object and motive(s), that is resulting a new, more functional or advanced activity system. Such a development and change is a long-term spiral process of appropriation and externalization that Engeström (1987) has called "learning by expanding" clarifying that

> "*The essence of learning activity is production of objectively, societally new activity structures (including new objects, instruments, etc.) out of actions manifesting the inner contradictions of the preceding form of the activity in question. Learning activity is mastery of expansion from actions to a new activity. While traditional school-going is essentially a subject-producing activity and traditional science is essentially an instrument-producing activity, learning activity is an activity-producing activity*" (Engestrom, 1987, p. 124-125, italics in original).

Concluding comments: Dealing with the complexity of mathematics classroom

Mathematics classrooms are complex systems by themselves or are essential constituents of broader activity systems in which many types of activities are interacting involving different subjects, objects, instruments, rules, communities and division of labours, having diverse motives and pursuing various outcomes.

An idea of the multiplicity of activity systems which coexist and interact in a mathematics classroom, except the two leading activities of learning for students and teaching for the teacher (both subjected to broader institutional, administrative and socio-cultural activity systems) may be given by the assumption that students, although being in the same class and formally attending the same course of mathematics, have different motives because of their different socio-cultural backgrounds and for that reason they may work towards different outcomes. As a consequence, they actually participate in different activities. Those students who consider schooling to be a very important step towards their future life are participating in the "activity of school learning", appreciate school achievement and work for achieving good examination grades. On the contrary, other students who consider school attending as a formal obligation prior to their working life and are totally unconcerned with mathematics learning they actually participate in a traditional "school-going activity", a different and in many aspects conflicting activity to "school learning". As mathematics teachers know by their experience, such a conflict between two or more activities taking place in the same classroom generate contradictions which undermine and in many cases degenerate the intended learning activity.

Conceptualizing mathematics classrooms as complex systems of activities and considering their interactions with other activities within educational system as well as outside it in its broad societal and cultural context may offer a new look at the challenges of mathematics education faced both by researchers and teachers of mathematics. Activity theory provides, in my view, a powerful lens through which we may conceive mathematics classrooms not simply as complex systems of activities but, at the same time, as socio-cultural terrains on which both individual and social levels interact shaping practices and processes.

Assuming that mathematics classrooms are social formations and students and teachers act primarily as social beings and secondary as individual persons is not a new premise in mathematics education. However, for most researchers in the field this assumption means in fact that individuals involved in mathematics learning, teaching and schooling live in a social world or they bring into play socially inherited forms of thinking and acting.

On the contrary, activity theory takes for granted that individual thinking and acting are products of social and cultural processes. Claiming that human cognition is "in a very fundamental sense a cultural and social process" (Hutchins, 1995, p. 353) and is mediated by the tools and resources used (Wertsch, 1994), activity theory put emphasis on the social, cultural and historical influences of the institutions and contexts in which students engage in learning and teachers commit teaching mathematics. In the complex systems of activities which are structured and developed in mathematics classrooms, students and teachers thinking and acting as social beings embody institutional influences shaping the teaching of teachers, the learning of students, the organisation of schooling and, above all, the complexity of their mutual interactions.

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A Mathematical activity for the training of In-Service Primary school Teachers using a Systemic Approach

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Abstract

The present article studies the influence of learning mathematics through the use of an activity called "The target number" which was proposed to in service primary school teachers within the context of an ongoing training program and also focuses on the description of the activity used by the teachers and trainers. Its application is using a scenario of consecutive scenarios-phases which examines the learning process by focusing attention on a group of trainees-classroom- as a system and the interactions that are developed. The systemic concepts of framing, interaction and co-construction constitute as the core body which analysis the activity mentioned above, whilst at the same time the concept of division is repositioned within the tool – object dialectic.

Keywords: Systemic approach, Mathematics Education, Framing, Interaction, Co-construction, In service Primary school Teachers

1. Introduction

The systemic approach is a new scientific paradigm and an alternative proposal in the field of psychosocial and educational practices. Systemic perception focuses on the complexity of relationships between individuals and/or ideas and reminds us that we are all parts of an interconnected whole (Bateson, 1972).

The educational framework is a privileged thinking space for

Systemics who focus on: a) the relationship dynamics within the school, b) the ways of cooperating with teachers regarding problems that emerge with students, c) the cooperation with families and school, d) the cooperation of teachers and health professionals, e) the school failure etc.

In this presentation we will attempt to approach the area of education and in particular the teaching of mathematical concepts and the corresponding learning process from a systemic perspective (Karkazi & Nikolantonakis, 2009, 2011; Nikolantonakis, 2010). Valuable assistance in this process will be the involvement of a class of in service primary school teachers and an instructor-trainer whom will use a mathematical problem within the context of the ongoing training program.

Before we move onto the systemic analysis of the process that will be followed, we would like firstly to make a point of the context in which the teaching takes place: a class of in service teachers. We will consider the class as a system and we will refer to some aspects of this system such as its structure and function and then we will focus our attention on the 'status quo' of this class using as a reference the systemic concepts of ‘framing’, ‘interaction’ and ‘co-construction’.

Didactics of Mathematics and Constructivism

Constructivism is determined by the following principles (Von Glaserfeld, 1988):

1. Knowledge is a process of adaptation to the natural and social environment, and it is not the discovery of a pre-existing world regardless of the person.
2. Knowledge is constructed actively by the person and not passively understood from their environment.
3. Knowledge serves to organize our world and not the 'objective reality' meaning that the aim is to organize our learning experiences and to give them purpose and meaning.

The difficulties faced by a teacher when attempting to apply the principles of constructivism in mathematical education are various: conceptual, educational, cultural and political (Koleza, 2006).

The application of constructivism in practice requires from teachers’ to have the knowledge of teaching principles and at the same time to have an "appropriate" teaching approach. Constructivist teaching is based on students' activities, on problem solving, on exploratory projects and innovative ideas. Using these kinds of activities the teachers must not only be familiar with the theoretical principles underlying a particular subject (content knowledge), but also to be able to adopt a variety of approaches to the pertaining subject (pedagogical content knowledge).

Von Glasersfeld (1988) has identified five consequences of educational practices arising from radical constructivism.

First consequence: Learning which aims at understanding is separated from learning which is aimed at training (whereby repetitive processes are used and aim at the behavior). The contrast between concepts of understanding and behavior coincide with the traditional view of teaching whereby training aims at acquiring knowledge and beliefs and not to challenge repetitive behaviors.

Second consequence: Procedures arising from students (from the analysis of students' answers) are the interesting material. The behaviorist teacher is trying to see 'through' the apparent behavior of his/her students. The constructivist teacher is trying to see from 'within'.

Third consequence: Verbal communication becomes a process for guided learning for the student, and not a process for transferring knowledge.

Fourth consequence: Students deviating from teachers expectations becomes a means for teachers' to understand students' efforts towards understanding.

Fifth consequence: Teaching and using also interviews leads the trainers not only to conclude students' cognitive structures, but also to define them.

The theory of constructivism includes, besides the above, three other basic ideas:

a) Students devise personal methods for solving mathematical problems

b) Mathematical learning occurs through problem solving

c) The role of the social group is crucial in order to obtain knowledge.

Therefore, the basic pursuit of teaching, according to the constructivist theory, is to provide opportunities and to encourage the student to construct his/her own mathematical knowledge through exploration, experimentation, hypotheses' formation, generalization, justification, etc. This is the only way to consolidate their understanding and for effective learning to occur. In relation to how environmental learning occurs, many constructivist theories share four fundamental hypotheses which should be taken into account. These hypotheses can be described as knowledge construction, collaborative learning, self-regulation and the use of carefully formed authentic problems (Otting & Zwaal, 2007).

Von Glasersfeld concludes that Radical constructivism: a) is not a dogma, b) does not claim to be 'true', c) it is a way of thinking and d) one should think if it is useful in his/her field.

2. The School classroom as a system

A system is a whole with defined limits, consisting of people who interact, evolving over time, organized in relation with its environment and its intentions. All the elements of this definition are important, but we will focus on the key concepts of 'limits' and 'intentions'.

Limits

Interactions take place within the limits of a particular system and can be understood in connection within this framework. There are limits such as the limit 'of belonging' stating who belongs and who does not belong to this system, the limits that concern place and time etc. When one draws limits, it benefits the organization of the system, and in doing so favors certain stability. It is important for trainers to show interest in what state the organization of the system-class is because an organized group becomes a good context for learning (Curonici & McCulloch, 1997; Pauzé & Roy, 1987).

Intentions

One of the tasks of the trainer is to keep their focus on the initial intentions of the school which in this particular case is the ongoing training program and to create conditions with the students–trainees to manage it. With a systemic approach, the concept of the project plays a key role: the behavior of a system is interpreted in relation to its project and not to its structures. Its regulation and adaptation are designed to satisfy the project; its performances are defined in relation to the observed behaviors and intentions.

In our case, the activity "The target number" was the project which the group-class of primary school teachers focused on within the context of the ongoing training program. This project will serve as a reference point throughout the whole of this paper and which will allow us to see the course of the system-class and the co-constructions of the protagonists regarding a mathematical game-problem.

Table 1: *The Activity "The Target Number" [4]*

Each student forms 10 cards numbered from 0 to 9. Each player keeps the cards in his hands or placing them prominently in front of him on the desk. Students are divided into groups of 3 people. The teacher - trainer gives to all teams the command "I will suggest a number that will call target - number". Each of you, without looking at the choices of the other players and without talking with them, will choose a card, you will place it on the desk in the center and the purpose of each group will be the following: the sum of the three cards of its members to give the target number.

"The target number" as a game-activity is a good example of the theory of a-didactic situations [1] of Guy Brousseau (Brousseau, 1998), it is a field for repositioning the concept of division within the tool-object dialectic [2] of R. Douady (Douady, 1987) and implements the concept of the conceptual field [3] (Vergnaud, 1990).

Evequoz (1990) from his point of view insists that a class is an artificial system whose history is shorter than that of a physical system. It is based on a hierarchical structure on two levels (that of the teacher-trainer and that of the students-trainees). We observe two types of interactions: vertical interaction, between the teacher-trainer and the students-trainees and horizontal interaction, between the students-trainees. The teacher-trainer is supposed to undertake the function of the 'navigator'.

3. Group-class and the concept of 'framing'

'Framing' is a key concept in our analysis regarding the group-class and the framing of trainees from the trainer. The concept of framing was developed by a group of family therapists and researchers in Lausanne (Fivaz, Fivaz & Kaufmann, 1982). It is a concept which is applied in a therapeutic, educational and parental relationship. The operation of framing is to favor the development and autonomy of the sub framing system. The system which frames holds a superior hierarchical position in relation to the sub framing system and is characterized by having greater stability over time and by its ability to adapt to the sub framing system (Curonici & McCulloch, 1997, 2004).

The framing of trainees by the trainer takes diverse forms during the Scenes of the Scenario in the game-activity "The target number".

Table 2: *Scenario of the activity [4]*

Scene 1	Try once. We set this question. Are you ready? This question can make them think that something needs to be prepared. If not we draw a few rounds of cards and then they will probably begin to perceive and request a few minutes to talk within their group.
Scene 2	Teacher proposes a few rounds to be dealt and to put small and large numbers, and in multiples of three.
Scene 3	The teacher asks each group to write on a paper their strategy. Different groups will exchange their strategies. In this phase we will repeat the game with the new strategies which arise from the exchange between the different groups.
Scene 4	We record on the board the different strategies and we start a dialogue on whether there is a better strategy than the others, and what are the characteristics of a good strategy.
Scene 5	We play the same game in groups of 4, 5, 6 or 7, etc. players.
Scene 6	In this phase the same game is repeated but using the cards from 0 to 7 or from 0 to n, where n <9. At the end we play all together in a group.
Scene 7	The teacher - trainer gives the task to research the following problem. We have cards from 0 to l. The players are divided into groups of k players. The target number is n. Explain in this case the strategy accepted during the previous phase.

Firstly, the trainer is obligated to impact on the core of the learning system in order to create: 1.The setting for the trainer and the trainees, 2.The teaching and learning times, 3. The objects of the situations which create the environment and 4. The organizational relations with respect to these objects.

The trainer's actions define and provide the framework within which the trainees and the trainer will interact so that trainees will be able to negotiate the given problem.

Indicatively in Scene 1 we pose the same question every time, "Are you ready"? This question makes them think that something needs to be prepared. If this is not the case, another round is dealt and hopefully they will begin to perceive and to request a few minutes to talk within their group in order to agree on an action project.

In Scene 2, the trainer regulates the situation where the ultimate goal of its implementation is to produce a profitable strategy from students, even if at that time, the purpose is relatively distant. An important part of the trainers' activeness consists of a constructive contract, to dissipate the enunciations and to ensure their treatment and their discussion in class.

In Scene 3, the trainer defines the duties of each group while simultaneously gives the necessary space and assigns the trainees to record their strategy, to apply it in another group, to receive the rule of another group and to execute the game-activity with a new strategy.

Within the context of a teaching course the concepts of 'framing' and 'navigation' from the trainer implies that the trainer is the one who handles the process of learning through a linear relationship with his students? In other words, mathematical knowledge is a product that is transmitted from A to B and the perception of teaching is identified with the direct transmission of knowledge?

According to Fivaz et al. (1982) the operation of framing can favor the autonomy of the sub framing system: the autonomy of the trainees in relation to the trainer.

In Scene 4, the trainer provokes the trainees to listen and to study. They create conditions of conflict and dialogue. The trainer is interested in what the trainees do, he/she listens to them not because he/she expects a precise answer from what the trainer asked (classical function of the trainer in the didactic contract), but because they are capable of having a say and can participate under conditions of conflict. This discussion can lead closer to the real profitable theory. Also in this scene the trainer's question "So do you believe this is it?" seeks confirmation and urges the group of trainees to construct a logical argument. The phrase which is often repeated "I do not know, we must see..." is a statement of ignorance which also constitutes an incitement for them to research. The award is a process through which the trainer expresses a form of symmetry with the trainees during the learning project, which results to the onset of research and thereby legitimizes the trainees' ignorance and encourages them to invest and to research (Sensevy, Mercier & Schubauer-Leoni, 2000).

Consequently, the context of framing, the student-trainee from the teacher-trainer does not mean that the student-trainee doesn't act by him/her-self. In contrast, the active subject is interposed between the stimulus and the response. For Piaget (1937), intelligence organizes the world by its self-organization that means that learning is a continuous mental reorganization process. This organization and reorganization takes place through an experiential and reflective process in which the main emphasis for the interpretive process is attributed to the subject.

In Scene 2, the students-trainees play as partners. The plan that they have agreed on is that every time they participate in the game at the same time the project poses confirmation or rejection of their position. The aim of each team is to win, that means to construct collectively and cumulatively the desired number. By repeating another round, the group develops strategies as a means or reason as to why each member chooses to play a particular number rather than another. The students - trainees in each group construct a representation of the situation and use it as a model in order to be able to make decisions. We should of course note that the interaction with co-trainees for this specific problem regarding its resolution is the key way that affects the learning process.

4. Learning and interaction

In the systemic approach the concept of interaction, feedback, (Watzlawick, Bewin & Jackson, 1972) is the cornerstone of how a system functions and its members within this system. In the systemic consideration, the human systems are ‘open systems’ that means that there is a continuous flow of exchange (matter, energy or information) with the environment (Bertalanffy, 1973). The interaction between the elements of a system is characterized by the circularity that is a complex feedback and reciprocal process. In the case of the system ‘the class’ during the game-activity "The target-number" various subsystems composed of students-trainees. The trainees participate and discuss within these sub systems/groups that are formed.

In Scene 4, the conditions consist of communication between trainees who propose and oppose. The trainees are in symmetrical positions and in relation to: (a) the means of action to the environment, (b) the information and (c) the rules of the dialogue. The trainees receive feedback from 1) the environment for action and 2) the view of the interlocutor. The trainees interact with the environment and the exchanged messages are theorems and proofs developed or in a process of development.

If we invoke the First Cybernetic period of systemics and stay close to the systemic concept of feedback (Bertalanffy, 1973), we could say that the person receives feedback through information that relates to the consequences of his/her actions and based on this information he/she can tackle past information, can make new conclusions and be orientated towards the path of change (solve the problem). This is not achieved by ‘sake of grace’ but because of the interactions–feedback of the trainees.

Maturana and Varela (1980) working on perception, invite us to simultaneously consider the organization as a system endowed with its own internal logic and as a unity with numerous interactions. Simultaneously Maturana develops a theory of self-organization or autopoiesis. The human as a system but also other micro-societal systems, such as the family and the school are subjected to a process of autopoiesis by exchanging energy and information with the environment. This process is necessary for their development and takes place without threatening their identity (the human system is regarded simultaneously as both an open and close system).

In Scene 2, we ask from students to have feedback with the environment and to repeat the game in order to obtain knowledge (rule, technique, strategy) and in order to 'pass' from basic strategy to excellence. 'To pass' means the reduction and/or elimination of the student's–group of trainees', uncertainty for the choices that need to be made. The group of trainees reflects on their choices and decisions, through their actions with the environment. Feedback is considered to be the influence of the environment onto the group, which is perceived by the group as a reward or rejection related to their actions, allows for the correction of the groups actions, to accept or to reject a hypothesis and to choose between several solutions.

Patricia McCulloch (1994) has studied in particular, the horizontal interactions between students and has highlighted the huge potential for the entire class system. Within this, there is also a place for the teacher-trainer assuming they have relevant control of the class which McCulloch prefers to designate as a framing position. The teacher encourages the exchange between co-trainees in a classroom or members of a team of professionals (in service teachers). Consequently we are moving away from a linear perception of teaching (A to B) and we pass onto a constructivist perception of mathematical teaching, whereby the process of construction emerges in order to obtain the intended knowledge. Each student-trainee actively participates in the learning process and co-constructs with other students-trainees to find the solution to the mathematical problem. This construction is co-woven through the interactions of the trainees in the working groups where the inter subjectivity is desired, and therefore is not a problem anymore.

Heinz von Foerster (1988) considers that there is no distinction between observers and observed systems. He proposes to replace the epistemology of description with the epistemology of construction.

We can therefore say that the proposed exercise to the students by the teacher is based on the constructivist organization and teaching which ensures on a cognitive level, self-motivated learning through collectivity.

5. Systemic and Learning

We believe that mathematics in a school frame is first and foremost a social activity and not only an individual one. The transition from natural thought such as that of mathematical syllogisms is followed by constructions and rejections, and by the use of different means of proof. In scene 4, the students make declarations (an integrated form or partially integrated form of strategy solution) examined by their interlocutors for consideration. They can refuse a logic as wrong and then to prove in their turn.

Table 3: *Catalogue of strategies [4]*

(1) Use of the division by 9

This is the procedure used and accepted at the end. We divide n by 9. We took n = 9p + u. The first p players play 9, the next u and the following ones 0.

Very rarely it is expressed with this form. It frequently founded in the following form:

- between 0 and 9, the player A plays n, n<=9, the players B and C play 0
- between 9 and 18, the player A plays 9, the player B plays n-9 and the player C plays 0
- between 18 and 27, the players A and B play 9 and the player C plays n-18

With k players and cards from 0 to l, you divide n with l and you have n =lp +u. The first p players play l, the player on the position p+1 play u and the others 0. The number k is used for the determination of the interval of target-numbers. The difficulty is to give numbers to the players and to correspond the number in relation with p. At the end of the scene 5 the generalization is used to share the roles to everyone, with the target-number, that means if the target-number is before, inside, or after his/her interval.

(2) Use of the division by 3

We divide n by 3 and we have n = 3p + u.

If u = 0, every player plays p, if u = 1, two players play p and the other plays p +1. If u=2, one player play p and the other two play p+1.

This strategy can also be generalized. With k players and cards from 0 to l, we divide n with k, and we take n = km +u. The u first players play m+1 and the others m. The number l doesn’t play any role to calculations, but only to the calculation of the interval of possible numbers.

(3) Another strategy
If n<=18
If n is an even number, the players A and B play n/2 and the C player plays 0.
If n is an odd number, the players A and B play (n-1)/2 and the C player plays 1
If n>18, the players A and B play 9 and the C player plays the rest.

Bateson (1972) distinguishes four learning levels: the first corresponds to behavior of reflective type, the second consists to learning, the third to learn how to learn and the last level could be defined as the level where one learns how he has learned to learn or even to find the reasons of his own reason.

In Scene 2, the trainer repeats the phrase "what should one do in order to always find the requested target-number" and after various attempts from the different groups “what happened that you found it this time?" or to another group "what did you do so that you may always find it?" urges them to recognize what they are doing when they are doing it. With the phrase that is expressed to each group separately and in total to the class "what strategy is necessary (mathematically) to follow so that you win every time?" helps repeat to the groups the initial general guidelines. We observe at this point which way the teacher-trainer prompts the students-trainees to recognize their actions making use of the techniques of identification-reconstruction of suitable indications that characterizes their action.

However, we would like to focus on the collective dimension of the last phase of the learning process where all the different subsystems of the class (teacher/different working groups) interact and decide on the 'appropriateness' of the problem solution. The students-trainees put forward the theorems of each group which agreed to open dialogue within the class. In Scene 4, often the trainer-teacher asks a question between students’ enunciations “is it possible that all could be right?". The students’ enunciations are appropriate from the fact that they can develop the whole class as collective thinking unit.

Thus, students-trainees submitted interesting arguments that led all the trainees to accept the use of division (division by 9 or division by 3 or some minor variations) as part of the solution to the problem. General overview of Scene 4 is that "Doing mathematics doesn’t mean that I only accept, learn and correctly transmit mathematical messages. To formulate a theorem does not mean to give information, but what is true is within a system". Von Glasersfeld (1988) refers to “areas of consensus” in relation to the production of knowledge and not to one reality with a capital R. The formalization is generally regarded as a fundamental dimension of the productive work of a class, a group etc. The teacher-trainer and the students-trainees are structured as a collective of thinking to generate knowledge whilst at the same time allows for the evaluation of this production. Specifying the ways of execution and the structure they are forming, acknowledges their legitimacy. Summarizing in terms of the a-didactic situations theory we could note that the students-trainees are put in front of a problem and should try to solve it and in the midst of their efforts they are utilizing their resources of available knowledge, the ways and action systems that have been successfully used up till then so as to have the best result.

During these activities, the decisions that the student-trainee will make for the solution of the proposed problem will lead to the concept that is required for its efficient use. After the student-trainee passes through an action situation, he/she seeks ways of formulating the operations he/she has done in order to be able to maintain them in a formulated summary in his/her memory and to communicate them to his/her classmates. The environment of the student-trainee requires from him/her to be able to be in a position to convince the effectiveness of the solution he/she put forward. He/she needs to convince the correctness of his/her decisions and his/her proposals. We have a validation situation. Upon the completion of the search, we arrive at an accurate and concise formulation of the concept that was constructed by the student-trainee or the team of students-trainees in the format of an action rule that can be used in such situations. We are talking about formalization of situations where a concept that is constructed and used by the student-trainee acquires with the teacher's intervention the character-status of established knowledge.

6. Epilogue

Collective learning is inherently systemic in the sense that it is an interactive process and is condensed so that one may learn to act in a complex system where other players also act, so that one learns to juxtapose and combine representations and to process with others common representations. Consequently, the systemic approach can help develop such learning processes, mainly by suggesting open reporting models and representations, more reflectively rather than regulatory.

In regards to the constructivist teaching organization, we would like to point out that it enables and supports the development of students-trainees on multiple levels. It ensures self-motivated learning on a cognitive level, it enables the student-trainee how to learn to use himself/herself in complex interactive situations, achieves cooperativeness within the group-class and the implementation of his/her self.

The teacher-trainer can be the guarantor of the framing process for the development of students-trainees. In what way? With the double movement of stability and adaptation of the trainer (framing system) that allows students-trainees (sub framing system/s) to experiment the facts, to be recognized, to be listened to according to their rhythm, their difficulties, their strong points, their qualities and at the same time to be associated with their peers or colleagues who will take care to maintain the objectives, requirements; a dynamic tension of an interactive relationship of students-trainees to within the context of the exercise that was given to them.

In other words, a teaching technique is formed whereby the teacher organizes the confrontation of students' – trainees' enunciations and their adjustment. (Sensevy et al., 2000). From its success is dependant the transformation of the environment (Comiti & Grenier, 1997). An important part of the trainers' actions suggests the construction of a contract, to dissipate the enunciations and to ensure the treatment of students' discussion in class. In the case of the game-activity "The Target Number" we are confronted with the recognition of the need ("So we find things") which seem to form not only the mathematical and epistemological project of the teacher-trainer but also a teaching mean "if things are found" this means for students-trainees that their recognition is likely to lead them to win. So we acknowledge the fundamental function of the construction of a dialogue at multiple levels (teacher, student - trainee, class).

Endnotes

[1] The theory of a-didactic situations expresses and examines the genesis conditions of mathematical knowledge that is formed between a teacher and his/her students. In this we distinguish a social project which aims at learning knowledge from the learner and constitutes a list of conditions that must be satisfied in order to improve the student's ability so that they may succeed in solving a problem.

[2] During the school period of a pupil division takes successively the following statuses: implicit tool for solving problems, teaching object,

explicit tool to solve problems. This change of status (tool/object, implicit/explicit) corresponds to a macro scale, that Regine Douady has called object-tool dialectic. We say that a concept is a tool when we focus our interest on the use that is made to solve a problem. For object we mean the cultural object which has its place in a larger edifice that is the knowledge learned at a certain point, socially recognized. The tool-object dialectic has borned from a reference to the activity of the mathematician. Regine Douady talks of tool-object dialectic to distinguish the process of change of status of the concepts that are found in particular at the didactic transposition level. She distinguishes for the same concept three statutes: object, implicit tool, explicit tool. She organizes more widely these three statuses in a tool-object dialectic.

[3] A mathematical concept is a carrier of many meanings enclosed in many different situations. We can be sure that the student has understood the whole concept when they have tackled all the different situations which represent the concept. A set of problems-situations need to be constructed and their treatment will lead to the integrated concept.

[4] The description of this activity was given during a Master course on Didactics of Mathematics in Grenoble by Claude Comiti (1991-1992).

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Improving the teachers' understanding of complex systems through dynamic systems modelling and problem solving

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Abstract

Teachers' training is a fundamental factor for any effort to introduce students to complex systems, through mathematics or any other learning path. In this paper, we argue that dynamic systems modelling can support the meaningful learning of complex systems concepts in teachers' training. System dynamics concerns the study of the non-linear behavior of complex systems. In this research, teachers of various subjects and grades, approach complex systems by solving sustainability problems on the field of ecosystems during their training for the Education for Sustainability. The research by design and the case study methodology were applied to investigate the effectiveness of a specially devised teaching intervention participants' experiences, perceptions and conceptions of the complex systems and systems modelling based learning approach were studied. The present paper proposes an effective learning design for the introduction of complex systems concepts to teachers, combining systems dynamics, authentic problem solving and Digital Games Based Learning.

Keywords: Sustainability, systems modelling, complex systems, system dynamics, DGBL, teachers' training

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1. Introduction

The modern systems thinking (Bertalanffy, 1973; Wiener, 1961) has stirred the interest of the educational world mainly due to the awareness of the restrictions of linear causal analytic approach in the study of complex dynamic systems that surround us (Groff, 2013; Klopfer, Scheintaub, Huang, Wendel & Roque, 2009; Sterman, 2001). Forrester (1992) proposes system dynamics in combination with the learner centered learning as a framework for achievement of cohesion, meaning and motives in the modern education of all ranks. Traditional curricula teach static snapshots of the real world while the world's problems are dynamic. The human mind perceives these static snapshots as photographs, maps and static relationships in a wonderful and effective way. However, in systems with interacting components that change over time, the human mind is an inadequate "simulator" of their behavior. Even a simple social system cannot be solely understood by intuition (Forrester, 1992). The penetration of systems thinking in the curriculum of primary and secondary school is fairly low. While simulations are widely used in education to actively involve students in self-guided discovery procedures, the modelling, which opens the "black box" of simulations and allows learners to create or modify their own models, have not become an integral part of the educational process (Klopfer, Colella & Resnick, 2002). The idea of introducing system dynamics in the education has inspired important educational projects such as the Creative Learning Exchange (CLE) (http://www.clexchange.org) or the work on Starlogo (http://education.mit.edu/starlogo; Colella, Klopfer & Resnick, 2001). Projects like the "Adventures in Modelling (AIM)" using StarLogo aim to introduce students and teachers to the process of designing, creating, and analyzing their own models. Moreover, the participants explore models in any domain in order to develop a deeper understanding of patterns and processes in the world around them (Klopfer et al., 2002). StarLogo allows students and teachers to see the "invisible" and examine complexity in ways that were previously impossible to do (Blauvelt, 2001).

Because of its interdisciplinary nature, the utilization of dynamic systems modelling in the education is of particular interest for several subjects including mathematics, science, ecology, social sciences, management and technology. Specifically, for the mathematics education, the dynamic systems modelling provides an additional domain of applications of mathematics as a formal mean for the description and the study of the world, in a manner that utilizes the Information and Communication Technology (ICT) as a cognitive tool. The research concerning the introduction of students in systems thinking through computer modelling in various subject matters is advancing internationally (Gkiolmas, Karamanos, Chalkidis, Papaconstantinou & Stavrou, 2013; Leiba, Zuzovsky, Mioduser, Benayahu & Nachmias, 2012). The effective training of teachers constitutes necessary prerequisite for the utilization of systems thinking and the study of complex systems in K-12 education. In the present study concerns the investigation of teachers' ideas and misconceptions, about fundamental concepts of complex systems, along to the development of the understanding of systems thinking concepts. The authors aim to facilitate teachers' meaningful learning (Ausubel, 1963; Ausubel, Novak & Hanesian, 1978), as well as their conceptual understanding of complex systems, through a thoroughly designed instructional intervention. The formal systems and complexity theories can be introduced more smoothly and incrementally, building upon the infrastructure of the understanding of the key concepts. In this context the formal description of systems emerges initially as a tool for their clear description which serves the needs of human-human and/or human-computer communication.

2. Theoretical framework

2.1 The sustainability as key concept in the context of the intervention

For the design of the teaching intervention in an authentic context, a familiar, interesting, and accessible for teachers, problem domain, rich in complex systems, is needed to be selected. Ecology constitutes a well-known example of a complex systems domain (Anand, Gonzalez, Guichard, Kolasa & Parrott, 2010; Grimm, 1994). The structure of an ecosystem is determined by the interactions between populations of animals, birth and death rates, the quantities of food and other more specific variables of different ecosystems (Forrester, 1992). System dynamics modelling offers a method for the understanding of the behavior of such complex systems, by representing them in executable computer models which simulate the ecosystems taking advantage of the many simultaneous computations (Forrester, 2007). Ecosystems were also selected in the present research because the participant teachers (of various subject matters) were attending postgraduate course concerning the education for the sustainability and the environment. The research focus on the key concept/problem of the sustainability of an ecosystem, while systemic concepts emerge and develope in relation to the key concept.

In the context of environmental education, sustainability concept is often determined by the interaction of three components, namely the *environment*, the *society* and the *economy* without explicit signification of its relation to complexity. For Pittman (2004) there is a dynamic dimension to the concept of sustainability which embodies the ideal of change towards being viable. Sustainability is an ongoing process of a dynamic equilibrium of behaviors and conditions. There have been proposed many ways of representing sustainability (Todorov & Marinova, 2011) with most popular the Venn diagram of the three overlapping circles (economy, society, environment). In this paper, we adopt a quantitative model approach for the sustainability concept in the field of Education for Sustainability (EfS) using the STELLA modelling environment as a cognitive tool. The proposed conception of sustainability is more or less that of the general systems theory in which a system is sustainable for a time interval if it is has not exhausted the required resources for its operation (Dexhage & Murphy, 2010). UNESCO has been invited from the United Nations Commission on Sustainable Development to make a significant effort to help educators worldwide: not only to understand sustainable development concepts and issues, but also to learn how to cope with interdisciplinary values-laden subjects in an established curriculum (UNESCO 2005b; UNESCO, 2010). In that direction the ecosystems' computational modelling activities could play a significant role, helping teachers to develop their conceptual understanding of complex systems to innovate Mathematics and Science education enhanced by ICT.

2.2 Computer modelling of system dynamics as a learning approach

For the needs of the research, participant teachers are engaged in computer modelling of dynamic systems and the use of game simulations. It constitutes a widespread conviction in the field of technology enhanced learning that development of computer models supports the comprehension of the modelled system. The developer of a system model engages in experimentation and reflection circles repeatedly, during which he/she improves his/her understanding of how and why the specific system operates. This claim is supported by research findings (Confrey & Doerr, 1994; DiSessa, 1986; Talsma, 2000) while being the starting point for the creation of several cognitive learning environments and educational programs relative to systems thinking and the complex systems in particular (Colella et al., 2001). Computer modelling value as learning tool and epistemological method justifies the integration of computer programming in education as the pioneer Papert (1980) defended with, his brainchild, LOGO programming language. More recently, several computing environments were developed especially for studying of dynamic modelling and complex systems, such as Stella (Richmond, Peterson, & Vescusco, 1987), StarLogo (Resnick, 1994), Model-It (Jackson, Stratford, Krajcik & Soloway, 1995), and NetLogo (Wilensky 1999; Wilensky & Resnick, 1999). These environments enable the design and the development of complex systems models, not only by researchers but also by teachers and students. The modelling activities contribute to the understanding of complex concepts such as sustainability. Students who explore models exhibiting complex behavior of real world systems, may observe the same behavior in other systems, internalizing this idea as part of their normal thinking (Senge, 2000). Even if some students will not construct models later in their lives, they should be aware of the nature of these models that will be built by those whose propose changes in economic and social policies and will be available for public inspection.

In parallel, over the past few years the Digital Games Based Learning (DGBL) has become recognized as distinguishable pedagogical model (Connolly et al., 2012; Gee, 2003) which could enhance students' engagement with complex learning content. Learning games are considered as interactive simulations enriched with game features (Klopfer, 2008).

2.3. Review of related works

In this section selective review of educational research on dynamic systems modelling is given. Surveys of Hogan and Thomas (2001), as well as of Nuhoglu (2008) focused on the nature of the reasoning of secondary school students in modelling dynamic ecosystems with STELLA software through the identification of cause and effect relations. Hogan et al. (2001) research have been discussed srudents' difficulties on development quantitative models, where it deemed necessary to provide continuous Scaffolding. The Joolingen and Bollen (2013), and Jong, Joolingen, Savelsbergh and Borkulo (2012) present the research results of the use of two learning modelling environments in young groups. This is SimSketch, which allows modelling development through an informal way of design and the Co-Lab**,** a collaborative inquiry-based learning environment. The findings of these researches showed that while there was no significant difference in the groups creating and investigating models with the different software, there was a shift of participants from the representation of models accurately (morphic analogy) to a more functional representation (validation of models based on their results rather than their structural similarity). This shows improvement in the participants' ability of abstract perception of the systems.

More recent research combines meaningful learning and conceptual understanding with the system dynamics, integrating conceptual modelling (concept maps) in the modelling software. The Zuzovsky and Mioduser (2012) work is a typical example in this category of research, which concern a modern pedagogical approach to support students in the study of complex systems through the conceptual modelling environment Dynalearn (www.dynalearn.eu). According to the researchers, the *concept mapping* functionality enabled direct and intuitive construction of representations of the systems under study. Furthermore, concept mapping supported the analysis of students’ ideas about the structure and the cause-effect relationships in the systems under study. Similarly, studies of Leiba et al. (2012), and Souza, Sa, Costa e Silva, Wilhelms and Salles, (2011), concerning the Dynalearn modelling software evaluation, claim that students increased their ability to represent structural and functional properties of systems. The combination of conceptual mapping (qualitative modelling) with the simulation of dynamic systems (quantitative modelling) seems to facilitate the understanding of complex systems by students.

Summarizing this review, many researchers note that there are promising results and significant progress in the design of dynamic systems modelling environments for learning internationally (Komis, Raptis, Politis, & Dimitrakopoulou, 2004). Despite this fact, dynamic systems modelling environments (such as Stella, Modellus, Vensim, iThink and DynaLearn) are not used extensively in educational practice. Moreover, the research that concerns the teachers’ understanding of complex systems, as well as suitable teachers’ training in dynamic systems modelling for learning are limited. Since teachers’ training is a decisive precondition for the success of any educational innovation, we focus on the need for study the issue of teachers’ training for the conceptual understanding of complex systems concepts and their potential in education.

2.4. Research rationale, and research questions

The world we live is full of complex systems with circular relations and the simple linear deterministic reasoning of traditional mechanics is simply not enough to deal with them. Subsequently, systems thinking and complex systems dynamics constitute significant capabilities that could be developed by the education system. Many researchers have already studied the students’ exposure to systems dynamics modelling with encouraging results. Furthermore, international organizations advocate the cultivation of systems thinking in K-12 education. Sophisticated and user friendly ICT modelling environments for complex systems dynamics, of discrete events or continuous time, are available widely along with teaching and learning material and support. Systems thinking and the study of complex systems dynamics could extend the frontiers of mathematics education and its applications to the teaching practices of other knowledge fields, in combination to the use of ICT modelling tools. The integration of complex systems modelling in K-12 education depends crucially on the suitable training and professional development of the teachers. In this work we focus on the conceptual understanding of complex systems concepts and especially on the concept of sustainability. The aim of the research is to improve complex systems understanding by the teachers using a specially designed teaching intervention in which they solve authentic sustainability problems using simulations and systems dynamics modelling.

Thus, an integrated instructional intervention (series of learning activities) was conducted aiming to assess teachers' conceptions of the Sustainability concept and related concepts of complex systems, addressing the following questions:

RQ1. Can dynamic system modelling environments be used as tools for meaningful learning (conceptual understanding) of complex systems concepts?

RQ2. Can the concept of sustainability be approached through the study of a dynamical system of an ecological problem?

RQ3. How effective are the processes of modelling as teaching and learning methods?

RQ4. Does the proposed instructional intervention and learning activities contribute to the improvement of conceptual understanding of the participant educators?

3. Methods and procedures

3.1 Methodology

A combination of research by design and case study methodologies was selected to approach the posed research questions. More specifically, the exploratory case study methodology (Yin, 2011) was selected because it allows a better understanding of complex phenomena and, in-depth observation of authentic conditions, in specific cases. In an exploratory case study, it is not required hypotheses declaration a priori about the results, but in this case the authors expected positive effect of interventions. The research by design is a relatively new methodology that is suitable for ICT enhanced learning interventions development and evaluation. Research by design incorporates an iterative process of intervention design and implementation in order to promote both theoretical understanding and educational practice (DiSessa & Cobb, 2004). The design element is just as important as the research-experimental element. In the present case the main purpose of the research by design was the design of innovative instructional intervention supporting the improvement of the teachers’ understanding of complex systems concepts.

3.2 Sample and procedures

The research was conducted to 8 teachers (seven females and one male) who were postgraduate students attending the program "Environmental Education for Sustainable Development" of the Department of Philosophy, Pedagogy and Psychology of the University of Athens, during the academic year 2013-2014. The teaching intervention (delivered by the second author) was implemented in two three-hour sessions.

3.3 Research data collection instruments

A full and detailed description of interventions with worksheets and questionnaires exceeds the limits of this article, the interested reader can find detailed information at Kyrodimou (2015). Three research data collection instruments were used:

1) Q1. Initial investigation questionnaire of ideas-perceptions of teachers of the concepts: system and sustainability. The initial investigation questionnaire was designed to explore the initial ideas of the participant teachers about complex systems and sustainability/viability concepts (see §5.1).

2) Q2. Final investigation questionnaire of ideas-perceptions of teachers and evaluation of the intervention. The final questionnaire was given to the participants after the teaching intervention in order to: a) detect any conceptual change from their original ideas for the concepts of system and sustainability, b) to record the opinion of the teachers about whether and to what extent can the modelling environments be used as tools in understanding the dynamic concept of sustainability and whether they contribute to the conceptual understanding by the students.

3) Three learning activities work sheets (WS1-WS3): for introduction/familiarization, design/development and simulation/analysis of a dynamic system model with qualitative-quantitative modelling software Stella (see §4.4).

3.4 The teaching intervention

In the teaching intervention, teachers are getting familiar with the modelling and simulation of dynamic systems through three learning activities from the ecology field. More specifically the teachers are called to give sustainable solutions in environmental issues. Through the investigation of the long-term effects of different policies for population control in an eco-system, the participant teachers are expected to improve their understanding for the concepts of sustainability and complex dynamic systems because they are going to use them as authentic problem solving methods (tools).

Learning Activity 1. Familiarization with STELLA software. A step-by-step guide to modelling a predetermined ecosystem.

The purpose of the first learning activity is to familiarize the teachers with the STELLA modelling software. The teachers initially introduced to basic concepts of STELLA (e.g. entities, stocks, flows, connectors) and then were asked to create a model population of hares in an eco-system following a step-by-step guide (Figure 1). By defining the equations, a STELLA model will follow from the structure. The model will illustrate the system's dynamic behavior and the hare population change over time. Adding the hunting as a factor of depopulation of the hares, educators are seeking regulatory policies to bring the population in dynamic equilibrium. By simulating the regulatory policies of the model, the linking of the time-paced evolution of the population with the concept of sustainability and dynamic system emerged (Figure 2).

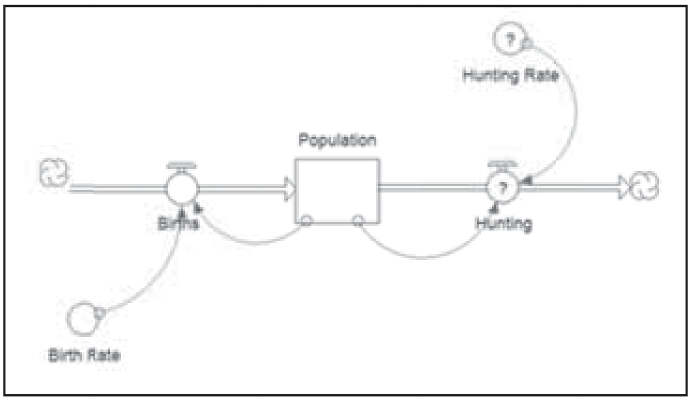


Figure 1. The model of the hares' population in Stella environment.

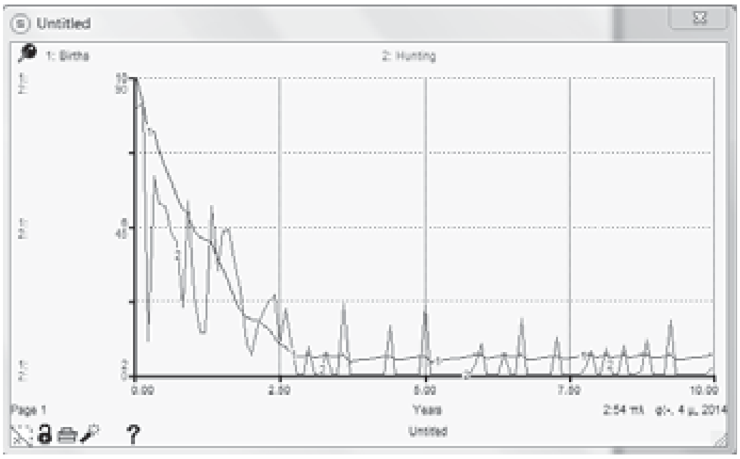


Figure 2. The graph of the population of figure 1 model in dynamic equilibrium

Learning Activity 2. Learning-Study complex systems through a computer simulation game

The second learning activity aimed at development of intuitive teachers' conceptions of complex system and sustainability through a simulation game. Table 1 contains an excerpt from the learning activity worksheet. Learning activity 2, uses the Fish Game, by the Cloud Institute for

Sustainability, that simulates the human fishing activity on a lake, in order to ask teachers to discover a sustainable fishing practices (Figure 3). Teachers provided descending guidance to explain the rules of the game while they were recording the results of the fishing activity before and after the implementation of fishing policies.

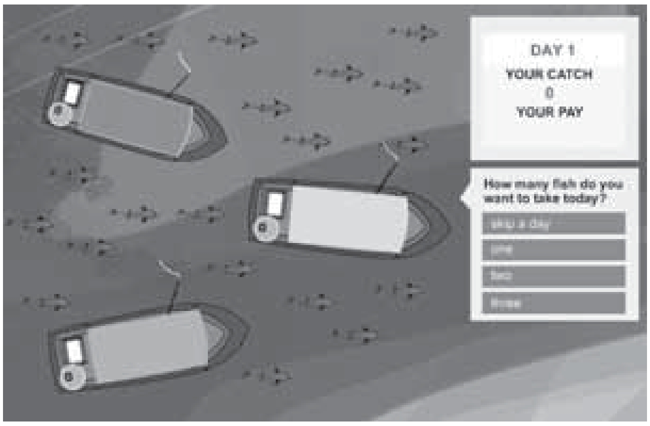


Figure 3. Screenshot from the Fish Game, of Cloud Institute for Sustainability

Table 1. The directive of the Fish Game

You can be guided to the Fish Game through the website: http://cloudinstitute.org/fish-game/ You have 10 days to catch as many fish as you can. The money you make from these fish will need to support your family for the next month. Each fish nets €2. Your computer is your opponent. The lake in which you are fishing can only support 20 fish (that is the carrying capacity of the lake). Every night, the fishes that remain after a day of fishing will reproduce at a rate of 25% (for the purpose of this game, we round to the nearest whole number). However, the total number of fish can’t exceed 20. For instance, if there are 12 fish, they will multiply to 15 overnight. If there are 19, they will multiply to 20. For your 10 fishing days you can choose whether you want to take none, one, two, or three fish for the day. There are two other fisher folk also trying to catch as many fish as they can – they will follow your lead, and base their catch on yours.

Remember: The goal of the game is to win as many fish as you can.

You will play the game 3 times. Before starting each round, you have to choose a catch fish policy that can change every day, you will record the expected results and you will apply your policy to the game until the end of the 10 days.

For each game complete as below (e.g. Fisheries Policy: 1,1,2,3,0,1,1,1,1 and 1)

Learning Activity 3. The teachers develop systems model for resolving a historically recorded ecosystem management problem

In the third learning activity, the teachers apply what they have learned earlier to study a complex ecosystem associated with the historical problem of the collapse of the fishing activity of sardines in the Pacific Ocean in the 50's The same problem is used also on the board game Fish Banks Ltd. (http://www.systemdynamics.org/products/fish-bank/) which is a role-playing game developed by Dennis Meadows. Table 2 shows, the introductory excerpt of the 3rd learning activity worksheet.

Table 2. The problem which was raised to teachers in learning activity 3.

The seas are a source of wealth for centuries. Throughout the course of these centuries, the fishermen are in constant battle with the sea and in recent years with the development of higher harvest fisheries technology have prevailed in this long battle. The excerpt below illustrates the result of over-exploitation of the sardine population led to the collapse of the catching sector in 1950.

"The Pacific Coast sardine industry had its beginnings back in 1915 and reached its peak in 1936-1937 when the fishing netted 800,000 tons. It was first in the nation in numbers of pounds of fish caught, and ranked third in the commercial fishing industry, growing $10 million annually. The fish went into canned sardines, fish bait, dog food, oil, and fertilizer. The prosperity of the industry was supported by overexploitation. The declines in the catch per boat and success per unit of fishing were compensated for by adding more boats to the fleet. The fishing industry rejected all forms of regulation. In 1947-1948 the Washington-Oregon fishery failed. Then, in 1951, the San Francisco fleet returned with only eighty tons. The fishery closed down …".

Robert Leo Smith (in Owen, 1985)

The Tragedy of the Commons

The tragedy of the commons is an economic theory of a situation within a shared-resource system where individual users acting independently and rationally according to their own self-interest behave contrary to the common good of all users by depleting that resource.

Activity 1 – The model in Stella

You are hereby required to take the role of the manager of fisheries sector in the period of 1950 before the sardine collapse in the Pacific fishery. Studying the simulation model of the Fish Banks L.td. game you will be invited to submit proposals that should ensure the conservation of the fish population and fishing activity. The model is based on the Fish Banks L.td. game which was originally developed by Professor Dennis L. Meadows.

Through the interaction between the system fish population and the fishing companies, you are asked to analyze policies by changing its structure, having **responsibility** and **aim** to configure a regulatory policy to support a sustainable fisheries policy.

The goal of the 3rd learning activity is to permit teachers to use the historical authentic data of the collapse of sardine fishery to create a computerized simulation model for the prediction of the dynamic behavior of the corresponding complex system in order to be able to test the long-term effects of different management policies. Teachers open the "black-box" of simulation and assume the controller role in a sustainable fisheries policy facing a real situation. The problem solving ability of the participant teachers will reveal the level of understanding of the examined concepts. Problem data and the different policies are given to students with the assistance of authentic documents of the time. Specificly, in learning activity 3, the teachers study the behavior of the system (Figure 4) and give answers in some initial conditions of the fish population to be studied. Three different policies that affect the dynamics and sustainability of the system follow. In each of these, the teachers are asked to record their prediction before and after the execution of policy. The ultimate objective is or students to search and design their own fisheries policy for the continuation of fishing activity and ensuring the fish population.

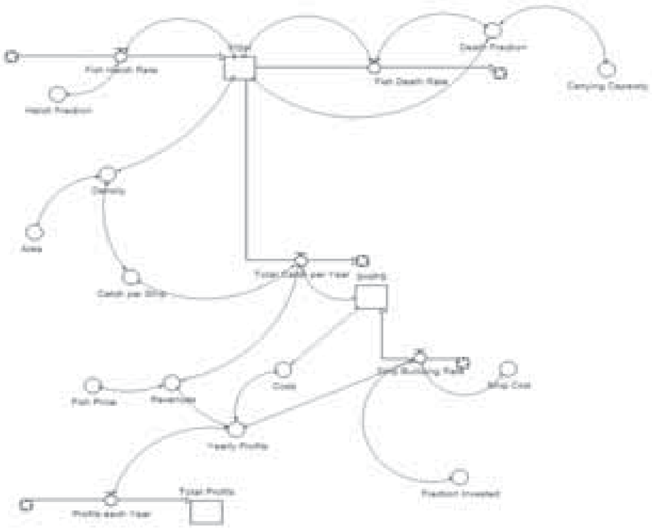


Figure 4. The model of the sardine population in STELLA.

4. Results

4.1 Questionnaires and learning activities

The key findings of the research questionnaires and learning activities implementation are presented in this section along with the answers the research questions.

Q1. Initial investigation questionnaire of ideas-perceptions of teachers of the concepts: system and sustainability. Examining the participant teachers' answers the question *Q1.1: "How you would evaluate your own understanding of the sustainability/viability concepts?"* supports to the estimation of how confident they were initially about their comprehension of the sustainability concept. Most of the teachers believed that they understand quite good (4/8) or very well (3/4) the concept of sustainability while one teacher answered "satisfactory". This was rather expected due to the fact that they attend to a relevant postgraduate program. Table 3 shows the answers to the question *Q1.2 "How do you perceive the concept of sustainability/viability?"*.

Table 3. Teachers descriptions of the meaning of sustainability

Teacher	***Answer***
T1	*"… I have combined the concept with growth, which is a development based on intergenerational justice and solidarity"*
T2	*"… is the development that meets the needs of the present without compromising the future of generations to come, taking into account environmental protection"*
T3	*"… is the human interaction with the environment and society"*
T4	*"… is the socio-economic development that has as a stock the quality life of future generations, as for the current generations in a climate of growth and solidarity"*
T5	*"… is the combination of social, economic, political, cultural considerations in relation to various environmental issues"*
T6	*"… highlights social - economic - political - ethical - environmental - educational dimensions. A driving force, effort for the perpetuity"*
T7	*" ... is the non-use of natural resources until exhaustion "*
T8	*"... is development that meets the needs of the present without compromising the needs of future generations..."*

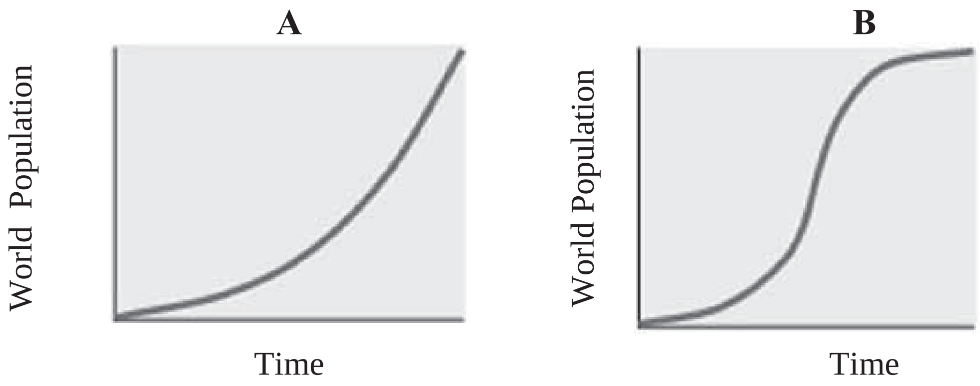
Teachers may elaborate the static conceptions of sustainability, affected by the basic theory of the environmental education which approaches the concept using the vein diagram of economy, society and environment concepts.

Table 4, shows the answers to the question *Q1.3. "What constitutes a sustainable or viable system in Ecology?"*. The answers of the teachers show that their conception of sustainability initially is quite confused without explicit reference to concepts such as: resources, exhaustion, time interval. Table 4 answers support the hypotheses that the participant teachers had incomplete sustainability conceptions.

Table 4 The sustainable system according to the participants' initial opinion

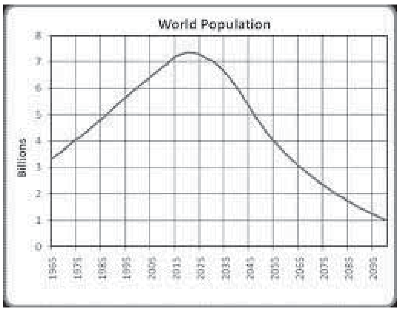
Teacher	***Answer***
T1	*"…is a system which is structured in such a way as to indicate the causes, the effects and possibly solutions of an environmental issue"*
T2	*"…we consider the way that the elements interact and the results of interactions are optimal as much for the person as for the society and the environment "*
T3	*"… is the maximum enjoyment of goods without affecting the future generations. Coexist economic, political and social systems"*
T4	*"… the avoidance of an environmental issue with aim of continuing the existence and maintenance of life and natural resources."*
T5	*"… is a system that combines economic-environmental and social factors "*
T6	*"… is the development of a system so as not to disturb their equilibrium"*
T7	*"… is the system that develops relationships with its parts"*
T8	*"… is the system that is in balance with the environmental, economic and its social factors "*

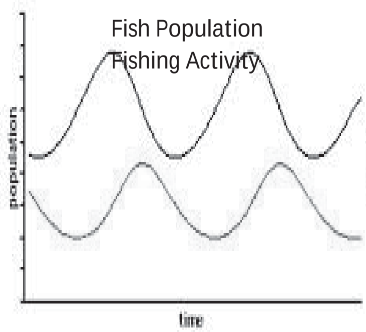
In addition, teachers asked to answer the question *Q1.4. "A system is often represented by a graph relating its input to its output over time. The graphs (Figure 5) represented this variation for some systems cases (e.g., world population, fish or hare population). Which of these graphs correspond to sustainable systems?".*



C

World Population





D

Figure 5. Identification of the graphs that correspond to a sustainable system.

The teachers studied the four graphic representation of the systems and identified which ones correspond to a sustainable system. The answers of the teachers were as follows T1:A & B, T2:A, T3: B & D, T4: A & B, T5:B, T6:B & D, T7:D, and T8:B. Only 2 out of the 8 teachers (T3 & T6) recognized correctly both sustainable systems -which reveals misconceptions in understanding of the general concept of sustainability in mathematical terms. This finding is consistent to the finding of the previous question Q1.3.

The initial investigation of the teachers' ideas and conceptions reveals misconceptions and incomplete ideas about sustainability despite the high confidence of the participant teachers about their understanding of the concepts. More specifically some teachers associated the concept of "sustainability" with the concept of development and management of natural resources, while others imprinted it as a way of human interaction with the environment. Teachers seem to perceive the concept of sustainability as an attempt to balance the environmental impact of economic and social development. The teachers' conception of sustainability is compatible to, and affected by, their "Education for Sustainability and the Environment" studies, and does not refer to the technical meaning of sustainability as understood in the context of systems theory. In systems theory, sustainable system is the system which does not exhaust (never or within a certain time interval) the essential resources of its operation. Mathematically, the variables that represent the resources should not be zero or get negative values. The initial investigation of misconceptions and difficulties of the teachers revealed that their understanding levels of the sustainability concept could be improved significantly by teaching interventions and/or learning experiences. The incomplete conceptual schemata of the teachers need to associate the sustainability concept with concepts of complex systems dynamics.

Analysis of Learning Activity 1. Familiarization with STELLA software. A step-by-step guide to modelling a predetermined ecosystem. The 1st learning activity was designed to familiarize participants with the STELLA Software and the concept of the systems dynamics modelling which was emerged during the simulation of hares' population in an ecosystem. All the participant teachers completed successfully the learning activity. After the completion of the activity teachers were asked to answer the question *"What is a dynamic system"*, the answers are shown on Table 5. The answers show the influence of the terms of dynamical systems, as these presented in Stella, in their discourse. This indicates that the participant teachers adopted easily the conceptual framework of the systems dynamics, despite the fact that they had no previous contact with it.

Table 5. Answers to the question what is a dynamic system

Teacher	**Answer**
T1	*" ...it's a system that changes over time, according to internal or ext/nal events"*
T2	*" ... it's a system that the variable population fluctuates depending on factors "*
T3	*"... a system of interactions where are changes in the long term"*
T4	*"... it's a system that we can define with some variables, and based on them we can change-affect the system"*
T5	*" ...It is a regulating mechanism that contributes to the maintenance of a system"*
T6	*" ...it is the system that changes over time "*
T7	*" ...it is a system of interaction with certain factors "*
T8	*" ... it is the system that includes variables that affect itself"*

Analysis of Learning Activity 2. Learning-Study complex systems through a computer simulation game. In the second learning activity of the instructional intervention, the teachers used the Fish Game to invent and test the consequences of alternative fishing policies in a lake. Fish game simulation presents complex behavior that is difficult to predict in the long term, despite the fact that it works according to very simple rules. This makes the specific game ideal mean for the introduction of the complex systems features and properties. The goal of the game for the participant teachers was to catch as many fish as possible in a given time interval of ten days without exhausting the stocks of the lake. After a short experimental familiarization with the Fish Game, the teachers had to determine three fishing policies and to declare the results they assume as expected for each proposed policy. Afterwards, they applied their proposed policies in the game and compared the real results to their forecasts. This way, the teachers are invited to devise intuitively a sustainable policy for the specific situation in order to have a firsthand experience of the difficulties this task has even for simple complex systems. Indeed, 7/8 (88.5%) of the teachers did not managed to finish all the rounds of the game without exhausting the lake fish stocks while only one was able to implement sustainable fishing policy. This was quite surprising for them, and as an event of cognitive conflict it caught their attention and prepared them to be aware and learn more about complex systems.

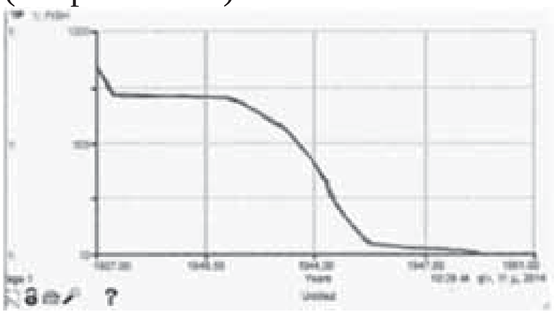
Table 6, shows the teachers' answers to the question: *" What conclusion can we draw for the evolution of a complex system, such as the above, when we apply simple and intuitive harvest rules? Eventually, is it easy to predict the time evolution of a system and to devise intuitively effective rules of sustainable management?"*, that was answered at the end of the learning activity.

Table 6. Teachers' opinions about intuitive management of complex systems

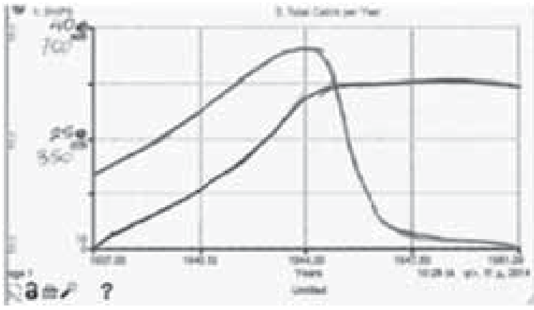
Teacher	Answer
T1	*" ...we can't have the best results for us and the environment using only simple intuitive reasoning. It's not easy to devise intuitive sustainable management rules."*
T2	*" ... with simple and intuitive rules it is not easy to make safe predictions..."*
T3	*"...it's not easy, needs understanding and developing adequate strategies..."*
T4	*" .. It is not easy to predict the evolution of a complex phenomenon only with the human brain ..."*
T5	*"... Intuition is not enough, there is difficulty in predicting..."*
T6	*"... There is a difficulty predicting the behavior of the phenomenon if you focus only the main problem..."*
T7	*"... sometimes the simple rules are more correct and have better results but greed did not let us think calmly and correctly "*
T8	*"... sometimes wants a deeper understanding of the system for prevention of its behavior "*

Responses on Table 6 show that the 2nd learning activity using the Fish Game effectively guided the participant teachers to realize the counter-intuitive behavior of complex systems which occurs even in those systems that are described by relatively simple rules. The teachers through the process of cognitive conflict and rebuttal of their own forecasts are prepared to engage in the next learning activity in which they create a dynamic system model to resolve a real ecological problem.

Analysis of Learning Activity 3. The teachers develop systems model for resolving a historically recorded ecosystem management problem. As already mentioned, in the 3rd learning activity, the teachers opened the "black" box of the simulation and assumed the role of the controller in a sustainable fisheries policy facing a real situation (the collapse of the sardine in the Pacific Ocean fishing activity in the 50's). The teachers study the dynamic model of the Fish Banks game and then they are called to propose fishing policies that should ensure the conservation of the fish population and fishing. To assess the effectiveness of the learning activity, the teachers are called to answer questions in specific steps of the process.Teachers initially prepare graphs of the evolution of the sardine population during the period 1937-1951, with the fishing policies that were implemented by those who were responsible at that time and then to retell the story of the problem using the graphic displays of fish population and ships purchased as a policy response to the reduction in fishing activity (Graph 1 and 2).



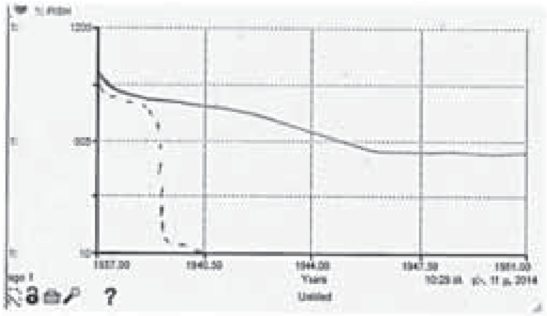
Graph 1. The population of sardines during the fishing period 1937-1951



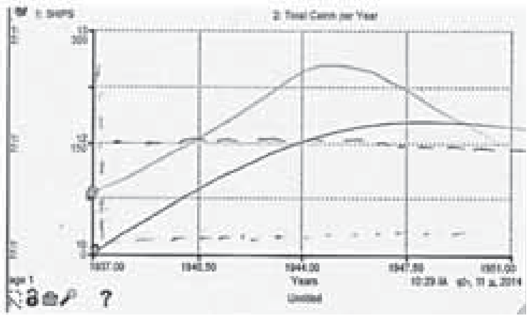
Graph2. The number of ships, stocks between 1937-1951

The participant teachers managed to produce correct graphs (in the sense that they depicted the historical data) and understood what happened as their storytelling reveals. Indicatively, the story of the sardines' population narrated by a teacher using Graphs 1 and 2, follows: *"The left graph shows the decline of fish populations because of overfishing and the right graph shows the decline in the fish population in relation to the growth of ships. When number of ships increased, the fish number dropped drastically"*.

The teachers are then guided to apply in the software model a 1st policy according to which tax is imposed for the new boats. Using the simulation of the model the teachers studied the policy repercussions in the sustainability of fish population and the piscatorial activity for the interval 1937-1951. Through the study of the tax implementation policy all the participants agreed that it was an acceptable sustainable policy (Graphs 3 and 4).



Graph 3. The population of sardines during the fishing period 1937-1951 under the tax policy



Graph 4. The number of ships, stocks between 1937-1951, under the tax policy

Subsequently, they were asked to extend the prediction (simulation time) of system behavior by the year 2030 and to answer what would happen if they applied the same policy that was recorded as sustainable. All the teachers (8/8) were surprised by the results of the simulation of the system behavior with new simulation time limit because they did not expect that the tax policy would be proved unsustainable in the long term. Then, the teachers were asked to consider a few more policies, concerning larger taxes amounts in order to conclude that the increase of tax increases the time interval of guarantee of fish catches, however, it is not capable to eliminate the collapse of population. Finally, the teachers were asked to propose and apply in the simulation their own plan of sustainable piscatorial activity. Seven out of eight (7/8) teachers were able to propose a sustainable policy that they validated through simulation and documented its viability using graphs.

From the analysis of the implementation of the second learning activity it seems that a) all the teachers completed the tasks successfully, b) they understood the functionality of the model, c) they became capable to transfer knowledge between the model and the real world system, and d) they were able to apply the concept of sustainability for the resolution of an authentic historical problem. The use of a historical ecological problem formulated a conceptual context in which the teachers managed to engage in mathematical and computational modelling of complex systems dynamics improving their understanding of relative concepts with a pleasant and effective way.

Q2. Final investigation questionnaire of ideas-perceptions of teachers and evaluation of the intervention. The final questionnaire aimed to detect any conceptual change about the complex system and sustainability concepts on the research participants. In addition, the questionnaire contains evaluation questions about the teaching intervention. The key findings of the questionnaire are presented in sections §5.2 and §5.3.

4.2 *Evaluation of the teaching intervention by the participants*

To the question*: Q2.1.* *"Did your attendance in the activities of* computational *modelling contributed in the better understanding of the concept of sustainability/ viability*?" most teachers found the contribution of the intervention satisfactory (4/8), enough (1/8) or very much (3), on a 5 points likert scale. The data shows a positive evaluation of the intervention contribution by the teachers'.

With question *Q2.2. "How successful was the approach regarding the concept of sustainability through the study of fishing activity Fish Game and the study of dynamic model simulation of the environmental problem of sardines in STELLA Software?"* the teachers evaluated the contribution of Fish Game and STELLA activities. The teachers found the contributin of Fish Game activity, Satisfactory (3/8), Enough (2/8) and Very much (3/8) on a 5 points likert scale. Similary, the teachers found the STELLA software contribution Satisfactory (2/8), Enough (3/8) and Very much (3/8). The answers support the hypothese that the teachers value positively the contribution of the second intervention that aimed to experiential exploration of the properties of complex systems such as the difficulty of forecasting their behavior with simple thinking approaches, and their counter-intuitive behavior.

In both the questions: *Q2.3 "Do you think the modelling activities constitute an* <u>*interesting teaching method*</u> *in education for the environment and sustainability?"*, and *Q2.4* "*Do you consider the modelling activities an* <u>*interesting approach*</u> *as an alternative way for learning in education for* *sustainability?"*, the teachers answered the questions similarly Very (4/8) and Enough (4/8). The responses show the participants' satisfaction and the possible intention to implement computational modelling in educational practice. This view is supported by the teachers’ answers in the next question: *Q2.5 "Do you think the modelling activities can be used to improve the understanding and other complex concepts in education?"*, in which all respondents answered positively.

In the question *Q2.6 "Express yourself freely for the Stella software"*, the teachers’ answers reveal that they appear to appreciate the possibilities of the specific modelling software in the understanding of the studied concepts and recognize a potential contribution in solving environmental and other problems. Educators expressed the need for more familiarization time with the Stella software.

4.3 Conceptual change detection

In the question: *Q2.7 "Did these activities influenced your thinking on how some changes in a complex environmental system can have long term effects on the system viability?"*. All the participants answered positively meaning that they perceived a change in their perceptions of complex systems dynamics, at least in terms of predictability of their long-term behavior. This is corroborated by the answers given in the next question: *Q2.8 "How do you think modelling activities can help to avoid making the wrong policy making decisions on real environmental issues?"*. The answers reveal that the teachers focused on the potential of long-term prediction, that is difficult for anyone to obtain with regular thinking (e.g. T1: *“They are very useful as they provide situations in the future that the human mind is impossible to do”*, T8: *“they can help by placing suitable variables in the model to increase our predictions on the behavior of a system in long-term")*. Teachers also mention the ability to evaluate alternative scenarios to assess and compare their effects (e.g. T4: *“before a policy decision is taken, the modelling can help those responsible to see expected results, advantages and disadvantages of their policies")*.

5. Discussion

An overview of teachers' participation in the research activities is that it was an interesting and unusual experience to them, through which they were puzzled about various environmental issues and developed their practices of critical and creative thinking. However, the allocated time is not considered enough and that more time is needed in order to understand and to consolidate the modelling activities. Most teachers focused on pleasant reflection and better understanding of the studied subject. This reinforces the view that the proposed instructive intervention can activate the conceptual understanding processes in the field of complex systems in a way that is pleasant and comprehensible to the educators. Consideirng these results with respect the research results the following answers are posited.

RQ1. Can dynamic system modelling environments be used as tools for meaningful learning (conceptual understanding) of complex systems concepts?

The implementation analysis of learning intervention shows that educators resolved with success the problems assigned to them. Moreover, the teachers believe that they actually benefited in understanding of complex problems (Q2.1, §5.2). Their statements regarding the intuitive addressing of complex systems is that we risk to make wrong prediction on how a system will behave in the future and the recognition for the need of computational dynamic systems modelling as alternative management policies (Table 6), shows that the intention were effectively enough.

RQ2. Can the concept of sustainability be approached through the study of a dynamical system in an ecological problem?

The utilization of ecosystems as examples for reasoning about complex systems was proved apt choice in the specific case. The ecological problems are familiar, interesting and authentic to the teachers. Ecosystems provide a conceptual framework for the introduction of abstract mathematical concepts of the general systems theory. The historical data and the available documentation on the ecosystems facilitate the creation of learning activities. The interdisciplinary nature of environmental problems makes them interesting means for the introduction to mathematical concepts.

However, an issue that remains open is whether that experience can be transferred to other cases and how to make the transition from the conceptual introduction to the formal mathematical description and study of complex systems.

RQ3. How effective are the processes of modelling as teaching and learning methods?

The results of the implementation of the intervention and the participants' opinions advocate that, in this research, computational modelling as a method of inquiry learning was effective and satisfactory. The teachers that participated, recognized clearly the effective and satisfactory levels of the method (Q2.3, Q2.4, Q2.5 §5.2) expressing their wish to implement this method.

RQ4. Do the proposed learning activities contribute to the improvement of conceptual understanding of the participant teachers?

The researchers, taking into account the findings, believe that the proposed instruction intervention and the use of modern learning approaches (authentic learning, learning through simulation, games based learning, problem based learning and learning by computational modelling) introduces successfully the study of complex systems dynamics and improves their conceptions of sustainability. The initial teachers' conceptions for sustainability were influenced only by the official general educational goal of environmental education. Through the teaching intervention the teachers enriched their cognitive schemata of sustainability with concepts of complex systems and the establishment of operating conditions without the exhaustion of essential resources. Moreover, the teachers realized that even simple systems that work with a few clear rules, can exhibit complex behavior. In addition, the participants learnt that the long term prediction of the complex systems behavior often exceeds the capabilities of the human mind. Consequently, the computational modeling and simulation of complex systems emerges as inevitable cognitive tool that serves the study of a large class of problems based on mathematical theory of systems. The success of the proposed teaching intervention, aiming to improve the understanding of complex systems of the participant teachers, is supported by their conceptual change, as it is revealed in their answers to the problems and the questions (compare Table 3, 4, and Q1.4 §5.1 to Table 5, 6, and Q2.8 §5.3) but also in their own opinion (Q2.1 §5.2).

Considering and summarizing the theoretical and the empirical parts of the research, it is supported that complex systems dynamics modelling, and learning activities can support students to understand better the problems of the modern world and may change the way students interact with their environment, adopting a more proactive attitude towards their reality. As Senge et al. (2000) claims, when a student has worked repeatedly with models that exhibit such behavior and design of such models incorporating various real observations, then the student may observe the same behavior in other systems of real life, internalizing this idea as part of normal thinking. Despite the considerable research efforts (Karamanos et al., 2012; Gkiolmas et al., 2013; Colella, Klopfer & Resnick, 2001; Komis et al., 2004) and the availability of advanced educational modelling environments, the systems thinking approach and the study of complex systems have not yet been adequately utilized in education. An important factor in this direction is the preparation and training of the teachers. The present study designed and applied experimentally a teaching intervention to improve conceptual understanding of complex systems and dynamic systems computational modelling based on the concept of sustainability in ecosystems.

The proposed intervention used thinking investigation tools for the initial expression and diagnosis of misconceptions of the teachers, as well as tools for the reflection upon their eventual misconceptions after the completion of the learning activities of the intervention. The learning activities of the proposed teaching intervention utilizes modern learning approaches such as learning through simulation and modelling, collaborative learning, authentic problem solving and Digital Games Based Learning. Despite the small number of participants in this case study and the fact that the results can't be generalized safely, the qualitative analysis of the research findings shows that participant teachers can a) build dynamic system models to represent real phenomena, b) explore the dynamic aspects of the corresponding systems through the simulation of the models and c) use the simulation of the models to consider forecasts and assumptions about the behavior of the represented systems. In other words, teachers using the teaching intervention can develop fundamental systems thinking capabilities. The authors believe that the proposed teaching intervention could be used as an introductory experience for conceptual understanding of key complex systems concepts in the context of environmental education or in any other context before formal training to systems thinking.

Furthermore, the participant teachers improved significantly their understanding of the sustainability concept and other related complex systems features and properties such as their difficulty on long term predictions and their counterintuitive behavior. More specifically, the participants seemed to have moved from their original static conception of sustainability to a more abstract, mathematical and dynamic meaning. Moreover, the participant teachers developed positive attitude and motivation to explore models further, to participate in computational modelling learning activities and to learn more about subjects that could be studied with the support of STELLA like software. The proposed learning activities offer an effective method for the teachers to develop further their conceptual understanding of complex systems and Education for Sustainability key concepts, enabling them to seize modelling activities in teaching and learning procedures.

6. Conclusion

The authors are convinced that the present research contributes to the preparation of teachers to improve their understanding of complex systems using systems dynamics modeling software in an authentic problem solving environment. The trained teachers are expected to be more amenable to utilize complex systems concepts in teaching and learning. More research, in the direction of the teachers' training relevant to systems dynamics modeling for EfS, is currently under design. Research will take into consideration the information collected from this first exploratory case study in order to produce more thorough results and document best practices.

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Systemic approaches to the complexity in mathematics education research

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Abstract

In this paper, we propose a systemic approach to the complexity of teaching and learning mathematics within the school unit. We hypothesise a self-similarity between the learning-teaching phenomenology (conceptualised as an emerging linking links process) and the school unit (conceptualised as a complex learning organization), introducing a co-developed methodological-theoretical framework to reveal implicit links in this complexity. Two empirical studies are discussed, investigating links within and amongst the system of scientific disciplines (mathematics in comparison with the other disciplines) and the system of school unit (mathematics as a school course in the actual lived school reality, in the desired reality and the perceived as normative reality), as experienced by the educational protagonists. The proposed approach and systemic instrument –also considering the ethical dimension of the interdisciplinarity and the multiplicity of choices– help in identifying a communication space amongst the seemingly incongruent experience spaces, thus facilitating the didactical planification towards to a meaningful learning as linking links.

Keywords complexity, system, mathematics discipline, school mathematics, views, beliefs, teaching-learning practices

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Marco Polo describes a bridge, stone by stone.
"But which is the stone that supports the bridge?" Kublai Khan asks.
"The bridge is not supported by one stone or another," Marco answers, "but by the line of the arch that they form."
Kublai Khan remains silent, reflecting. Then he adds: "Why do you speak to me of the stones? It is only the arch that matters to me."
Polo answers: "Without stones there is no arch."
Calvino (1976, p. 82)

1. Learning as linking links

The notion of linking seems to be at the crux of various conceptualisations of learning, including linking: a behaviour to a stimulus; a cognitive process with a certain task; cognitive processes with each other; cognitive and affective processes and experiences; neural and embodied experiences to cognitive and affective processes; intrapersonal, intersubjective and social experiences. The quantity and the quality of these links, as well as their interconnections within a learning network or a teaching design, characterise the quality of learning. The awareness of such linking of links, in the frame of complexity theories, is related to the concept of intelligence (Le Moigne, 1995).

In mathematics education, researchers have noticed the importance of the qualitative and quantitative differentiation of learning experiences, contrasting, for example: relational understanding and instrumental understanding (Skemp, 1976), conceptual knowledge and procedural knowledge (Heibert & Lefevre, 1986), processes, objects and procepts (Gray & Tall, 1994), deep, surface or achieving approaches to learning (Biggs, 2001; Entwistle, McCune, & Walker, 2001; Marton & Säljö, 1976). It seems that learning is characterised by the links made amongst various elements including the learners, the setting, the subject taught, the teacher. The complexity of the learning phenomenon appears to be in a direct relationship with the interconnection scheme of the links that a conceptualisation considers. Nevertheless, this scheme is characterised by its varied fragility: in the boundaries between the inside and the outside components the scheme seems to be sensitive to the change of the initial conditions, whilst at the same time the scheme seems to be resilient to the change of the conditions within which its identifiable learning or teaching style has been constituted (cf Moutsios-Rentzos, 2009; Moutsios-Rentzos & Simpson, 2011).

In this paper, in order to support a systemic instructional design for mathematics education, we focus on the mathematics learning phenomenon conceptualised through a soft systems theory approach, considering the links amongst different systems and roles, namely: amongst the system of scientific disciplines, the system of school unit system, the broader social system within the school unit functions, as well as the various roles that the teaching-learning subjects, individually and collectively, adopt (including, teachers, students, principals, parents etc). For this purpose, we introduce a co-developed methodological-theoretical framework, in order to identify the complexity of the phenomenon. Applications and implications of this framework will be discussed.

2. Systems and roles: mathematics learning in the school unit

2.1 The emerging importance of the school unit

Our approach places the school unit at the crux of the mathematics learning phenomenon, considering the school unit as a dynamic learning organisation which influences the learning process transcending the borders amongst the school, the family and the broader community. This approach is in (a seemingly) stark contrast with the official descriptions of the school unit objectives, structures, rules and social representations, which focus on the teaching processes, on the evaluation of learning by 'ordering' students according to their responses to tests and on the departmentalization of the cognitive contents. These official descriptions are linked with the centralised educational systems, in which the central planification is more important than the local dynamics and their interactions.

In order to introduce our systemic approach according to which the importance of the school unit emerges, we need to mention related core ideas of the 'system' conceptualisation. A system identifies a whole the parts of which are linked in ways that the constituted whole qualitatively differs from the mere bricolage of its parts. Aristotle explicitly identifies such wholes that are different than their constituting parts, contrasting them with the heaps of parts (*Metaphysics*, 1045a8-10). Bertalanffy (1968, 1975) posited the General Systems Theory, theorising the basic characteristics of systems, including purpose, structure, behaviour, connectivity, elements, subsystems, functions, interactions, boundary, environment. A system is defined by its objective and its boundary (the noematic purpose that identifies the system as a whole) and it may consist of elements (its parts) or other systems (subsystems). The dynamic links (as activated by the system's objective) amongst the parts of the system crucially determine its properties, which also crucially non-deterministically emerge as a result of the aforementioned links and interactions. Systems vary in their openness (referring to their level of interaction with its environment and other systems), complexity (referring to the number of parts and their links) and dynamic (referring to the volume and speed of systems' input and output).

In accordance with these ideas, the school unit may be conceptualised as a learning system (or learning organisation), consisting of subsystems (such as the school class; Cobb & Jackson, 2008) and elements that interact and are interdependent with the purpose to produce a multileveled and multifaceted educational outcome. At the same time, the school unit does not exist in vacuum; it lies within broader interacting social systems, including the immediate (geographically and administrative) social environment and the broader educational system. In Figure 1, a diagrammatic descriptive (albeit necessarily partial), representation of those links is presented.

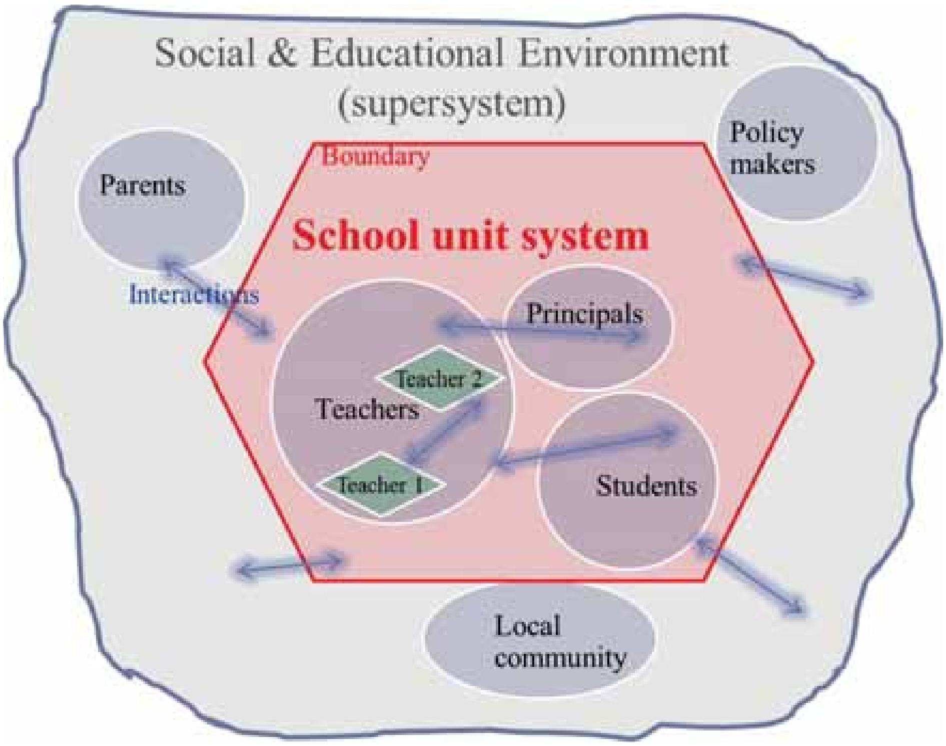


Figure 1. A descriptive approach to the complex school unit reality.

2.2 Mathematics and mathematics education in the school unit

Though not always explicitly mentioning the systemic 'language', mathematics education researchers have discussed the complex interaction between the school unit and the families, the curriculum, the socio-cultural environment, the beliefs and the stereotypes about mathematics that the protagonists (including teachers, students, parents) hold (Begg, 2003, 2005; Bouvier, Boisclair, Gagnon, Kazadi & Samson, 2010; Chen & Stroup, 1993; English, 2007, 2008; Kalavasis, Kafoussi, Skoumpourdi & Tatsis, 2010; Moutsios-Rentzos, Chaviaris & Kafoussi, 2015; Thornton, Shepperson & Canavero, 2007; Wittmann, 2001, 2005).

Following the conceptualisation of learning as linking and the aforementioned systemic systemic ideas, it is posited that learning signifies a systemic change, a disequilibrium of the till then status quo with regard to the relationships amongst the learning protagonists, the corpa (rather than corpus) of knowledge, as well as the interactions amongst and within the various interacting systems. The quality of learning is embodied in the new state of equilibrium that the system reaches. Crucial factor in the learning process and outcome is the individual and collective reflection upon each experience and/or action that allow the identification of the qualities of the occurring change. Through the attempts to communicate these reflections and to obtain a shared, intersubjective meaning, learning leaves the cognitive and affective ‘shadows’ and enters the individual and the ‘collective foreground’ (cf. the idea of a ‘collective learner’ in Davis, 2005).

In mathematics, such conceptualisations of learning may refer to the constant revisit of ideas, through new levels and/or new qualities of experience and generalisations (cf. the circle ‘experiencing’ – ‘questioning’ – ‘theorising’; Kalavasis & Moutsios-Rentzos, 2015), through re-analysing and re-synthesising old parts and wholes to new parts and wholes, and/or even to new old wholes and parts. The continuous communications (as conceptualised in Watzlawick, Beavin & Jackson, 1967) amongst the protagonists of the educational process (including, students, teachers, principals, school advisors, policy makers, parents, siblings, family, broader community etc) render the new inherently systemic mathematics learning apparent.

A snapshot of a self-similar approach to the structure of these communications, relationships and interactions is diagrammatically outlined in Figure 2 (adopted from Kalavasis, 2007, 2013). The endogenous relationships amongst the Protagonists (Prot) are affected by the relationships occurring in the School Unit (SchUn), with the pentagons defined by the diagonals identifying self-similarity and constructive interaction: each order and normality or disturbance and inadequacy of the inner pentagon is ‘reflected’ in the outer and vice versa.

By highlighting the importance of the school unit as a learning organisation rather than an educational structure, the unit develops epistemological and ethic exchanges with its members. This exchange, formal or not, exists and at the same time is at an antagonistic relationship with the ‘respective’ exchange of the other systems, including the family and the broader community. Thus, the system’s learning should appropriately address the linked (not necessarily aligned or even congruent) changes experienced by both the protagonists and the school unit. By conceptualising the protagonists and the school unit as complex adaptive, learning system (organisation), the communication space is at the crux of the learning phenomenon, thus rendering the interactions amongst the protagonists and the structures visible (Kalavasis, 2011).

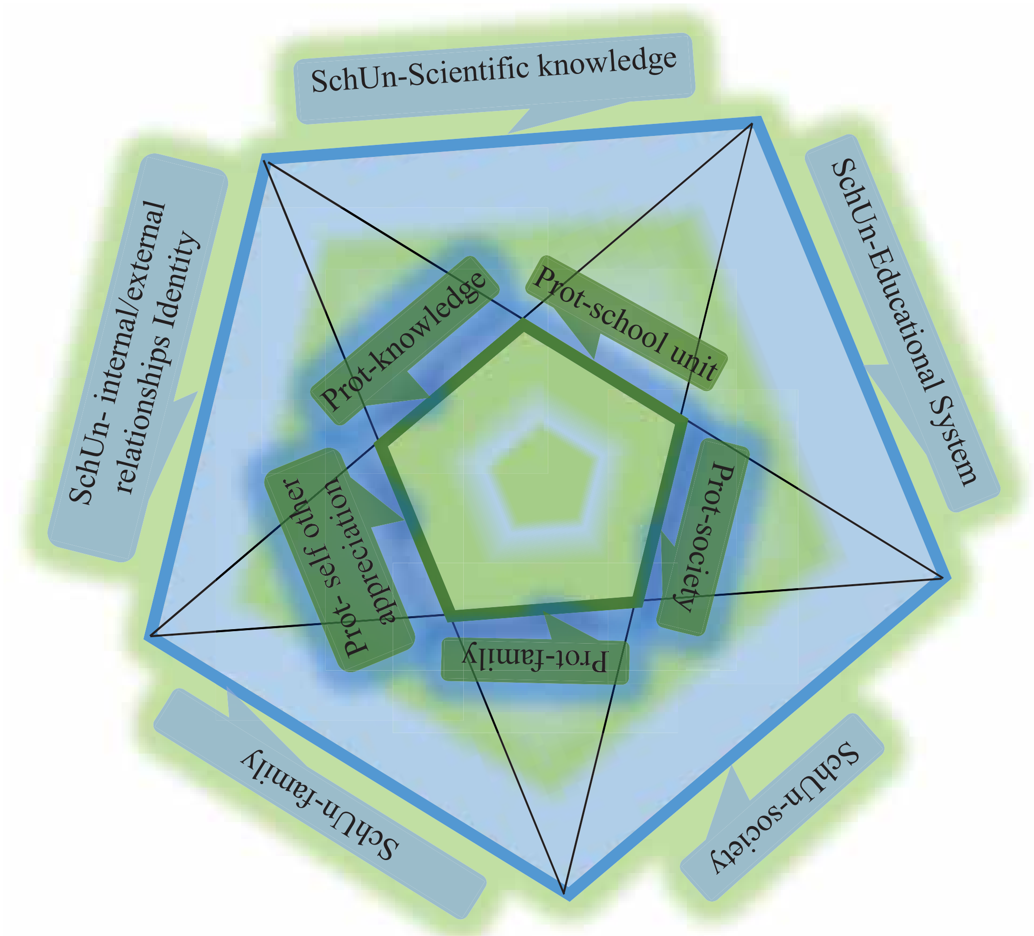


Figure 2. A self-similar approach to the Sch(ool) Un(it) – Prot(agonists) complexity.

2.3 A systemic framework for the teaching and learning of mathematics

We attempt to map aspects of the communication space by considering mathematics as element of interacting systems. In particular, in our approach we consider the *System of disciplines* and the *System of the school* *unit*. On the one hand, we investigate the relationships between the epistemic views that the protagonists (the teachers, principals, students, family etc) of the school unit hold about mathematics, whilst, on the other hand, we identify their pedagogical/educational views about mathematics as a school course. This investigation is multifaceted exploring within each of the two systems, as well as across their interactions.

It should be stressed that our interest concentrates in the individuals' perceived *views* and *experiences*, rather than in the observed experiences by an 'objective' observer. Our interest is in the realities experienced by the protagonists and the invisible realities that emerge through their interactions. Furthermore, we wish to differentiate our conceptualisation of view from the notion of beliefs in the sense that a 'view' implies a (potentially chosen) 'perspective' to something, rather than a more static stance about something usually linked with beliefs. Notwithstanding this conceptual difference, we agree with researchers (see, for example, the volume edited by Pepin & Roesken-Winter, 2014) who realised that the observed phenomena required the introduction of a more complex construct identified as belief systems (Green, 1971). For Green, beliefs form relatively isolated clusters that are internally structured through hierarchical relationships (primary or derivative; central or peripheral). Beswick (2006, 2012) builds on these ideas and on Davis' ideas about complexity (Davis, 2004, 2005; Davis & Sumara, 2006; Davis, Sumara & Luce-Kapler, 2008) to argue for a conceptualisation of belief system as a complex system with properties and characteristics that transcend the individual beliefs/agents that constitute the system.

Consequently, in our approach we conceptualise views akin to beliefs with respect to their formation and their clustering to form systems, but we consider views to be more of a matter of choice, thus more amenable to change and more dynamic. Importantly, each individual's view of the topic under investigation, implicitly or explicitly requires the individual to assume the aspect of the self that is required in order to be part of the system under investigation and to look into the topic from a certain perspective. By investigating different views, we essentially investigate different systems or subsystems, different and in differently ways related integrated wholes. Notwithstanding the theoretical differences, the proposed framework shares elements with the body of research looking into beliefs and belief systems that may affect educational practices.

In Figure 3, we diagrammatically present our intersystemic, multi-focussed approach.

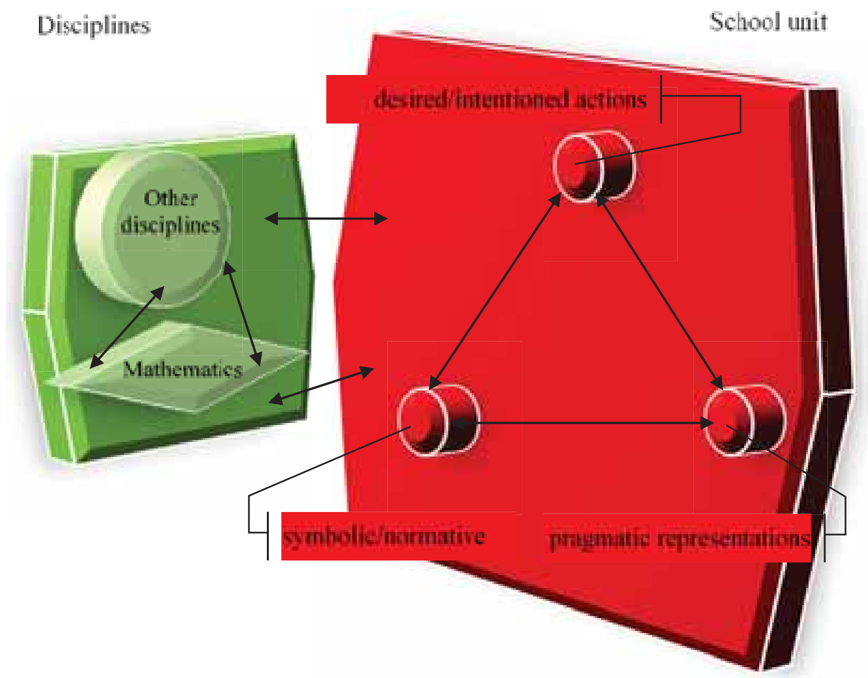


Figure 3. An approach to the complexity of school mathematics education.

First, we consider *mathematics within the system of disciplines*, essentially altering the focal point in order to identify epistemic views about mathematics in relation with the other disciplines: *from* the typical, neutral-point-of-view investigations about whether or not, for example, ‘mathematics is useful’ *to* an investigation about whether or not ‘mathematics is useful, in comparison with other disciplines’. Hence, the emphasis is on a relational/systemic view, rather than an ‘individual’ view/agent. Secondly, we consider the system of the school unit from three interacting and interrelated perspectives that each protagonist assumes: a) the *symbolic/normative* (the perceived official regulations), b) the *pragmatic representations* (the perceived current state of school practices), and c) the *desired/intentioned actions* (the personal hypothetical actions, assuming the power to implement them).

In this way, at first, we consider the protagonists' self as a tri-chotomised, yet integrated, whole that acts and interacts within a school unit. At the same time, the protagonists inherently and unavoidably bring within the school unit their own experiences and views of the broader social network, thus rendering the school unit open to such interactions. Moreover, the power equilibrium spans across this tri-focussed reality: a) *the official power structure* as described by the official regulations, b) the *actual power structure* as existing with each school unit, and c) the *desired power structure* that each protagonist dreams/hopes/aims for the school unit.

Though 'tri-chotomised self', 'power struggle and equilibrium', 'symbolic-pragmatic-desired' are terms that may resemble well-known theoretical frameworks and ideas of the 'French theory' (for example, Foucault, 1989; Lacan, 1982), it should be stressed that in our proposed approach these ideas are stemming from a systemic perspective and, hence, the ideas presented should be discussed within this framework (avoiding, for example, a Lacanian discussion about the 'symbolic' or Foucault's ideas about 'power'). Moreover, considering mathematics educational research, apparently similar ideas may be found; notably 'espoused beliefs', 'intentions of practice' and 'actual practice' as proposed by Liljedahl (2008), but our approach crucially differs in being systemic and 'top-down'. For example, though Beswick's (2006, 2012) discussion about belief systems and her research shares elements with our approach, we wish to note that we attempt to map a network of views for specific aspect of mathematics by employing an inter-systemic, multi-focussed approach, rather than first identifying views about mathematics considered in different systems and then link them (see §2.4-2.5).

2.4 A systemic research instrument

The aforementioned theoretical considerations have been co-developed with a compatible methodology. Our efforts stemmed from the following axes:

- The methods employed should be in line with the aforementioned inter-systemic, tri-focussed approach.
- Bearing in mind that the field is relatively under-researched, we preferred to a methodology that would appropriately and adequately map the existing school realities.
- The obtained data should be able to provide insights about the till then hidden convergences and divergences about these realities.
- The results should be relatively easily to be communicated to the protagonists, in order to maximise the effect of our approach.

We wish to stress that the latter may seem to be of minor importance, but it is at the crux of our approach, since we posit that it is exactly through the appropriate communication of the diverse co-existing realities that the communication space (which facilitates the systems' learning) emerges. The appropriate communication tool allows for the potential tenses within and amongst the protagonists to appear, thus allowing their efficient management. The purpose of our approach is to accept that we don't choose our differences, but we can choose to converse about them.

Consequently, a quantitative, questionnaire-based, approach is proposed:

a) Each system was investigated with different questionnaire sections
b) The disciplines system was investigated with items with appropriate wording in order to emphasise the relational and comparative nature of the questions asked. For example, "Do you think that mathematics more than other disciplines promotes the development of logical reasoning?".
c) The school system was investigated with a triplet of question for each topic of investigation, in accordance with the three foci of our approach. Hence, each topic was investigated with three items with the same ending phrase and three different beginning phrases. For example, for a topic investigating the school principal's allocation of the school budget three items were constructed with the end phrase "…a bigger part of the budget in materials for the teaching of mathematics in relation with other courses?" matched with three beginnings "According to your

opinion, should the official regulations allocate…", "Do you think that in reality in schools…", and "As a school principal and assuming you had the power, would you allocate …".

d) The analysis allows investigations between the two systems, as well as within the school system. In cases where more than one protagonists are included further inter-systemic and intra-systemic investigations may be conducted.

e) The results of the analysis are summarised in a hybrid symbolic-figural representation allowing both holistic and analytical interpretations (see Figure 4 and Figure 5).

2.5 Applications of the proposed approach

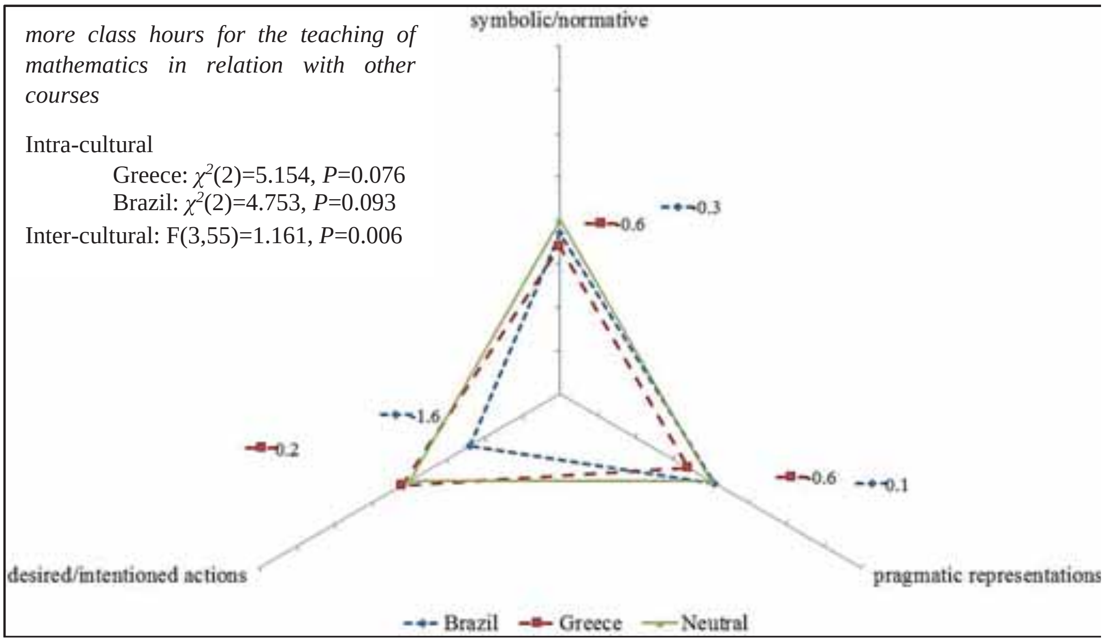
In order to gain deeper understanding in the proposed approach, two studies are briefly presented in the following section, with an emphasis on the school unit system.

2.5.1 One topic, two countries

The purpose of this study (discussed in detail in Moutsios-Rentzos, da Costa, Prado & Kalavasis, 2015) was to investigate whether or not sociocultural, economic and structural differences are evident in the professed views and practices of in-service principals. At the same time, we wished to investigate whether or not the proposed approach is useful in inter-cultural, comparative studies. Twenty-nine in-service school principals from Brazil and thirty from Greece (N=59 in total) participated in the study.

The results of the conducted comparisons suggested convergences and divergences in the epistemic views about mathematics held by the principals of the two countries, which were also evident in their views about mathematics as a school course, thus revealing intra- and inter- systemic interactions. For example, considering the system of all disciplines, the principals of both countries appear to consider that mathematics more than other disciplines promotes reasoning, that its epistemic value spans across the spectrum of disciplines, and that it has a real world value. Nevertheless, the Greek principals' 'mathematics-is-special' mix is skewed towards more absolutistic and utilitarian aspects (Ernest, 1991), whilst the Brazilian principals' 'mathematics-is-special' is characterised with a more fallibilistic aspect.

Considering the school unit systems and the intra-cultural inter-foci comparisons, in most cases it seems that the mathematics school course is considered to be 'special' in terms of the way that both the Greek and the Brazilian principals would intend to act assuming their having the power about: ways of assessment and of teaching, as well as about their own professional development. Focussing on the inter-cultural comparisons, divergences were found, amongst others, in the allocated class hours to mathematics and their professional development. In Figure 4 these two results are summarised with a symbolic-figural representation.



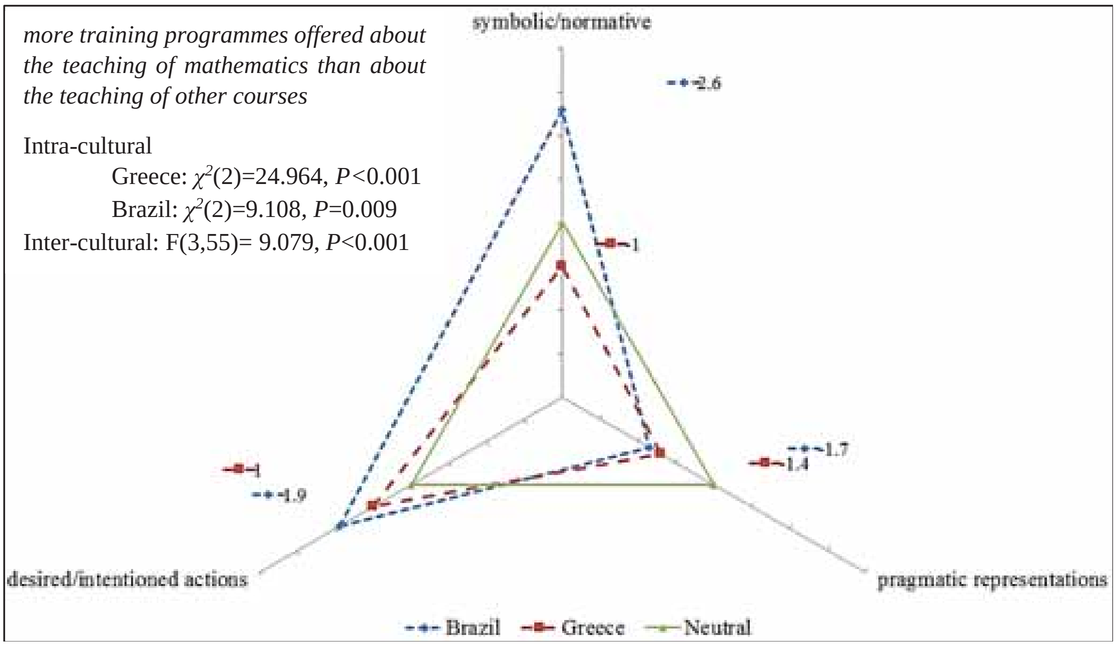


Figure 4. Mathematics as a course within the school unit system (both inter- and intra- cultural comparisons, values range '-4' to '+4'; Moutsios-Rentzos et al., 2015, p. 17).

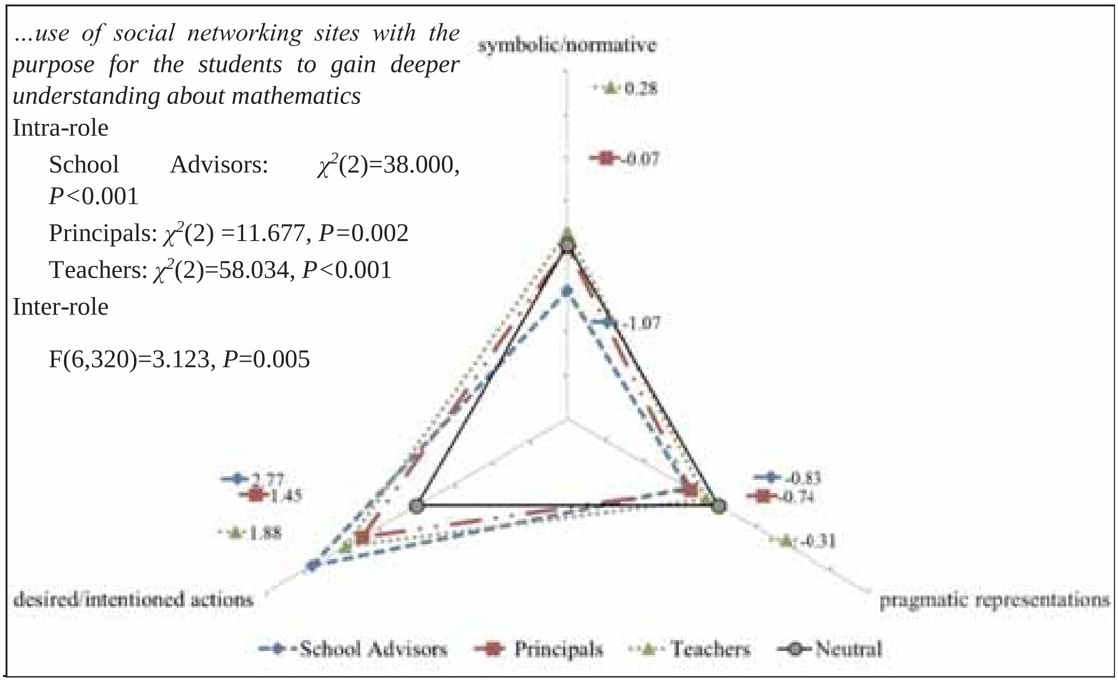
In Figure 4, the green triangle represents the *neutral border*: the ideal 'neutral' views on a subject for each of the three foci, since its vertices lie on the zero of each axis-focus. The views of each population (Brazil-Greece) are represented with a point in each of the three axes-foci, thus forming their triangular *experience space*. The comparisons between each experience space and the neutral border, as well as between the two experience spaces offer a wholistic, qualitative, yet structured view of the identified complexity. At the same time, we are offered a qualitative perspective of the communication space: the space within which interactions occur and upon which educational engineering may act. These qualitative aspects are coupled with quantitative statistical measures (tests) to investigate their statistical significance.

Overall, the proposed approach appears to be useful in an inter-cultural, comparative study, allowing our realising this complexity and meaningfully investigating the interactions amongst and within different systems, including the disciplines, the school unit, the two countries-systems.

2.5.2 One topic, three protagonists

In this study (discussed in detail in Moutsios-Rentzos, Kalavasis & Sofos, in press), we investigated the views that in-service primary school teachers and principals hold about the interrelationships of globalisation and internet social networks with the teaching of mathematics and with teaching in general. Importantly, we wished to explore whether or not our approach is useful in comparing the experience spaces of different protagonists. The sample included 108 in-service primary school teachers, 31 principals and vice-principals and 30 school advisors (N=169 in total).

Considering the system of disciplines, mathematics is considered by all three protagonists to hold a special place in comparison with other disciplines: in everyday life, in reasoning and in requiring systematic teaching. With respect to the school system, the participants of the study appeared to be willing to incorporate social networking sites in the school teaching, though they think that these sites are not actually used and especially that the formal regulation are against their use (see Figure 5).



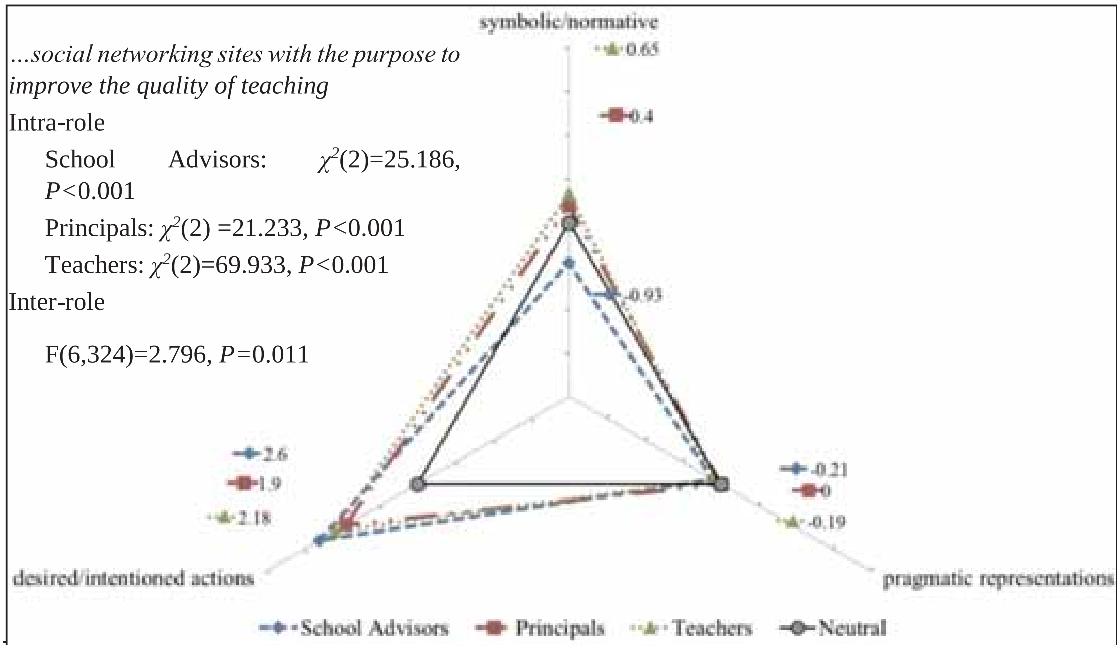


Figure 5. Mathematics within the school unit system (both inter- and intra-role comparisons, values range '-4' to '+4'; Moutsios-Rentzos et al., in press).

In conclusion, our approach seemed to help in more validly identifying the views and practices of the Greek teachers, principals and school advisors. Such information is crucial for all the protagonists for class or school level decisions, but we posit that it is especially useful for policy makers who wish to identify the communication space that essentially shows the systems' potential receptiveness to change, as well as the potential direction of that change.

3. Concluding remarks: systems, roles and complexity in mathematics education

The observer as projected in the act of observation becomes unavoidably a part of the observed phenomenon, thus participating in its evolution. In our approach, the phenomenology of the teaching and learning mathematics is interconnected with the development of the school unit as a learning organisation. We conceptualise the learning of mathematics, as an emergent continuously re-negotiated equilibrium, stemming from continuous reconstructions of the cognitive, sentimental and social links between each protagonist and mathematics.

This multi-construction is influenced by respective links and relational constructions (bridges) concerning mathematics between the school unit, the family and the broader community. Inversely, the internal links and the external relationships of the school unit are influenced by the dynamics of the relationships (within the spectrum ranging from antagonism to cooperation) between the educational protagonists’ roles, as well as by each protagonist’s emerging internal reflective equilibrium about mathematics.

In this frame of complexity, the mathematics is involved both as a discipline and as a school course. The valorisation of the school course may be related with the beliefs about the importance of the discipline and may be observed in, amongst others, the number of the allocated teaching hours in the curriculum, in the placement in the school horary, in the importance and in the sense of responsibility given by the protagonists (including students, teachers, principals, advisors, policy makers, parents and others). The research instrument that we propose may help to make visible significant aspects of the mental procedures constituting the equilibrium of the complex construction. By being visible, it is possible for the construction to be the object of our conscious, intentional thinking (cf. Husserl’s conceptualisation of intentionality; for example, Zahavi, 2003). Hence, it is possible for the educational designers to manage a multiplicity of options and actions, in alignment with the ethical imperative, as stated by von Foerster (1988) to act in a manner that increases the number of possible choices. It makes possible to re-think about and reflect upon the epistemological obstacles, the social representations, the alternative learning networks and/or the fragmented constructions, as well as to re-organise teaching practices, in order to construct the much needed teaching bridges linking the seemingly incongruent learning paths (Moutsios-Rentzos, Kalavasis & Sofos, in press).

The inter-systemic and intra-systemic investigation spans across the individual, the community and the structure, with the purpose to crucially identify a communication space amongst the diverse experience spaces, the lived realities. Within these realities, both visible and invisible interconnections emerge amongst disciplines acknowledging their dual, co-existing yet discrete, construction as a scientific domain and as a school course. In this context, the notion of interdisciplinarity helps in approaching the complex educational systems, in particular as a type of ethics of the didactical complexity. Interdisciplinarity involves the concepts and methods of each discipline within a dialectic and transforming interconnection, enriching each discipline (Piaget, 1974) through a cooperating experience crucially inclusive to the diversity of the communicating disciplines. Hence, interdisciplinarity traces the direction of the research of the complex phenomenon, by the revalorisation of each disciplinary in the emerging frame of interdisciplinarity.

Learning mathematics is an inherent interdisciplinary phenomenon that emerges through the collective mind's ability to continuously reflect upon experience, with the purpose for the experience to disappear *with* a trace that is the initial sketch of the 'mathematical idea'. Intentionalities and necessities seem to be the force that drives our species to actions and endeavours that go beyond the externally set biological survival, to the internally and socio-culturally developed internal consistency and communication (Skemp, 1979; Moutsios-Rentzos, 2009; Moutsios-Rentzos & Simpson, 2011). We posit that the empirical investigations that drew upon our proposed approach suggested both its theoretical and practical usefulness for the involved protagonists, notably the learners, the teachers and the policy makers. Complexity may be impossible to be modelled in the traditional sense, but we hope that the proposed approach is in line with the effort to reconcile 'ingenio', designo' and ethics (Le Moigne, 2013).

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Second part

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Values about mathematics learning: focusing on Greek high school students

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Abstract

Value in itself as a culturally-specific notion has a vital role to play in classroom mathematics learning. In this paper, we report on a study that explored the structure of the Greek students' values through the use of a values questionnaire in order to have a better understanding of what the students find important in mathematics learning. The nature of the various mathematics and mathematics education value dimensions were validated and fine-tuned using an exploratory analysis. The data analysis revealed nine key factors valued by Greek students in their mathematics learning. Importantly, the analyses revealed inter-cultural aspects of mathematics values and specific to the Greek students' value aspects. The effect of grade level and gender was also considered to further our understanding of the students' values structure.

Keywords: mathematics values, values classification, value structure

1. Introduction

Following a socio-cultural perspective, according to which learning is a collaborative social endeavour, values are considered as the window through which a person views the world, “situated as being characteristic of particular sociocultural contexts, drawing their form and meaning from the discourses, practices and norms of participants and of the interactions amongst themselves” (Seah & Wong, 2012, p. 36). In the learning process, values determine the students’ ways of utilising their cognitive abilities and their affective dispositions for learning, contributing to the students’ practices; their decisions, actions and evaluations (Andersson & Seah, 2012).

It is sensible to argue that the successful in-class communication of values depends of the teachers’ valid knowledge regarding which values should be fostered for their students. The teachers’ professional classroom experience will provide then with this information. Nevertheless, to the extent that the identification of these values is accurately and timely realised (for example, in the beginning of the academic year), the potential value-related conflicts may jeopardise the teachers’ design with respect to both its cognitive and affective learning aspects. Thus, it is crucial to accurately and validly identify and explore the students’ values.

The nature of values may be the reason why the various educators choose to methodologically approach values research through a time-consuming, qualitative perspective, usually including observations and interviews (Law, Wong, & Lee, 2012; Yauch & Steudel, 2003).

In this study, we analyse and discuss aspects of the findings of an international research project entitled ‘What I Find Important (in mathematics education)’ (WIFI) focused on Greek high-school students. WIFI is an international project that was conceptualised in 2010. The participating countries are the following: China, Hong Kong, Taiwan, Greece, Australia, Sweden, South Africa, Japan, Malaysia, Thailand and Turkey. The research instrument was a values questionnaire that the researchers of the WIFI project constructed with the purpose of utilising it with different students in different cultures. It is argued that such an instrument allows the effective and valid identification of the students' values about learning mathematics. Furthermore, by establishing a satisfactory cross-cultural reliability and validity of the questionnaire, comparative studies may be conducted amongst different parts of the world in order to gain deeper understanding about the cultural effect on the students' values about learning mathematics.

2. Theoretical framework

2.1. Values and beliefs

Beliefs and values are constructs which are closely related. Krathwohl, Bloom and Masia (1964) taxonomy positions values as having developed from beliefs. Clarkson, Bishop, FitzSimons and Seah (2000) expressed the relationship in terms of the volitional aspect of values:

> 'values are beliefs in action'. That is, the values that teachers are teaching in the mathematics classroom are not only beliefs the teacher holds, but their behaviour in the classroom actually point to these beliefs. (p. 191)

However, one may ask if values are necessarily expressed as actions. Might it be that in some cultures, what is valued may not be expressed as an action since there are more important values which are prioritized?

Yet, another perspective emphasizes the difference in nature between beliefs and values, although each affects the development of the other within an individual. According to Seah, Atweh, Clarkson and Ellerton (2008), beliefs relate to what is considered to be true (or false), whereas values relate to what is considered important (or unimportant). Thus, two teachers may value, say, information and communication technology *ICT*, but the beliefs that they have can be very different. One teacher may value the use of four-function calculators in the early years so as to free up more time for students to think. The other teacher who also values *ICT*, however, may feel very strongly against this belief, and instead subscribes to another belief that the adoption of data-loggers facilitates the collection of authentic data. Certainly, some different values are represented by these two beliefs as well. While the teachers' common valuing of *ICT* might have helped develop the two different beliefs, it can be observed that the belief statements support the valuing of *authenticity* in the second teacher (for example).

2.2. Classifying mathematics values

Bishop (1996) theorised that the students' values about school mathematics could be classified in three categories: a) *mathematical values*, b) *mathematics educational values*, and c) *general educational values.*

The 'mathematical values' concern the degree that mathematics are values in the Western culture. In earlier expositions of his views, Bishop (1988) theorised the existence of three complementary pairs of mathematics: *rationalism* and *objectism*; *control* and *progress*; *mystery* and *openness*. Following these, it was revealed that different students may, for example, value mystery and openness in different degrees.

The 'mathematics educational values' express the degree that the various aspects of classroom mathematics teaching practices are valued. For example, different teachers may vary in their valuing of different teaching techniques considering the importance that they ascribe to each technique (and the values they hold) with respect to the teaching of mathematics. These values have been investigated through several studies coordinated by the Third Wave Project (Seah & Wong, 2012), a consortium of international research groups comprising eleven countries/regions, including such as Australia, Hong Kong, Singapore, Malaysia and Sweden. These studies have been interested in understanding how values and valuing shape mathematics pedagogy (Law, Wong & Lee, 2012; Seah, 2011). The underlying values of such 'moments of effective learning' were examined with the students through interviews. These interviews had led to the identification of the following mathematics educational value continua: *Ability – Effort*; *Wellbeing – Hardship*; *Process – Product*; *Application – Computation*; *Facts – Ideas*; *Exposition – Exploration*; *Recalling – Creating*; *ICT – Paper-and-pencil*.

Bishop's (1996) third category of values, 'general educational values' concerns the values that characterise the students as they experience school education. They may include cultural values (such as honesty, politeness and collaboration), while in some school settings and cultures they may also include religion-related values.

The aforementioned values categories are of educational character, in the sense that that they reflect what is considered to be important in the scientific, educational and social context of the school experience. The school class is settled within a sociocultural context: the teachers’’ practices and the students’ learning are affected by the evaluations and the assessments of the parents, the state and the broad social context. It is posited that the teachers; values are affected in a similar manner. The social interaction within the school class is part of the micro-context and is directly observable, whereas the interaction with the broader sociocultural context is part of the macro-context, thus demanding a wider investigation in order to be identified. And it is through this investigation of the dialectic between the two contexts where mathematics learning occurs that the advancement of the mathematics education is realised (de Abreu, 2000).

Seah (2005) suggested that a societal category could be added as a fourth category of values in the mathematics classroom, to allow us to fully account for the principles and convictions that are valued and co-valued amongst the players within the classroom. In this respect, it has been useful to refer to Geert Hofstede’s proposal that each culture (which he defined generally to include classroom cultures as well) can be uniquely defined in a five-dimensional space (Hofstede, 1997). There are five cultural dimensions, namely: *power distance*, *collectivism*/*individualism*, *femininity*/*masculinity*, *uncertainty avoidance*, and *life orientation*. Take *power distance*, for example, in which a country’s index score shows the extent to which subordinates and the less powerful members of the community expect and accept that power is shared unequally. Thus, the level of power distance associated with any country is characteristic of that country, and reflects the cultural tradition of the region in which the country is located. In this sense, then, it is a societal value.

2.3. Values, grade level and gender

Educational research in general and mathematics education in particular are used to adopting the factors of age and gender as two important demographic variables (Jabor, Machtmes, Kungu, Buntat & Nordin, 2011) with which their effects on some educational outcomes (such as self-perception and achievement in subjects such as Mathematics) are investigated.

From the developmental perspective, age and/or *grade level* is an important issue of concern in our investigation into the value structure as exhibited by Hong Kong students. On the one hand, as the students get more matured, their beliefs and values about mathematics became stabilised. On the negative side, when the students accumulated more and more unpleasant experience with mathematics, the belief that 'math is not for me' becomes crystallised (McLeod, 1992). The more they begin to think along those lines, the less effort they are willing to invest, and the poor result that is subsequently obtained would reinforce the above beliefs, creating a vicious circle. On the other hand, when one moves up the grade levels, mathematics presented in the school curriculum moves gradually from mathematics encountered in real-life experiences to mathematics as a discipline.

Mathematics as a discipline is more symbolic, abstract and formal. The students would naturally begin to decide whether they will select the subject of mathematics (or how much mathematics) in their future studies (for example, in college and university).

All these might have an impact on the values they attach to mathematics. This was confirmed in a large scale study of some 10,000 students in Hong Kong, which found that the students' interests about mathematics dropped significantly from Grade 3 to Grade 6. A marked proportion of students become aware of learning difficulties in Grade 9, while the number of students thinking of giving up Mathematics notably increased in Grade 10 (Wong, Wong, Lam & Zhang, 2009).

G*ender* is another issue of concern in mathematics education research which can be traced back to the 1990s or even earlier (Fennema & Leder, 1990). Findings from TIMSS and PISA have also stimulated much more recent research in this area (Else-Quest, Hyde & Linn, 2010). Although there are broad similarities between boys and girls in mathematics achievement, the intricate link between the valuing of mathematics achievement and its effect on the formation of positive mathematics attitudes as a values component deserves further investigations. Such a comment is indeed consistent with the literature (Leder, 1992; Leder, Forgasz & Solar, 1996) that urges us to pay particular attention to the affective constructs and values, including attitudes, beliefs, confidence, attribution of mathematical success and how the intersection of these notions demonstrate a complex interaction among themselves.

Attempts have been made in explicating how these factors would have influences on the learning of mathematics with regards to differences in gender. These include the studies using biological, school, teacher, and parent as variables, as well as those with the focus being drawn on the effect of affective factors on problem-solving heuristics and the influence of teachers (Gunderson, Ramirez, Levine & Beilock, 2012; Leder, 1992). While the findings of these studies revealed the trend that there seems to be a narrowing in mathematics achievement with gender differences, it remains unclear what exactly the students value in terms of their mathematical engagement in the activities.

2.4. Research question

Following these, in this study we address the question: *What are the values regarding mathematics and mathematics learning that characterise the Greek students?* Drawing upon the aforementioned discussion, we considered the students' *gender* and their *grade level* as factors that may affect the students' values about mathematics learning.

3. Methodology

This is a cross-sectional, quantitative study, part of an international collaboration. A questionnaire consisting of 64 5-point Likert type item was employed for the purposes of the study (see Appendix). The students indicate the degree of importance they ascribe to each item, ranging from

'1' ('absolutely important') to '5' ('absolutely insignificant'). The questionnaire is the result of discussions involving all the countries/regions participating in the project, in order to strengthen the cross-cultural reliability and validity of the instrument and to respect the cultural diversity. 725 high-school students (13-15 years old) participated in the study (397 boys and 328 girls), studying in the second grade (Grade B) and third grade (Grade C) of the Greek Gymnasio in schools located in the Attica region of Greece (which includes the capital Athens). The quantitative data analyses were conducted with SPSS 22, including: Principal Component Analysis (PCA), Cronbach's alphas and Analyses of variance.

4. Results

4.1. Validity and reliability of the Greek version of the questionnaire

The conducted PCA with Varimax rotation resulted in a 9-component solution with eigenvalues greater than 1 (in line with the Scree plot), accounting for 57.20% of the variation (first component 12.32%). After examining the 51 items (with loadings over 0.40) retained for each component, the following titles were assigned (see Table 1): *C1 Problem solving with mathematical understanding; C2 Feedback and interaction; C3 ICT in mathematics; C4 Communication (exploration – output); C5 Routine problem solving; C6 Mathematics and mathematicians' practices; C7 Practice and evaluation; C8 Real-life mathematics; C9 Communication (collaboration – input).*

Component Item	Loading
C1 Problem solving with mathematical understanding	
Q64 Remembering the work we have done	0.683
Q58 Knowing which formula to use	0.654
Q56 Knowing the steps of the solution	0.636
Q63 Understanding why my solution is incorrect or correct	0.620
Q59 Knowing the theoretical aspects of mathematics	0.603
Q54 Understanding concepts / processes	0.584
Q33 Writing the solutions step-by-step	0.522

Component Item	Loading
Q32 Using mathematical words	0.515
Q52 Hands-on activities	0.502
Q55 Shortcuts to solving a problem	0.494
Q51 Learning through mistakes	0.472
Q38 Given a formula to use	0.468
Q53 Teacher use of keywords	0.434
Q28 Knowing the times tables	0.425
C2 Feedback and interaction	
Q41 Teacher helping me individually	0.626
Q47 Using diagrams to understand maths	0.551
Q49 Examples to help me understand	0.548
Q48 Using concrete materials to understand mathematics	0.543
Q44 Feedback from my teacher	0.540
Q46 Me asking questions	0.512
Q35 Teacher asking us questions	0.452
Q45 Feedback from my friends	0.445
C3 ICT in mathematics	
Q23 Learning maths with the computer	0.799
Q24 Learning maths with the internet	0.749
Q22 Using the calculator to check the answer	0.741
Q4 Using the calculator to calculate	0.712
Q27 Being lucky at getting the correct answer	0.452
C4 Communication (exploration – output)	
Q21 Students posing maths problems	0.583
Q30 Alternative solutions	0.560
Q19 Explaining my solutions to the class	0.493
Q29 Making up my own maths questions	0.421
Q40 Explaining where rules / formulae came from	0.400
C5 Routine problem solving	
Q15 Looking for different ways to find the answer	0.560
Q8 Learning the proofs	0.501

Component Item	Loading
Q14 Memorising facts	0.463
Q13 Practising how to use maths formulae	0.447
Q2 Problem-solving	0.414
C6 Mathematics and mathematicians' practices	
Q61 Stories about mathematicians	0.765
Q60 Mystery of maths	0.621
Q17 Stories about mathematics	0.610
Q18 Stories about recent developments in mathematics	0.559
C7 Practice and evaluation	
Q37 Doing a lot of mathematics work	0.788
Q57 Mathematics homework	0.712
Q62 Completing mathematics work	0.523
Q36 Practising with lots of questions	0.427
C8 Real-life mathematics	
Q12 Connecting maths to real life	0.680
Q11 Appreciating the beauty of maths	0.520
Q10 Relating mathematics to other subjects in school	0.514
C9 Communication (collaboration – input)	
Q7 Whole-class discussions	0.592
Q3 Small-group discussions	0.587
Q5 Explaining by the teacher	0.420

Table 1: PCA results and item loadings.

The identified value structure is in line with studies conducted in other participating countries/regions (including China, Hong-Kong and Taiwan). Importantly, through the number of the identified components differs (nine instead of six), it appears that there was a conceptual correspondence amongst the different studies. For example, considering the results presented by Seah, Zhang, Barkatsas, Law and Leu (2014), most of the items appear to either load in conceptually related components (for example, their 'ICT' and our 'ICT in mathematics') or to load in conceptual sub-components (for example, their ‘achievement’ is broken down to our ‘Problem solving with mathematical understanding’ and ‘Routine problem solving’).

The reliability (internal consistency) of each of the nine factors was investigated through the computation of Cronbach’s alphas (see Table 2). Most of the components showed acceptable internal consistency (>0.60), except for ‘Real-life mathematics’ and ‘Communication (collaboration – input)’, which nevertheless were found to be less reliable in the respective study in Hong-Kong (unpublished data analysis results in a study conducted by the second author).

Component	Greece	Hong-Kong
C1 Problem solving with mathematical understanding	0.859	0.91
C2 Feedback and interaction	0.769	0.85
C3 ICT in mathematics	0.756	0.86
C4 Communication (exploration – output)	0.646	0.79
C5 Routine problem solving	0.597	0.79
C6 Mathematics and mathematicians’ practices	0.719	0.79
C7 Practice and evaluation	0.761	0.82
C8 Real-life mathematics	0.560	0.72
C9 Communication (collaboration – input)	0.446	0.70

Table 2: Internal consistency of the questionnaire (including Hong-Kong comparisons).

4.2. The grade level and the gender effect

Considering the fact that we would investigate the effect of gender and grade level, further reliability analyses were conducted (see Table 3). It was revealed that the lower reliability of the internal consistency of ‘Real-life mathematics’ and ‘Communication (collaboration – input)’ was mainly due to the younger students of Grade B, while the younger girls appeared to be the cause of the marginally acceptable reliability of ‘Routine problem solving’.

Component	Girls		Boys	
	Grade B	Grade C	Grade B	Grade C
C1 Problem solving with mathematical understanding	0.787	0.873	0.876	0.901
C2 Feedback and interaction	0.674	0.816	0.770	0.818
C3 ICT in mathematics	0.765	0.783	0.646	0.830
C4 Communication (exploration – output)	0.637	0.722	0.644	0.583
C5 Routine problem solving	0.422	0.642	0.634	0.690
C6 Mathematics and mathematicians' practices	0.638	0.763	0.713	0.763
C7 Practice and evaluation	0.749	0.749	0.753	0.792
C8 Real-life mathematics	0.502	0.521	0.598	0.620
C9 Communication (collaboration – input)	0.352	0.550	0.351	0.532

Table 3: Internal consistency of the questionnaire (the gender and grade level effect).

Subsequently, the students' mean responses for each component were computed to conduct gender and grade level comparisons. In Table 4, we outline the mean scores (with standard deviations) and the results of the two-way Analyses of Variance (considering gender and grade level). Notice, that the mean scores range from '1' to '5' and that the *lower the score, the higher the students' agreement with a component*.

	Girls				Boys				Analysis of Variance: effects		
	Grade B		Grade C		Grade B		Grade C				
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	Gender	Grade level	Gender x Grade Level
C1	1.88	0.47	2.03	0.61	1.82	0.65	1.85	0.63	**	*	
C2	2.25	0.51	2.42	0.65	2.16*	0.67	2.24	0.62	**	**	
C3	2.85	0.92	3.14	0.86	3.13	0.88	3.04	0.84			**
C4	2.42	0.66	2.58	0.69	2.41	0.67	2.51	0.63		**	

C5	2.24	0.57	2.29	0.61	2.10	0.66	2.10	0.65	***
C6	3.08	0.88	3.16	0.86	3.01	0.89	3.08	0.87	
C7	2.28	0.90	2.16	0.85	2.14	0.87	2.13	0.89	
C8	2.69	0.92	2.76	0.81	2.63	0.84	2.69	0.88	
C9	2.26	0.72	2.27	0.72	2.02	0.57	2.05	0.67	***

*p<0.05 **<0.01 *** p<0.001

Table 4: Mean responses for each component (the gender and grade level effect).

First, it is noted that the students seem to adopt the vast majority of the values expressed in the questionnaire (as indicated by the lower than 3 identified mean scores). Moreover, it appears that the students agree more with the values that are linked with C1 (‘Problem solving with mathematical understanding’) and less with the values expressed by C3 (‘ICT in mathematics’) and C6 (‘Mathematics and mathematicians’ practices’). Furthermore, the order of the mean scores for each component remains roughly the same in each combined category of grade level and gender.

The Analyses of Variance revealed statistically significant gender differences with respect to C1 (‘Problem solving with mathematical understanding’), C2 (‘Feedback and interaction’), C5 (‘Routine problem solving’) and C9 (‘Communication (collaboration – input)’), with the boys appearing to be agree more than the girls with the expressed values. Moreover, considering the grade level effect, the Grade B students appeared to agree with the values expressed in components C1 (‘Problem solving with mathematical understanding’), C2 (‘Feedback and interaction’), and C4 (‘Communication (exploration – output)’) than the Grade C students. Finally, the effect of the interaction of gender and grade level was statistically significant only in C3 (‘ICT in mathematics’) with the girls’ agreement as they progress from Grade B to Grade C appearing to diminish, while the boys’ agreement increases.

5. Discussion and concluding remarks

The conducted analyses revealed that component structure of the questionnaire corresponds with the structure found in studies conducted in other countries/regions collaborating in the project. The internal consistency of seven out of the identified factors was acceptable, with the two components with lower reliability (C8 and C9) to also show comparatively lower reliability in the other studies. Moreover, the descriptive statistics (means and standard deviations) support that the Greek students in general agree with the values expressed by the questionnaire components.

Bearing in mind that this was not a comparative study per se, the fact that the identified value components correspond well with the components found in the studies conducted in other countries/regions participating in the project may also be interpreted as indication of the existence of a core of mathematics values that pertain different cultures. At the same time, the fact that differences amongst the various studies are evident suggests and, importantly, identifies the cultural effect on the students' value structure. For example, the 'achievement' component identified in the studies in mainland China, Taiwan and Hong Kong (Seah et al., 2014) was divided in two components, which embodies the fact that for the Greek students the different qualitative characteristics of 'achievement' that are incorporated in each component are valued differently. This is further supported by the gender and grade level contrasts, which differ in these two conceptually linked (yet valued differently) components. Nevertheless, specially designed comparative studies should be conducted to investigate the veracity of these claims, in order to identify the specific and the general (if any) of the mathematics values conceptualisations.

Notwithstanding the aforementioned concerns, these results may be linked with the current curriculum and the mathematics teaching practices in Greece, which though claiming to adopt contemporary mathematics education findings, the everyday classroom teaching practices has not been clearly affected yet. The broader mathematics goals set within the school context are concentrated in fostering 'surface' mathematical abilities favouring procedures and rote learning (for the sake of national exam success), rather than 'deeper' metacognitive, social and affective abilities. The teaching techniques in most cases remain compatible with traditional teaching models, without utilising novel teaching approaches (Kasimatis & Gialamas, 2001).

In regards to gender and grade level differences, the analyses revealed that the boys and the younger students agree more in some components than respectively the girls (in C1, C2, C5 and C9) and the older students (in C1, C2 and C4). Considering the interaction of gender and grade level, it was revealed that agreement of boys and girls for C2 ('Feedback and interaction') follows opposite directions by revealing an increase in the boys' agreement and a decrease in the girls' agreement. These findings may be linked with special characteristics of each grade, since in Grade C the students first encounter important mathematical results in a more traditionally identified as 'mathematical' way, which, combined with a traditional teaching model, may cause the students; valuing less these three components (C1, C2 and C4) that express 'deeper' mathematical values (such as problem solving with mathematical understanding, feedback, interaction, exploration and communication). At the same time, these 'deeper' mathematical values appear to be adopted by boys more than the girls which may be linked with the broader socio-cultural *stereotypical* view of boys outperforming girls and/or having deeper mathematics understanding than the girls (Chronaki, 2009). These results may be also linked with developmental factors or with the students' social experiences, including the school unit, their family and the broader socio-cultural context (Kafousi & Chaviaris, 2013; Moutsios-Rentzos, Chaviaris & Kafousi, 2015; Moutsios-Rentzos, Kalavasis & Sofos, 2013). Nevertheless, these claims require further studies to be conducted (including qualitative and/or longitudinal designs).

Finally, though this study was conducted within a specific region in Greece, we maintain that the adopted quantitative methodology appeared to help in our gaining deeper understanding about the values that the Greek students hold about mathematics and mathematics learning. The gender and grade level differences revealed aspects of the development of the Greek students' values useful to teachers, researchers and curriculum designers as they highlight the nature of their effects on the students' values about mathematics.

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Appendix

For each of the items below, tick a box to tell us how **important** it is to you when you learn mathematics

	Absolutely important	Important	Neither important nor unimportant	unimportant	Absolutely unimportant
1. Investigations					
2. Problem-solving					
3. Small-group discussions					
4. Using the calculator to calculate					
5. Explaining by the teacher					
6. Working step-by-step					
7. Whole-class discussions					
8. Learning the proofs					
9. Mathematics debates					
10. Relating mathematics to other subjects in school					
11. Appreciating the beauty of maths					
12. Connecting maths to real life					
13. Practising how to use maths formulae					
14. Memorising facts (eg Area of a rectangle = length X breadth)					
15. Looking for different ways to find the answer					
16. Looking for different possible answers					
17. Stories about mathematics					
18. Stories about recent developments in mathematics					
19. Explaining my solutions to the class					

20. Mathematics puzzles					
21. Students posing maths problems					
22. Using the calculator to check the answer					
23. Learning maths with the computer					
24. Learning maths with the internet					
25. Mathematics games					
26. Relationships between maths concepts					
27. Being lucky at getting the correct answer					
28. Knowing the times tables					
29. Making up my own maths questions					
30. Alternative solutions					
31. Verifying theorems / hypotheses					
32. Using mathematical words (eg angle)					
33. Writing the solutions step-by-step					
34. Outdoor mathematics activities					
35. Teacher asking us questions					
36. Practising with lots of questions					
37. Doing a lot of mathematics work					
38. Given a formula to use					
39. Looking out for maths in real life					
40. Explaining where rules / formulae came from					
41. Teacher helping me individually					
42. Working out the maths by myself					
43. Mathematics tests / examinations					

44. Feedback from my teacher					
45. Feedback from my friends					
46. Me asking questions					
47. Using diagrams to understand maths					
48. Using concrete materials to understand mathematics					
49. Examples to help me understand					
50. Getting the right answer					
51. Learning through mistakes					
52. Hands-on activities					
53. Teacher use of keywords (eg ‘share’ to signal division; contrasting ‘solve’ and ‘simplify’)					
54. Understanding concepts / processes					
55. Shortcuts to solving a problem					
56. Knowing the steps of the solution					
57. Mathematics homework					
58. Knowing which formula to use					
59. Knowing the theoretical aspects of mathematics (eg proof, definitions of triangles)					
60. Mystery of maths (example: 111 111 111x111 111 111=12 345 678 987 654 321)					
61. Stories about mathematicians					
62. Completing mathematics work					
63. Understanding why my solution is incorrect or correct					
64. Remembering the work we have done					

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Inspiration for Elementary Mathematics Descriptions from a "Heritage"[1] Reading of *On the Nonexistent* by Gorgias

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Abstract

Lessons for elementary mathematics concepts may be developed from a heritage reading of an early Greek text, *On the Nonexistent*, by the fifth-century B.C.E. Greek Sophist philosopher and *rhetor* Gorgias. The history versus heritage source distinction made by Ivor Grattan-Guinness defines the novel approach taken. It is argued here that this work by Gorgias, not considered as a historical mathematics text by historians of mathematics, can be probed successfully for rational language and teaching ideas useful to elementary mathematics education. However, it must be understood that, as explained by Grattan-Guinness, such heritage use does not mean similar math lessons were taught by Gorgias. The historical origins of math concepts are not the concern of this paper. The goal is to encourage

[1] In the sense of Grattan-Guinness; see section 1.

elementary math teachers to read Gorgias's text for the potential to improve the descriptions they use in several challenging math lessons.

Therefore, descriptions for math concepts that were inspired by reading *On the Nonexistent* by Gorgias are shared by the author with colleagues in the hope that reading Gorgias could offer them similar intellectual invigoration. These descriptions are as follows: (1) Language terms used by Gorgias, such as "nonexistent," "beginning," and "continuum," can help to describe the rationale of the basic lesson in math that a continuum may have a beginning point which potentially extends to infinity. A teacher's reflection on how to explain the concept of a beginning point may also help students learn how to determine whether the set of natural numbers begins with 0 or 1; (2) The historical term "nonexistent," as used by Gorgias some 2,500 years ago, is discussed in terms of its contemporary meanings, which include null, nil or zero. In addition, the word "magnitude" as used by Gorgias is related to contemporary use of the word in the math concept orders of magnitude; (3) The idea of the nonexistent is considered for its potential to aid in describing the division of fractions. A statement by Gorgias, that anything that exists is not indivisible, inspires further thoughts about a fraction that is considered to be not allowed or to have no well-defined meaning, namely, why zero cannot be the denominator of a fraction; (4) Gorgias explains in *On the Nonexistent* that there are distinctions to be made about what is existent, in terms of container and contained, and the conceptualization of such measurement. He demonstrates that a body is three-dimensional because it has length, breadth and depth. [2] In math teaching in our era, instruction compares and contrasts the formulae for a right-angle rectangular prism or parallelepiped and for volume measured in cube units, which can be one and the same, namely Length x Width x Height. [3] Lessons incorporating the descriptive language of Gorgias, as well as the Classical heritage as the context for such language, could provide *mnemonic* associations that could aid in teaching essential formulae for geometry, volume, and the cube unit.

[2] As an expert reviewer notes, the term used by Gorgias in Greek means "triple," although the term is translated as "three-dimensional" in English versions of the text.

[3] It has been suggested by an expert reviewer to include "parallelepiped." I use "rectangular prism" or "right-angle rectangular prism" because I think their volume formula of Length x Width x Depth is used by Gorgias. While the parallelepiped formula encompasses this formula, it is given as Base x Height, which is broader in its definition and allows for polygons whose base area is not found only through Length x Width. But for universal clarity, I have included the term parallelepiped.

1. Introduction

The work of philosophy known as *On the Nonexistent*,[4] by the fifth-century Greek Sophist philosopher Gorgias (circa 485–380 B.C.E.), is recommended reading in the search for fresh ideas about language and proofs to use when teaching early mathematical concepts. This paper invites elementary school teachers who seek intellectual engagement with rigorous thought to imagine how this ancient text can be plumbed for inspiration.

Gorgias is believed to have moved from Leontini, Sicily, to Athens in 427 B.C.E., where he subsequently achieved fame as a teacher through his public speeches. He is considered a transitional figure in the history of fifth-century Athens, because his speeches are known to be the first Sophist works recorded and handed down to posterity as texts. Studied as a historical source in rhetoric and composition studies, his explanations of rational language (*logos*, "rational language" being only one of the meanings the word has in Greek)[5] and his rhetorical demonstrations or proofs are still given considerable scholarly attention. *On the Nonexistent* also continues to be important in the study of philosophy as a source work for the exploration of philosophical problems such as the meaning of existence.

2. The 'Heritage' Approach of Ivor Grattan-Guinness

If this paper were to be criticized, or rejected, as an anachronistic attempt to suggest that Gorgias might have done mathematical thinking or teaching of elementary mathematics concepts, it should be understood that I qualify the suggestion to read Gorgias for the purpose of improving mathematics education in light of the insights offered by the historian of mathematics Ivor Grattan-Guinness, who distinguishes between "history" and "heritage" studies of any historical texts. Although Grattan-Guinness never discusses Gorgias as a history or as a heritage source, as I do herein, his approach shapes this paper. Apart from the history of mathematics,

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[4] The text of *On the Nonexistent* by Gorgias of Leontini is widely available in many languages. In English translation (see "References"), the title is sometimes translated as *On Nature* or *On Negation*. The ancient Greek original text can be found in Daniel W. Graham (Ed.), "*The Texts of Early Greek Philosophy: The Complete Fragments and Selected Testimonies of the Major Presocratics.*" Cambridge University Press, 2010, Part I, Ch. 16.

[5] As an expert reviewer of this paper elaborates, "*logos* in ancient Greek has (and still retains in Modern Greek), several meanings (among others; word, speech, talk, oration, discourse, ratio, logic, cause, rationale) thus making it a key concept of philosophical thinking."

within the field of the history of rhetoric studies, the scholar Edward Schiappa has suggested Gorgias may be a fifth-century exemplar of “predisciplinary” scientific thinking; Schiappa’s suggestion has also informed my decision to read and think about the ideas in *On the Nonexistent* for inspiration when teaching elementary mathematics (Schiappa 1999, p.12).[6]

The heritage approach defined by Grattan-Guinness enables a teacher to think about teaching *with* a text as a source of inspiration today rather than *about* a text as a historical foundation. Therefore, to be clear, I am not attempting to suggest that Gorgias provides a foundation for mathematical theories that appeared later, or that his works should be considered a new addition to mathematics history, to be “laid down as the platform” upon which mathematics theory should be built; that would be an entirely inappropriate and mistaken view of this analysis (Grattan-Guinness, 2004(a), p.171). No claim whatsoever is being made herein that the historical author Gorgias was a knower of terms in use by contemporary mathematicians and math educators, such as sequence, set, number line, natural numbers, orders of magnitude, fractions, volume, rectangular prism, or parallelepiped.

Why, then, this paper? It is offered because, I argue, that as a heritage text, Gorgias’s *On the Nonexistent* provides much useful “description” language for teaching, but that such language must be distinguished from mathematics “explanation” (Grattan-Guinness, 2004(a), p.173). Grattan-Guinness thinks that too often math teachers teach math history by going “*backward* in time,” so to be clear, this paper does not constitute a plea to include Gorgias in the modern math curriculum (Grattan-Guinness, 2004(a), p.171). Nonetheless, Grattan-Guinness notes that giving attention “to the broad features of history may well enrich the inheritance” of mathematics education (Grattan-Guinness, 2004(a), p.168). Talking about the fifth century of Athenian civilization as a place and time where math concepts were part of the Greek cultural *zeitgeist* would be, for some students, like medicine that’s easier to swallow when coated with honey. As I consider reasons to use Gorgias’s language and proofs in the modern math classroom, my goal is to give credit to the civilization that fostered and recorded many basic mathematical, philosophical, and rhetorical ideas. But since Gorgias is not numbered among ancient Greek mathematicians, I only

[6] Schiappa’s suggestion that the text reveals predisciplinary scientific thinking is considered controversial by many scholars within the field of rhetoric studies.

recommend reading what Gorgias teaches in *On the Nonexistent* because I believe the text’s contents can refresh and inspire a teacher’s thoughts about *how* to teach certain math lessons, not *what* to teach. I think there may be many possibilities for such results when reading Gorgias, beyond those I feel most confident about giving here.

Therefore, I am making the case that math teachers, when reading Gorgias, can attain an overall positive result, without going beyond mathematics in education and trespassing into the history of mathematics, the history of Sophist philosophy, or the history of Sophist rhetoric. Teachers should, however, be warned by Grattan-Guinness’s concerns about notions “*photocopied* onto the past” (Grattan-Guinness, 2004(a), p.165). If mathematics educators do as I suggest and delve into Gorgias’s intricate thoughts, they must be content to find their own intellectual rewards, rather than any historical basis in fact for their mathematics lessons, when grappling to understand this great Sophist teacher.

3. ‘Heritage’ Math Language and Proofs Inspired by *On the Nonexistent*

3.1 While equating the use and meaning of the word “nonexistent” with the use and meaning of zero taught today, I am not suggesting that Gorgias was teaching anything original about zero, a math concept whose origin and use in ancient times has been identified in many cultures. But I do think that his explanation of “nonexistent” can help a teacher clarify the use of zero in the universal math curriculum. The historical term “nonexistent,” however it was understood and used by Gorgias some 2,500 years ago, includes in its contemporary synonyms the words null, nil and zero. My conclusion is that his acclaimed text can, therefore, be accessed as a heritage source for descriptive language when teaching when to use zero.

Gorgias states that

> everything which is generated has some beginning, but the eternal, being ungenerated, did not have a beginning. And not having a beginning it is without limit. (Section 69)

This statement can help a teacher to describe the basic idea of a sequence, whose beginning must always be determined and which continues indefinitely unless an end point is determined. Elementary mathematics education nowadays begins with the study of numbers placed in sequences and sets. Teachers may appreciate Gorgias’s statements about everything having a beginning, by comparing them to their own descriptions about where a number line may begin. In elementary math classrooms, the number line is advised to be visible and used frequently. While Gorgias himself did not teach about "number line" in his text, teachers may appreciate the challenge of thinking about his explanations of "beginning" and "continuum" when demonstrating how to determine the beginning of a sequence, set, or number line.

The question of whether there is a "nothing" that exists as a beginning point is conceptualized by Gorgias as follows:

> If it exists, it is either one or many. But it is neither one nor many, as will be set forth. Therefore, the existent does not exist. For if it is one, it is an existent or a continuum or a magnitude or a body. But whatever of these it is, it is not one, since whatever has extent will be divided, and what is a continuum will be cut. And similarly, what is conceived as a magnitude will not be indivisible. (Section 73)

An elementary mathematics educator may find it useful to take the word *continuum* and then draw upon contemporary knowledge of the goals of math education to develop lessons that describe how the natural numbers must begin with either zero or one. The word *continuum*, in any heritage dictionary, always includes the definition that it is a mathematics term for *the set of real numbers*. Real numbers include 0 and negative numbers. A set of numbers that represents only zero, {0}, "is neither one nor many"; although there is one number in that set, the number does not represent the existence of one or many things that exist. To be a real number, it can be included to represent the existent that does not exist in a place, sometimes termed the place-holder number. To be a natural number, however, requires understanding what exists or does not exist as the beginning number in the set of natural numbers.

Now, it is a fact that there is sometimes a question in the contemporary math curriculum about when to include 0 in the set of natural numbers. Although 0 is always in the set of real numbers, 0 is not necessarily the number that should begin the set of natural numbers. When deciding whether to begin the set of natural numbers with 0 or 1, students must be taught to ask and then determine, which natural number is needed as the beginning, from the set {0, 1}? I suggest that the experience of reading and grappling with Gorgias's difficult philosophy will end up helping the primary or elementary grades teacher describe this problem of how and when to choose whether to begin the set of natural numbers with 0. Teachers must teach and students must learn when zero must be the beginning point. They must find the beginning place for that nonexistent, uncounted, number, in the set of natural numbers. When a number is determined to be the beginning point, it is never a matter of random choice between the two numbers {0, 1}, as if a student is asked to take a side and choose a number in a dispute about the beginning point, but rather a decision that requires knowing when and why to select one of the two numbers from the beginning set {0, 1}. While Gorgias may not be speaking about this decision, comprehending the word distinctions he makes in his text may help a teacher describe when the nonexistent, or zero, exists as the beginning point in the set of natural numbers.

Students are shown how when there is no count, no amount, or nothing that will exist in number at all, the null set {0} represents this math fact. If the beginning point is 1, such as for counting in the earliest arithmetic lessons, or there is only the possibility of 1 as a number, then the set of natural numbers begins with 1, or {1}. However, when they are ready to learn measurement, they must comprehend that all measurement begins at 0 and not at 1 and so the set of numbers, such as on a one-meter ruler, for example, will be {0 . . . 100}. The confusion to avoid is when students see these numbers as equivalent to the total. Even though the ending number of centimeters is 100, if a student were to count the numbers in the set they would total 101; yet, regardless of the total of the numbers in the set, there are not 101 available to arrive at the total number of centimeters to be measured. Students are not asked to measure 0 centimeters, rather 0 represents that nothing exists to be measured. This might seem self-evident to most students. Nonetheless, I have seen instances where learners do count the beginning number as they move along the ruler, pointing at 0 and saying or counting 1. But beyond that inaccuracy, some students find it difficult to determine any other number as being a conceptual beginning point equivalent to 0, even though that conceptualization is the basic requirement for measurement math. For example, in a beginning problem where a student uses a ruler to measure the distance between 1 and 3 centimeters, the total may be answered as 3, rather than 2 centimeters, because the measurement ends on 3. So it must be demonstrated to students how to make the distinction between a beginning point that literally exists as 0, or a point that is designated as the beginning point 0 from which to move to 1, realizing that movement from the first number to the next is counted as 1. This counting occurs no matter what the actual numeral may be to label that beginning point along the ruler, the line, or the continuum. The total is determined by where the measurement ends, or borrowing from Gorgias, to be cut, by moving from one number to the next, or from point to point along the continuum. The total does not come from counting static numbers or points along the way. Such counting and measurement concepts are demonstrated in pre-school and kindergarten, taught before addition or subtraction are introduced; however, I have found that without demonstrating that the beginning point in measurement is conceptually always zero and that the measurement moves in a continuum from the designated beginning point, some students unfortunately lag in future lessons involving any measurement much later in elementary school.

To summarize, it can be difficult for a beginning student to understand how to distinguish between a point that does not exist as a number to be counted (the conceptual 0 beginning point) and a point that does exist as a number (the conceptual beginning point of 1). Wrestling with how to explain a continuum in the relative terms provided by Gorgias could help teachers clearly describe and demonstrate this distinction about whether zero or one begins the working set of natural numbers.

3.2 Mention of the word “magnitude” in Section 73 of *On the Nonexistent* can be resourced by elementary math educators for good language to utilize when they teach about orders of magnitude. Again, what follows is a heritage use of the historical text. But, it is argued, by studying Gorgias, an elementary school teacher can develop stronger descriptions of how and when zero can exist as a power within the orders of magnitude. For example, any number to the zero power, or x^0, always equals one, $x^0 = 1$, except when the number is 0, in which case 0^0 is explained as having no well-defined meaning or as being undefined.

That Gorgias was or was not teaching a lesson about orders of magnitude becomes a moot point. As a math educator, I have been seeking ahistorical inspiration, rather than the historical basis, for good, clear description to use when teaching universally valid concepts. As a teacher, I have developed better language to describe orders of magnitude, after having muddling through sections of *On the Nonexistent*, such as Section 73. When I can show my students the places in the text where Gorgias uses the word “magnitude” or I can more generally give credit to “Athens in the fifth century B.C.E.” as a place and time where the word “magnitude” was in educational use, my perception is that math students will enjoy learning a word with multiple meanings like “magnitude” is a word known to the ancients, with a specific example from an early text from ancient Athens. Whatever Gorgias thought was to be “conceived as a magnitude,” the fact today is that the word “magnitude” is used to describe earthquakes, star brightness, amplitude of sound, as well as to describe powers or orders of numbers in mathematics. I think it helps teachers to teach a complex concept like orders of magnitude by keeping an eye out for such opportunities for broader cultural literacy. But, most of all, it's the ideal language, such as the descriptive phrase "conceived as a magnitude," that I like to use in my own teaching.

Since orders of magnitude are multiplied, and therefore divisible in an inverse operation, I would also credit Gorgias for his creativity in writing the succinct phrase, "what is conceived as a magnitude will not be indivisible" (73). Magnitude, that which "will not be indivisible" (73), can then be taught as a word that is, in fact, a synonym for multiplication, whenever the inverse relationship between multiplication and division is under discussion.

3.3 Teachers who write lessons about fractions, such as what constitutes the set of real numbers to use for numerators and denominators, or how the multiplication and division of fractions are inverse operations, may benefit from considering Gorgias's thoughts about how "whatever has extent will be divided" (73). After all, a fraction conceptualizes a division, with the fraction bar denoting a division symbol. Furthermore, it would be helpful to borrow the language of Gorgias and describe how since 0 represents the nonexistent, it represents what has no extent and therefore can neither be divided nor divide.

In fractions, distinctions involving 0 in the numerator and denominator must be taught. These will include a lesson on numerators about how, when a numerator is nonexistent, i.e., the number 0, then it does not exist as a fraction of the denominator. As a result, students proceed from what they have learned about how to set up a division equation, such as when, if they see a 0 in the dividend, then the quotient is always 0, i.e., $0 \div x = 0$. No matter what x is as the divisor, the presence of the 0 as the dividend demands the answer for the quotient be 0. Progressing to fractions, in a problem where the numerator of a fraction is 0,$\frac{0}{x}$, then students must learn that the answer is expressed simply as 0. The number 0 only represents the nonexistent part that exists in relation to an existent x. Therefore, the sets of real or natural numbers available for numerators, just as for dividends and quotients in division, may be described as including 0.

Next, just as students have learned in division lessons that they can never set up an equation such as $x \div 0$ because 0 is never a divisor and so there is no potential whatsoever for any quotient, students learn that in fractions the equation $\frac{x}{0}$ is not allowed. When the whole is nonexistent or 0, then there is no relationship or ratio of parts to a whole. There is no possibility of dividing any numerator x by a denominator of 0, as it would be absurd to say there could be any parts of a nonexistent whole. The sets of either real or natural numbers available for divisors in division and for denominators in fractions do not include 0.

Gorgias can be accessed even further for concise language that helps to conceptualize another rule that must be taught in fractions. His text notes, "Of course, if the existent is the same as the nonexistent, it is not possible for both to exist" (76). For fractions lessons, a teacher must describe how, when the numerator and the denominator are the same number, they always form the number one, $\frac{x}{x} = 1$, because $x \div x = 1$, *except x cannot be 0.* Although the numerator and the denominator in a fraction relationship always equal 1, i.e., they no longer exist as two separate numbers but as the number 1. If 0 were taught as existing in a relationship to itself, or as a numerator 0 that exists in a ratio to a denominator of 0, $\frac{0}{0}$, it would be absurd to state that this relationship equals 1, the rationale being that zero as the numerator cannot be divided by zero as the denominator.

3.4 Finally, studying *On the Nonexistent* could help an elementary mathematics teacher to describe how to make the distinction between container and contained through measurement, how to describe three-dimensional shapes, and how to measure in cube units.

Gorgias explains that there are distinctions to be made about the question of whether the existent can be contained in a container:

> It [the existent] is not contained in itself. For in that case, container and contained will be the same, and the existent will become two things, place and body (the place is the container, body the contained). But that is absurd. Thus, the existent is not in itself. . . . (Section 70)

This descriptive language could be useful for elementary school teachers, if only to know that such distinctions were being made in fifth-century Classical Greece. Teachers, I suggest, are continually seeking to refresh their teaching about how containers and contents are measured in different units, such as when the container is measured in cube units and the contents are measured in solid or liquid weights. In this age of premeasured contents and uniform-size packages, the opportunities to explain these differences are usually "off-the-shelf" demonstrations. To describe a Greek bath or a Greek urn allows for many creative history lessons to shore up necessary lessons. Furthermore, studying how Gorgias used such logic to make distinctions could be extended to what can be a difficult lesson for some students the differences between mass and weight or other differences. Again, I caution that I am not saying Gorgias was discussing the different units of measurements that distinguish between container and contents or anything to do with mass or its measurement. But I think that the terminology Gorgias used, "container and contained," is language relatable to distinctions that a mathematics teacher must make, so consequently, any teacher delving into this ancient thought will undergo a challenging yet invigorating reading and thinking experience that may inspire ideas about how to go about demonstrating and explaining such distinctions in the classroom**.**

Gorgias provides an explicit proof on how to measure space, as follows: "If it is by chance a body it will be three-dimensional, for it will have length, and breadth and depth. But it is absurd to say that the existent is none of these things. Therefore, the existent is not one" (73). Gorgias's explanation, that a *three-dimensional* body is (Length) (Breadth) (Depth), is usually expressed in elementary school mathematics by saying that it occupies a region of space measured by its width, length, and height. Then, this is expressed quantitatively by **defining**, as a measure of the space occupied, a **unit** cube, the product of its Length × Width × Height. This is equivalent to the conceptualization of volume of a right-angle rectangular prism or a rectangular parallelepiped, expressed as the number of unit cubes occupying it completely, from which follows the usual formula for its volume: (L)× (W) × (H). This passage from Section 73 can also inspire a lesson for a concept that can be difficult for many young students to grasp, which is how the volume of a unit cube, namely 1^3, while resulting numerically in a product of 1, is actually different from 1, in the sense that (1 unit of) Length × (1 unit of) Width × (1 unit of) Height = 1 unit of volume, which is **conceptually different** from 1 unit of length (or any number of points on a continuum or existing as a multiple on a plane). While all of these explanations belong entirely to mathematics today, Gorgias should be invited into the classroom for credit for an early textual mention of the equation for volume and parallelepiped and for the early use of describing a *body* as three-dimensional (literally speaking, Gorgias argues that a (solid) body "is triple," in the sense of having length, width, and depth). [7]

[7] I have incorporated the editing suggestions made by the expert reviewer throughout section 3.4, which I must credit for making this section much clearer.

4. Conclusion

This paper has discussed whether the heritage use of a historical text might possibly help mathematics educators describe complex concepts like zero, when studied in terms of the heritage versus history distinctions made by Grattan-Guinness. For an intellectual adventure, rather than a trudge into the past, elementary mathematics teachers should consider exploring the text *On the Nonexistent*. As readers and thinkers, educators can decide for themselves whether Gorgias, one of the great teachers in Classical Greek antiquity, has any relevance for mathematics in education today.

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An Application of Fuzzy Logic for Learning Mathematics according to the Bloom's Taxonomy

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Abstract

In this paper we apply an improved version of the Trapezoidal Fuzzy Assessment Model (TRFAM) to evaluate the students' progress for learning the topic "Real numbers" with respect to the principles of the Bloom's Taxonomy. The TRFAM is a new original variation of the Center of Gravity (COG) defuzzification technique, which has been properly adapted in earlier papers by the present authors to be used as an assessment method. The central idea of TRFAM is the replacement of the rectangles appearing in the graph of the membership function of the COG technique by isosceles trapezoids sharing common parts. In this way one treats better the ambiguous cases of student scores being at the boundaries between two successive assessment grades. Our model is validated by comparing it with traditional assessment methods (calculation of the means and GPA index), based on principles of the bivalent logic.

Keywords: *Bloom's taxonomy, Real numbers, Fuzzy Logic, Center of*

Gravity (COG) defuzzification technique, Trapezoidal Fuzzy Assessment Model (TRFAM).

1. Introduction

Fuzzy logic, the development of which is based on fuzzy sets theory, provides a rich and meaningful addition to standard Boolean logic. Unlike Boolean logic, which has only two states, true or false, fuzzy logic deals with truth values which range continuously from 0 to 1. Thus something could be *half true* 0.5 or *very likely true* 0.9 or *probably not true* 0.1, etc. In this way fuzzy logic allows one to express knowledge in a rule format that is close to a natural language expression and therefore it opens the door to construction of mathematical solutions of computational problems which are inherently imprecisely defined. New operations for the calculus of logic were also proposed and fuzzy logic showed to be in principle at least a generalization of classic logic [15, 16]. For general facts on Fuzzy Sets and Logic we refer to the book [5].

The methods of assessing the individual skills usually applied in practice are based on principles of the bivalent logic (yes-no). However, these methods are not probably the most suitable ones in ambiguous cases characterized by a degree of uncertainty. In Education, for example, the teacher is frequently not absolutely sure about a particular numerical grade characterizing a student's performance. Fuzzy logic, due to its nature of including multiple values, offers a wider and richer field of resources for this purpose.

In earlier works the present authors have properly adapted the corresponding fuzzy system's uncertainty (e.g. [10, 11], etc) as well as the popular in fuzzy mathematics Center of Gravity (COG) deffuzification technique (e.g. [6, 12, 13, 14] etc) to be used as assessment methods of individual skills. In this paper we apply a trapezoidal fuzzy model (TRFAM) for assessing the student success for learning mathematics in accordance to the Bloom's taxonomy. This taxonomy, which has been applied in the USA by generations of teachers and college instructors in the teaching process [2], refers to a classification of the different learning objectives serving as a way of distinguishing the fundamental questions within the educational system.

The rest of the paper is organized as follows: In Section 2 we present the fundamentals of the Bloom's taxonomy. In Section 3 we develop our fuzzy model. In Section 4 we present an application of this model connected to the teaching of the real numbers. Finally, Section 5 is devoted to our final conlusions and a short discussion on future perspectives of research on this subject.

2. The Bloom's taxonomy

In 1956 Benjamin Bloom with collaborators Max Englehart, Edward Furst, Walter Hill, and David Krathwohl published a framework for categorizing educational goals, the *Taxonomy of Educational Objectives* [3]*. Although named after Bloom, the publication of the taxonomy followed a series of conferences from 1949 to 1953, which were designed to improve communication between educators on the design of curricula and examinations. A revised version of the taxonomy was created in 2000 by Lorin Anderson [1], former student of Bloom. Since the taxonomy reflects different forms of thinking and thinking is an active process, in the revised version the names of its six major levels were changed from *noun* to *verb* forms. The six major levels of the revised taxonomy are presented in Figure 1, taken from [17].

* Bloom's taxonomy divides educational objectives into three domains: *cognitive* , *affective* and *psychomotor*, sometimes loosely described as "knowing/head", "feeling/heart" and "doing/hands" respectively. The volume published in 1956 [3] and the revision followed in 2000 [1] concern the cognitive domain, while a second volume published in 1965 on the affective domain. A third volume was planned on the psychomotor domain, but it was never published. However, other authors published their own taxonomies on the last domain. More details can be found in [17].

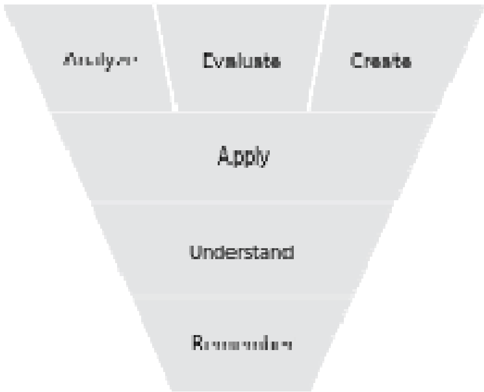

Figure 1: The six major levels of the Bloom's taxonomy

The above six levels in the taxonomy, moving through the lowest order processes to the highest, could be described as follows :

- *Knowing - Remembering:* Retrieving, recognizing, and recalling relevant knowledge from long-term memory. eg. find out, learn terms, facts, methods, procedures, concepts
- *Organizing - Understanding:* Constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining. Understand uses and implications of terms, facts, methods, procedures, concepts.
- *Applying:* Carrying out or using a procedure through executing, or implementing. Make use of, apply practice theory, solve problems, use information in new situations.
- *Analyzing:* Breaking material into constituent parts, determining how the parts relate to one another and to an overall structure or purpose through differentiating, organizing, and attributing. Take concepts apart, break them down, analyze structure, recognize assumptions and poor logic, evaluate relevancy.
- *Generating - Evaluating:* Making judgments based on criteria and standards through checking and critiquing. Set standards, judge using standards, evidence, rubrics, accept or reject on basis of criteria.

- *Integrating - Creating:* Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing. Put things together; bring together various parts; write theme, present speech, plan experiment, put information together in a new & creative way

Most researchers and educators consider the last three levels -- analyzing, evaluating and creating – as being parallel. It is obvious that using Bloom's higher levels helps the students become better problem solvers.

Teaching a topic, the teacher should arrange his/her class work in the order to synchronize it with these six steps of Blom's Taxonomy. The typical questions for evaluating the student achievement at the corresponding level are the following:

Knowing questions focus on clarifying, recalling, naming, and listing:
Which illustrates...?
Write... in standard form....
What is the correct way to write the number of... in word form?

Organizing questions focus on arranging information, comparing similarities/ differences, classifying, and sequencing:
Which shows... in order from...?
What is the order...?
Which is the difference between a... and a...?
Which is the same as...?
Express... as a...?

Applying questions focus on prior knowledge to solve a problem:
What was the total...?
What is the value of...?
How many... would be needed for...?
Solve....Add/subtract....Find....Evaluate....Estimate....Graph....

Analyzing questions focus on examining parts, identifying attributes/ relationships /patterns, and main idea:
Which tells...?

If the pattern continues,....
Which could...?
What rule explains/completes... this pattern?
What is/are missing?
What is the best estimate for...?
Which shows...?
What is the effect of...?

Generating questions focus on producing new information, inferring, predicting, and elaborating with details:
What number does... stand for?
What is the probability...?
What are the chances...?
What effect...?

Integrating questions focus on connecting/combining/summarizing information, and restructuring existing information to incorporate new information:
How many different...?
What happens to... when...?
What is the significance of...?
How many different combinations...?
Find the number of..., ..., and ... in the figure below.

Evaluating questions focus on reasonableness and quality of ideas, criteria for making judgments and confirming accuracy of claims:
Which most accurately...?
Which is correct?
Which statement about... is true?
What are the chances...?
Which would best...?
Which would... the same...?
Which statement is sufficient to proven...?

Bloom's taxonomy serves as the backbone of many teaching philosophies, in particular those that lean more towards skills rather than content. The emphasis on higher-order thinking inherent in such philosophies is based on the top levels of the taxonomy including analysis, evaluation, synthesis and creation. Bloom's taxonomy can be used as a teaching tool to help balance assessment and evaluative questions in class, assignments and texts to ensure all orders of thinking are exercised in student's learning.

3. The fuzzy assessment model

Reasoning with fuzzy rules is a forward-chaining procedure. The initial numeric data values are *fuzzified,* that is, turned into fuzzy values using the membership functions. Instead of a match and conflict resolution phase where we select a triggered rule to fire, in fuzzy systems, all rules are evaluated, because all fuzzy rules can be true to some degree ranging from 0 to 1. The antecedent clause truth values are combined using fuzzy logic operators. Next, the fuzzy sets specified in the consequent clauses of all rules are combined using the rule truth values as scaling factors. The result is a single fuzzy set, which is then *defuzzified* to return a crisp output value.

There are several defuzzification techniques in use, the most popular being probably the *centre of gravity* (COG) method [9]. According to this method the fuzzy data is represented by the coordinates of the COG of the level's section contained between the graph of the membership function involved and the OX axis.

Here we shall apply an improved form of a recently developed [7, 8] variation of the above assessment method that we have called *Trapezoidal Fuzzy Assessment Model* (TRFAM).

Let G a student group participating in a certain activity (learning, problem-solving, etc) and let A, B, C, D and F be the linguistic labels of excellent, very good, good, fair and unsatisfactory performance respectively with respect to this activity.

Set U = {A, B, C, D, F}. Then G can be expressed as a fuzzy set in U in the form

G= $\{(x, m(x)): x \in U\}$, where y=m(x) is the corresponding membership function. The main idea of TRFAM is the replacement of the rectangles appearing in the graph of the COG technique (e.g. see Figure 1 of [13]) by trapezoids.

Therefore, we shall have five such trapezoids in the resulting scheme, each one corresponding to a students' grade (F, D, C, B and A respectively). Without loss of generality and for making our calculations easier we consider isosceles trapezoids with bases of length 10 units lying on the OX axis. The height of each trapezoid is equal to the percentage of individuals who achieved the corresponding characterization for their performance, while the parallel to its base side is equal to 4 units.

We allow for any two adjacent trapezoids to have 30% of their bases (3 units) belonging to both of them. In this way we treat better the ambiguous cases of individuals' scores being at the boundaries between two successive grades. For example, in students' assessment it is a very common approach to divide the interval of the specific grades in three parts and to assign the corresponding grade using + and - . For instance, we could have 75 – 77 = B-, 78 – 81 = B, 82 – 84 = B+. However, this consideration does not reflect the common situation, where the teacher is not sure about the grading of the students whose performance could be assessed as marginal between and close to two adjacent grades; for example, something like 84 - 85 being between B_+ and A_-. The TRFAM fits better than the COG technique to this kind of situations.

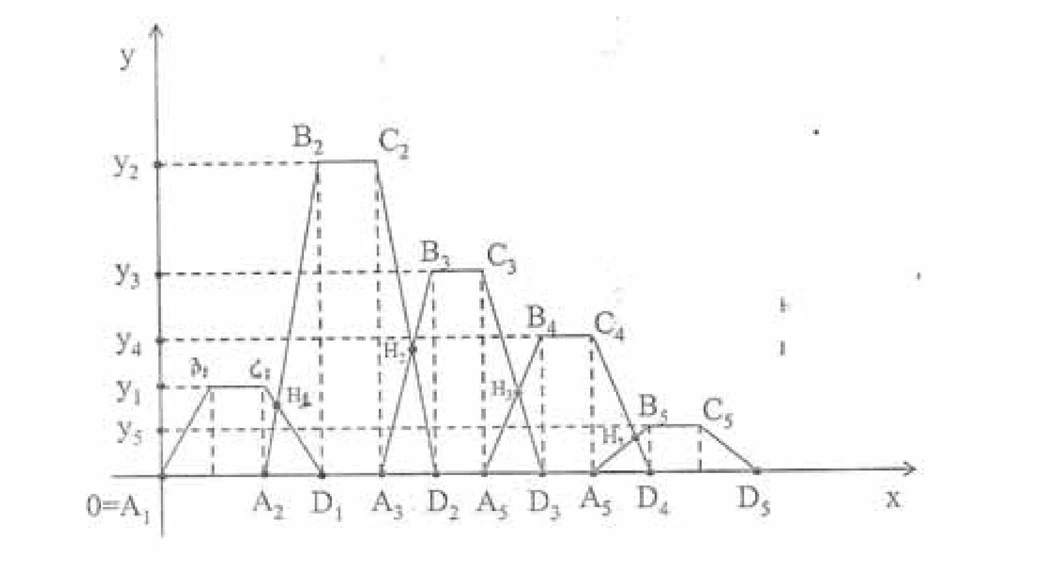

Figure 2: The TRAFM's scheme

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In TRFAM an individuals' group can be represented, as in the COG method, as a fuzzy set in U, whose membership function *y=m(x)* has as graph the line $OB_1C_1H_1B_2C_2H_2B_3C_3H_3B_4C_4H_4B_5C_5D_5$ of Figure 2, which is the union of the line segments OB_1, B_1C_1, C_1H_1,…….., B_5C_5, C_5D_5. However, in case of the TRFAM the analytic form of $y = m(x)$ is not needed for calculating the COG of the resulting area. In fact, since the marginal cases of the individuals' scores are considered as common parts for any pair of the adjacent trapezoids, it is logical to count these parts twice; e.g. placing the ambiguous cases B+ and A- in both regions B and A. In other words, the COG technique, which calculates the coordinates of the COG of the area between the graph of the membership function and the OX axis, thus considering the areas of the "common" triangles $A_2H_1D_1$, $A_3H_2D_2$, $A_4H_3D_3$ and $A_5H_4D_4$ only once, is not the proper method to be applied in the above situation.

Instead, in this case we represent each one of the five trapezoids of Figure 2 by its COG F_i, i=1, 2, 3, 4, 5 and we consider the entire area, i.e. the sum of the areas of the five trapezoids, as the system of these points-centers. More explicitly, the steps of the whole construction of the TRFAM are the following:

1. Let y_i, i=1, 2, 3, 4, 5 be the percentages of students whose performance was characterized by F, D, C, B, and A respectively; then $\sum_{i=1}^{5} y_i$ =1 (100%).

2. We consider the isosceles trapezoids with heights being equal to y_i, i=1, 2, 3, 4, 5, in the way that has been illustrated in Figure 2.

3. We calculate the coordinates (x_{c_i}, y_{c_i}) of the COG F_i, i=1, 2, 3, 4, 5, of each trapezoid as follows: It is well known that the COG of a trapezoid lies along the line segment joining the midpoints of its parallel sides *a* and *b* at a distance *d* from the longer side *b* given by $d=\frac{h(2a+b)}{3(a+b)}$, where *h* is its height

(e.g. see [18])..Therefore in our case we have $y_{c_i} = \quad = \frac{y_i(2*4+10)}{3*(4+10)} = \frac{3y_i}{7}$. Also, since the abscissa of the COG of each trapezoid is equal to the abscissa of the midpoint of its base, it is easy to observe that $x_{ci}=7i-2$.

4. We consider the system of the COG's F_i, i=1, 2, 3, 4, 5 and we calculate the coordinates (X_c, Y_c) of the COG F_c of the whole area S considered in Figure 2 by the following formulas, derived from the commonly used in such cases definition (e.g. see [19]):

$$X_c = \frac{1}{S}\sum_{i=1}^{5} S_i x_{c_i}\ ,\ Y_c = \frac{1}{S}\sum_{i=1}^{5} S_i y_{c_i} \quad (1).$$

In formulas (1) Si, i= 1, 2, 3, 4, 5 denotes the area of the corresponding trapezoid. Thus, $Si=\frac{(4+10)y_i}{2}=7y_i$ and $S = \sum_{i=1}^{5} S_i = 7\sum_{i=1}^{5} y_i = 7$. Therefore, from formulas (1) we finally get that

$$X_c = \frac{1}{7}\sum_{i=1}^{5} 7y_i(7i-2) = (7\sum_{i=1}^{5} iy_i) - 2\ ,\ Y_c = \frac{1}{7}\sum_{i=1}^{5} 7y_i(\frac{3}{7}y_i) = \frac{3}{7}\sum_{i=1}^{5} y_i^2 \quad (2).$$

5. We determine the area where the COG F_c lies as follows: For i, j=1, 2, 3, 4, 5, we have that $0 \leq (y_i - y_j)^2 = y_i^2 + y_j^2 - 2y_iy_j$, therefore $y_i^2+y_j^2 \geq 2y_iy_j$, with the equality holding if, and only if, $y_i=y_j$. Therefore $1=(\sum_{i=1}^{5} y_i)^2 = \sum_{i=1}^{5} y_i^2 + 2\sum_{i,j=1,\, i\neq j}^{5} y_iy_j \leq \sum_{i=1}^{5} y_i^2 + +2\sum_{i,j=1,\, i\neq j}^{5}(y_i^2+y_j^2) = 5\sum_{i=1}^{5} y_i^2$ or $\sum_{i=1}^{5} y_i^2 \geq \frac{1}{5}$ (3), with the equality holding if, and only if, $y_1 = y_2 = y_3 = y_4 = y_5 = \frac{1}{5}$. In the case of equality the first of formulas (2) gives that $X_c = 7(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5}) - 2 = 19$. Further, combining the inequality (3) with the second of formulas (2) one finds that $Y_c \geq \frac{3}{35}$ Therefore the unique minimum for Y_c corresponds to the COG $F_m(19, \frac{3}{35})$. The ideal case is when $y_1=y_2=y_3= y_4=0$ and $y_5=1$. Then from formulas (2) we get that $X_c = 33$ and $Y_c = \frac{3}{7}$. Therefore the COG

in this case is the point $F_i\,(33, \frac{3}{7})$. On the other hand, the worst case is when $y_1=1$ and $y_2= y_3 = y_4= y_5=0$. Then from formulas (2), we find that the COG is the point $F_w(5, \frac{3}{7})$. Therefore the area where the COG F_c lies is the area of the triangle $F_w\,F_m\,F_i$ (see Figure 3).

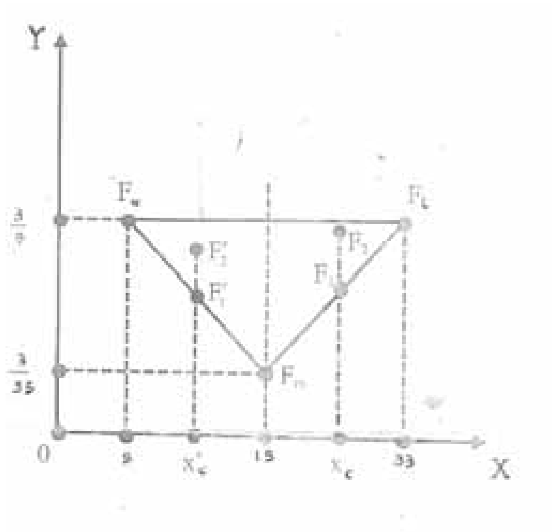

Figure 3: The area where the COG lies

6. We formulate our criterion for comparing the performances of two (or more) different student groups' as follows: From elementary geometric observations (see Figure 3) it follows that for two groups the group having the greater X_c performs better. Further, if the two groups have the same $X_c \geq 19$, then the group having the COG which is situated closer to Fi is the group with the greater Y_c. Also, if the two groups have the same $X_c<19$, then the group having the COG which is situated farther to Fw is the group with the smaller Y_c. Based on the above considerations it is logical to formulate our criterion for comparing the two groups' performance in the following form:

- *Between two groups the group with the greater value of X_c demonstrates the better performance.*
- *If two groups have the same $X_c \geq 19$, then the group with the greater value of Y_c demonstrates the better performance.*

- *If two groups have the same $X_c < 19$, then the group with the smaller value of Y_c demonstrates the better performance.*

As it becomes evident from the above presentation, the application of the TRFAM is simple in practice needing no complicated calculations in its final step. Further, our criterion shows that the assessment of the student performance is based on the values of X_c. But, as it turns out from the first of formulas 2 calculating the value of X_c, greater coefficients (weights) are assigned to the higher scores. Therefore the TRFAM provides a weighted measure focusing on the student *quality performance.*

4. An application on teaching the real numbers

4.1 Description

The following application was performed with subjects two groups of students from two different departments (30 students in each group) of the School of Technological Applications (prospective engineers) of the Graduate Technological Educational Institute (T. E. I.) of Western Greece attending the common course "Mathematics I" of their first term of studies and having the same instructor. This course involves an introductory module repeating and extending the students' knowledge from secondary education about the real numbers. After the module was taught, the instructor wanted to investigate the students' progress according to the principles of the Bloom's Taxonomy. For this, he asked them to answer in the class the written test presented in the Appendix of this paper, which is divided in six different parts, one for each level of the Taxonomy. The students' answers were assessed separately for each level in a scale from 0 to 100 and the means obtained correspond to each student's overall performance.

4.2 Results

Denote by L_i, i=1, 2, 3, 4, 5, 6 the levels of Knowing-Remembering, Organizing-Understanding, Applying, Analyzing, Generating-Evaluating and Integrating- Creating respectively of the Bloom's Taxonomy and by P the student overall performance. Then the test's results are summarized in the following two tables:

Table 1: Results of the first department

Grade	L_1	L_2	L_3	L_4	L_5	L_6	P
A(85-100)	8	6	5	3	2	3	4
B(84-75)	9	11	10	8	7	8	9
C(74-60)	10	9	10	12	10	8	10
D(59-50)	3	3	3	5	7	8	5
F(<50)	0	1	2	2	4	3	2

Table 2: Results of the second department

Grade	L_1	L_2	L_3	L_4	L_5	L_6	P
A(85-100)	9	8	6	4	3	3	5
B(84-75)	6	7	9	7	7	6	8
C(74-60)	9	8	10	12	10	8	9
D(59-50)	6	7	4-	4	7	11	7
F(<50)	0	0	1	3	3	2	1

4.3 Evaluation of the results using the TRFAM

From Table 1 we obtain the following percentages for the level L_1: $y_1=0$, $y_2=\frac{3}{30}$, $y_3=\frac{10}{30}$, $y_4=\frac{9}{30}$ and $y_5=\frac{8}{30}$. Therefore, applying the first of formulas (2) one finds that $X_c=7(\frac{6}{30}+\frac{30}{30}+\frac{36}{30}+\frac{40}{30})-2=\frac{724}{30}\approx 24.13$. Similarly one finds the following values of X_c:
23.2 for L_2, 20.87 for L_3, 20.17 for L_4, 18.07 for L_5, 19 for L_6 and 20.87 for the student overall performance P.

In the same way one finds from Table 2 the following values of X_c: 23.2 for L_1, 22.73 for L_2, 22.5 for L_3, 20.17 for L_4, 19 for L_5, 18.3 for L_6 and 21.1 for P.

On comparing the values of X_c for the two departments and according to the first case of the criterion stated in section 3 one concludes that the first department demonstrated a better performance at the levels L_1, L_2 and

L_6 of the Bloom's Taxonomy, while the second department demonstrated a better performance at the levels L_3 and L_5. Further, the two departments demonstrated the same performance at the level L_4, while the second department demonstrated a better overall performance than the first one. In general, the overall performance of the two departments as well as their performance at each stage of the Bloom's Taxonomy can be characterized as more than satisfactory, since the corresponding values of X_c are in all cases greater than the half of its value in the ideal case, which is equal to $\frac{33}{2}$=16.5 (see Figure 3).

We also observe that the performance of each department is decreasing from level L_1 to level L_4, which was expected since the success at the higher levels is based on the lower levels. However, for the first department this does not happen for the last three levels, a fact which is compatible to the view that the three higher levels of the Taxonomy are parallel to each other (see section 2 – Figure 1).

4.4 Comparison of the TRFAM with the traditional assessment methods

Most of the traditional assessment methods, which are based on the principles of the bivalent logic, measure the students' *mean performance*. Therefore, the conclusions obtained by applying these methods may differ from the conclusions obtained by applying the TRFAM, which, as we have seen in section 3, measures the students' *quality performance* by assigning higher coefficients (weights) to the higher scores. For example, in the hypothetical case where the students of the last column of Table 1 obtained the highest scores of the corresponding grade (i.e. 4 students scored 100, 9 students scored 84, etc), while the students of the last column of Table 2 obtained the lowest scores of the corresponding grade (i.e. 5 students scored 85, 8 students scored 75, etc), calculating the means one finds an average score 64.51 for the first and 53.33 for the second department. Therefore, the first department demonstrates a much better mean overall performance than the second one, in contrast to their quality performance measured by TRFAM.

One of the few traditional assessment methods - very popular in the USA- which measures the students' quality performance is the *Grade Point Average* (GPA) index. In terms of the student percentages the GPA index is calculated by the formula [4]: GPA= $y_2+2y_3+3y_4+4y_5$ (4)
In the worst case (y_1=1 and $y_2=y_3=y_4=y_5$=0) formula (4) gives that GPA=0, while in the ideal case ($y_1=y_2=y_3=y_4$=0 and y_5=1) it gives that GPA=4. Therefore we have that $0 \leq \text{GPA} \leq 4$.

Applying (4) on the data of the first column of Table 1 one finds that GPA=$\frac{3}{30}+\frac{20}{30}+\frac{27}{30}+\frac{32}{30}\approx 2.73$ at level L_1 of the Taxonomy for the first department. Similarly one finds the GPA values 2.6 for L_2, 2.43 for L_3, 2.17 for L_4, 1.87 for L_5, 2 for L_6 and 2.17 for the overall performance of the first department. In the same way working with the data of Table 2 one finds the GPA values 2.6, 2.53, 2.5, 2.17, 2, 1.9 and 2.3 respectively for the second department. Therefore, the two departments demonstrate the same performance at level L_4, the first department demonstrates a better performance at levels L_1, L_2 and L_6, while the second department demonstrates a better performance at levels L_3, L_5 and a better overall performance than the first department. These findings agree with the corresponding ones obtained by applying the TRFAM. However, according to the GPA index the performance of the first department at level L_5 and of the second department at level L_6 were found to be less than satisfactory, since their GPA values are smaller than the half of its ideal value, which is equal to 2. This difference with respect to the TRFAM is due to the fact that, as it can be easily observed on comparing formula (4) with the first of formulas (2), the TRFAM assigns greater weights and therefore it is *more sensitive* than the GPA index *to the higher scores*.

5. Conclusion

In the present paper we developed an improved version of the Trapezoidal Fuzzy Assessment Model (TRFAM) and we applied it to evaluate the students' progress for learning the real numbers with respect to the principles of the Bloom's Taxonomy. In the design of the TRFAM the rectangles appearing in the graph of the membership function of the COG technique were replaced by isosceles trapezoids sharing common parts. In this way one treats better the ambiguous cases of student scores being at the boundaries between two successive assessment grades. Our model was validated by comparing it with traditional assessment methods (calculation of the means and GPA index), based on principles of the bivalent logic.

Our future plans include the application of the same model for studying the students' progress with respect to the principles of the Bloom's Taxonomy in other fields of knowledge (not only for mathematics). Also, since the TRFAM seems to have the potential of a general assessment method, our research perspectives focus on applying it to evaluate other kind of human activities in Science, games, decision making, etc.

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Appendix: The test used in our application

Questions

1. Knowing - Remembering:

Give the definitions and examples of a periodic decimal and of an irrational number (in the form of an infinite decimal).

2. Organizing:

Compare the set of all fractions with the set of periodic decimals. Compare the set of irrational numbers with the set of all roots (of any order) that have no exact values.

3. Applying:

Which of the following numbers are natural, integers, rational, irrational and real numbers?

-2 , $-\frac{5}{3}$, 0 , 9.08 , 5 , $7.333...$, $\pi = 3.14159...$, $\sqrt{3}$, $-\sqrt{4}$, $\frac{22}{11}$,

$5\sqrt{3}$, $-\frac{\sqrt{5}}{\sqrt{20}}$, $\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)$, $-\frac{\sqrt{5}}{2}$, $\sqrt{7}-2$, $\sqrt{\left(\frac{5}{3}\right)^2}$

4. Analyzing:

Find the digit which is in the 1005th place of the decimal 2.825342342......

Write the number 0.345345345... in the fractional form.

Compare the numbers 5 and 4.9999…

Construct the line segment of length $\sqrt{3}$ with the help of the Pythagorean Theorem. Give a geometric interpretation.

5. Generating- Evaluating:

Justify why the decimals 2.00131311311131111..., 0.1234567891011... are irrational numbers.

Construct the line segment of length $\sqrt[3]{2}$ by using the graph of the function f(x)= $\sqrt[3]{x}$

6. Integrating- Creating:

Define the set of the real numbers in terms of their decimal representations (this definition was not given by the instructor in the class before the test).