Open Schools Journal for Open Science

Vol 9, No 1 (2026)

Open Schools Journal for Open Science

Heisenberg's Uncertainty Principle

Marina Giannakopoulou

doi: [10.12681/osj.44027](https://doi.org/10.12681/osj.44027)

Heisenberg's Uncertainty Principle

A First Look at Quantum Mechanics and Its Extensions

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$$

M. Giannakopoulou

11th Grade, Athens College, Athens, Greece

Abstract

Based on the author's 2024-2025 Middle Years Program Personal Project, the present work introduces Heisenberg's uncertainty principle. It is a product of secondary research, which opts to render quantum mechanics accessible to high-school students. Emphasis is primarily given to wave-particle duality, the de Broglie Hypothesis, state vectors, wavefunctions, probability density, and the Bohr model.

Keywords

Heisenberg's uncertainty principle, Wave-particle duality, Bohr model

The Quantum World

Think of everyday life… Imagine you were a skillful baseball or tennis player, and you were about to hit the ball. Everything seems perfect, until… a gust of wind blows the ball, and you miss. You may be annoyed, but you are definitely not scared. Nothing was strange; had you considered all the variables, you would have predicted the trajectory of the ball.

That seems straight forward, right? And it is…

…in the classical world, in the macroscopic world where humans live. But you cannot say the same thing about the quantum world…

The quantum world exists on a very small scale; so small that the intuition we have developed regarding the interactions in the world around us cannot apply there. That is the main reason why everybody has a hard time understanding it, even great physicists!

[1]

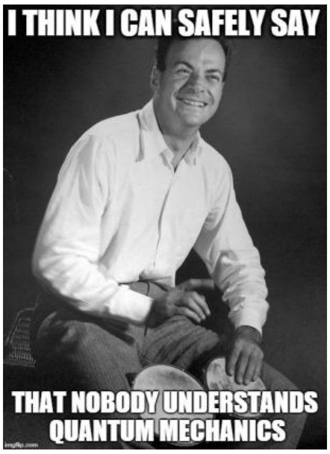


Figure 1: Yes! That Was Said by Richard Feynman, a Pioneer in Quantum Mechanics.

The quantum world is tricky and non-intuitive. Classical mechanics cannot describe it. There comes quantum mechanics, which is more fundamental than classical

[1] qonaom.wordpress.com/2016/05/02/anyone-can-understand-quantum-mechanics-part-1/

mechanics and studies the behavior of matter and energy on the atomic and subatomic scale.

Regarding the quantum world, there are four things we must remember:

1. **The smallest change**, even a measurement, **can affect the state** of a quantum system radically.
2. **Everything is probabilistic**.
3. **Particles can behave like waves one moment and like particles the next**.
4. **Quantities can only take definite values**. "Quantum" refers to *the smallest amount or unit of something*.

On our journey, we will first cross the untrodden valley of Heisenberg's uncertainty principle. Then, we will encounter our second challenge, the Bohr model (which you might have heard but pretend you have not. I am trying to create suspense.) Finally, we will examine their interesting connection. And do not forget that a new perspective of uncertainty awaits at the end of our journey!

Understanding Heisenberg's Uncertainty Principle

In 1927, German physicist Werner Heisenberg highlighted one of the main differences between classical and quantum mechanics.

[2]

Figure 2: Werner Heisenberg

According to classical mechanics, all physical quantities can be assigned exact values simultaneously. Consider a simple example: one of our friends is solving a physics exercise on linear motion (what an excruciating activity for him!). After performing calculations, he successfully finds the velocity v of an object at position x and knows the exact value of both velocity and position. Makes sense, right? This is what should happen. However, the same thing cannot happen in quantum mechanics!

[2] geoffneilsen.wordpress.com/navigation/twentiethcentury-draft/werner-heisenberg-1901-76/

Statement

A clue: *An observable is any physical quantity that can be measured*.

Heisenberg's uncertainty principle applies to conjugate pairs of observables $\hat{A}, \hat{B}$. These observables do not commute, i.e. $\hat{A}\hat{B} \neq \hat{B}\hat{A}$. There are several conjugate pairs: position $\hat{\boldsymbol{x}}$ and momentum[3] $\hat{\boldsymbol{p}}$, energy $\hat{\boldsymbol{E}}$ and time $\hat{\boldsymbol{t}}$, and more. We will consider the first pair.

The reason why $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ do not commute lies in that they are position and momentum **operators**. This means that they are not numbers but mathematical operations, and the order of their execution matters.

According to Heisenberg's uncertainty principle (for position and momentum), exact values cannot be assigned to the position and momentum of a quantum system simultaneously. These quantities can only be determined with **some uncertainty**. This means that their value is not precise but exists within a range of possible values. The tennis player's experience will help; the uncertainty in determining the bounce spot of the ball during an invisible, continuous rally is linked to the range of all spots within the court lines.

Back to uncertainty: Uncertainty **cannot be equal to zero** and **cannot take small values for both quantities** at the same time. Hence, the more accurately position is determined, the less accurately momentum is known, and vice versa.

No matter how many times you look at it, read it, or turn it around, this may be difficult to grasp at first, which is why our journey continues.

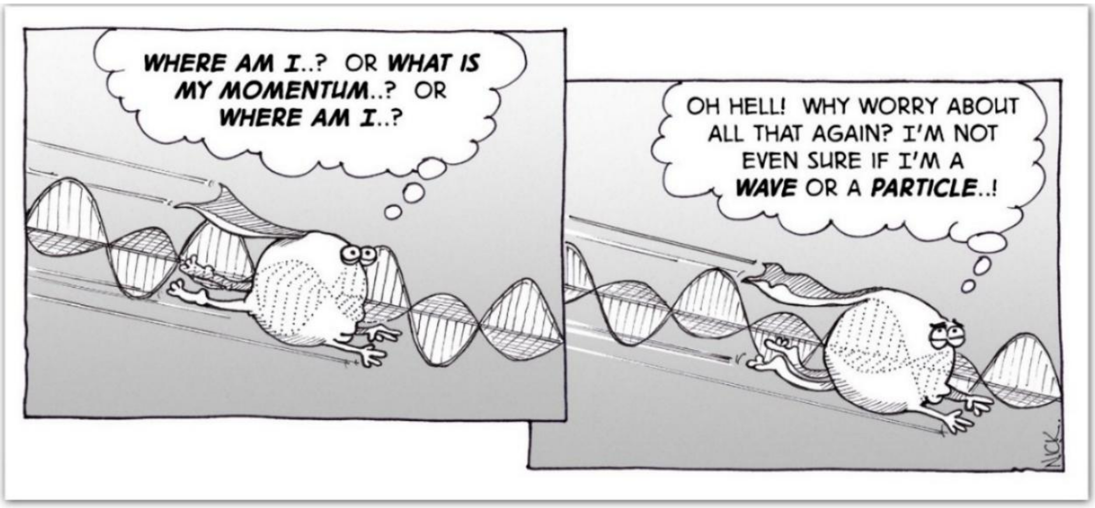
Wave-particle Duality

Let's try to explain what seems to be a "paradox." Remember that particles can behave like waves one moment and like particles the next? This is called wave-particle duality. It characterizes particles in the observable universe. However, on the macroscopic scale, the wave nature of matter is unperceivable.

Important notice: It is because of wave-particle duality, an innate property of matter, and not because of experimental limitations, that Heisenberg's uncertainty principle exists.

[3] Momentum is the product of mass times velocity: $p = m \cdot v$

[4]



Photon self-identity issues

Figure 3: Heisenberg's Uncertainty Principle and Wave-Particle Duality

The de Broglie Hypothesis

Because of wave-particle duality, every particle can be described by a wave. The de Broglie hypothesis relates the momentum $\boldsymbol{p}$ of a particle to its wavelength $\boldsymbol{\lambda}$:

$$p = \frac{h}{\lambda} \tag{1}$$

$\boldsymbol{h}$: Planck's constant, a very small number ($h \approx 6.63 \cdot 10^{-34}\, Js$.)

State Vectors, Wavefunctions, and Probability Density

This is a tricky part, yet everything will soon make sense.

Matter waves, the waves that describe particles, are not physical and cannot be directly measured; they are mathematical representations related to the likelihood of a particle acquiring specific values of observables, such as position, momentum, energy, etc.

To make better sense of matter waves, let's examine how they manifest in the double-slit experiment.

[4] newsletter.oapt.ca/files/Uncertainty-Experiment.html

[5]

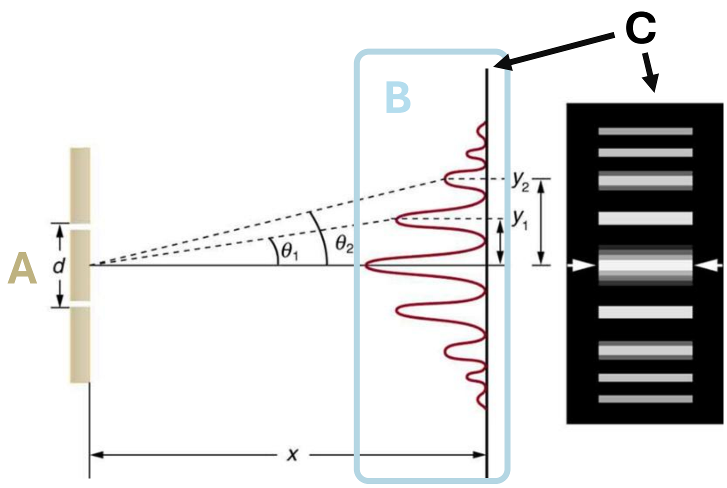


Figure 4: The double-slit experiment

In this experiment, electrons are fired towards a wall with two slits (Figure 4, area A), and their final position is marked on a screen (area C). After many measurements and while the electrons are not being observed, the concentration of the marks left behind by electrons varies and forms a **wave-like pattern** (area B). The pattern formed is the **probability density** for the position of the electron. Where the amplitude is larger, an electron is more likely to be found.

Probability density is closely related to wavefunctions (matter waves). However, it is **probability density**—not wavefunctions—that **is measurable**.

A wavefunction is a function usually denoted by the Greek letter $\boldsymbol{\psi}$. A wavefunction is derived from the so-called state vector $|\boldsymbol{\psi}\rangle$, written in Dirac notation. This vector contains all information about a quantum system, yet it is abstract. To acquire physical meaning, it must be expressed using a basis. Think of basis as a system of axes, where each axis is a different vector representing a unique direction, a unique state of the system (Figure 5). When the state vector is put in that basis, it acquires coordinates (c_1, c_2), which show how "close" the vector is to each state. In other words, the coordinates are related to the probability of the system being found in the respective state.

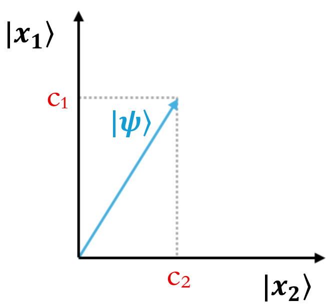


Figure 5: A State Vector Expressed in the Position Basis

Regardless of the basis used, the state vector remains the same; only perspective changes. To better understand this, imagine that different bases are different

[5] courses.lumenlearning.com/suny-physics/chapter/27-3-youngs-double-slit-experiment/

languages expressing the same idea, the same state vector, the same system. The most widely used bases are the position (Figure 5), and momentum basis.

Clarification: In any basis X, all observables can be examined when expressed in terms of X.

But… where does the wavefunction come from?

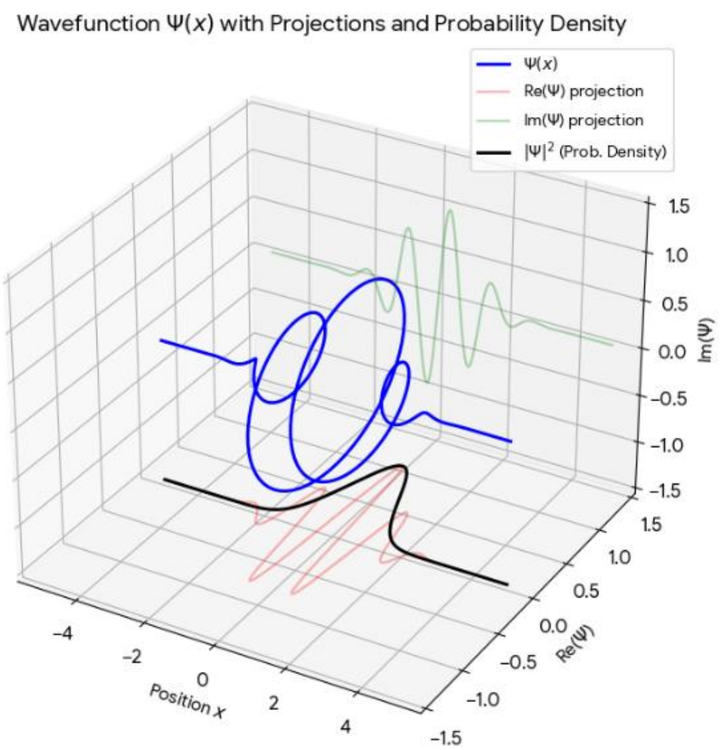
Since the state vector is located among all axes, it is a mixture of states, a so-called superposition. This is evident in its mathematical representation too:

$$|\psi\rangle = c_1|x_1\rangle + c_2|x_2\rangle + c_3|x_3\rangle + c_4|x_4\rangle + \cdots$$

From every $|\boldsymbol{x_n}\rangle$, we can get a number $\boldsymbol{x_n}$, which would be the position measurement if the state of the system was purely $|\boldsymbol{x_n}\rangle$. When $\boldsymbol{x_n}$ varies continuously, the system can be depicted using a function, which outputs the coordinate $\boldsymbol{c_n}$ for every number $\boldsymbol{x_n}$. This is the wavefunction.

Because $\boldsymbol{c_n}$ is complex and perhaps also negative, we must follow the **Born rule** to get the real, positive **probability density**, which is what is obtained experimentally after many measurements:

$$\boldsymbol{P = |\psi(x)|^2}$$



[6]

Figure 6: The Difference Between (Complex) Wavefunction and (Real, Positive) Probability Density

---

[6] Google Gemini. (2025). 3D Complex Wavefunction with Corrected Projections [Graph]. Generated using AI.

The de Broglie Hypothesis Applied to Wavefunctions: The Manifestation of Heisenberg's Uncertainty Principle

Let's compare the real part of the wavefunctions of the same system in the position and momentum basis.

[7]

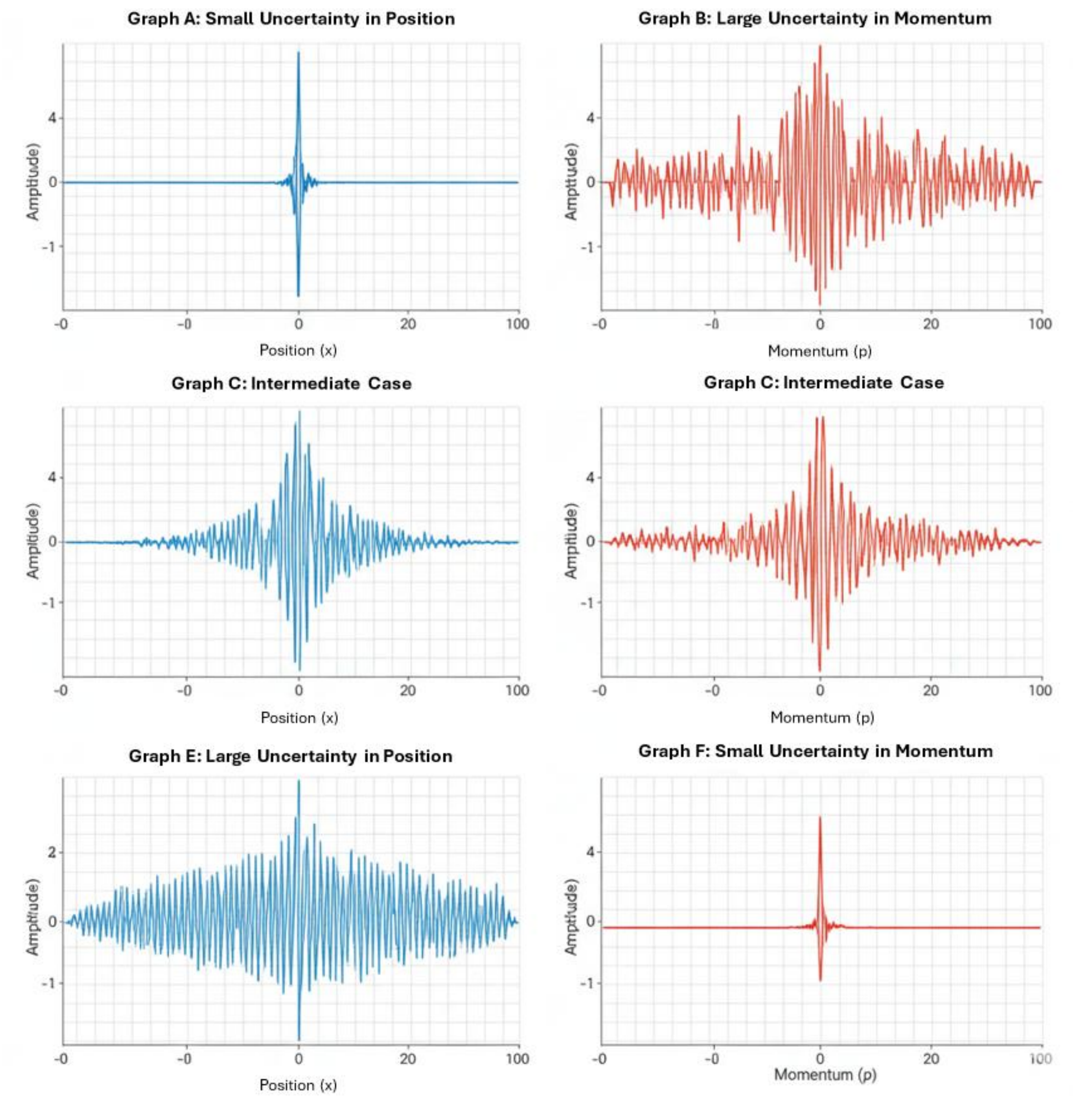


Figure 7: Wavefunction Comparison in the Position and Momentum Basis

In each row of Figure 7, two wavefunctions depict the same system at the same instant; they are different viewpoints of the same state vector.

For conjugate pairs, to get one wavefunction from the other, there exist operations: the Fourier transforms.

[7] Google Gemini. (2025). Wavefunction Comparison in the Position and Momentum Space: All Cases [Graph]. Generated using AI.

IMPORTANT! Localized ripples correspond to a decreased uncertainty in that basis because they yield localized probability density, thus the range of possible values shortens.

Graph A: Ripples are localized, thus uncertainty in **position** is small. However, due to the few ripples resulting from the composition of many wavelengths, wavelength cannot be calculated. Consequently, the de Broglie hypothesis cannot be used to calculate momentum and uncertainty in momentum is large. This is evident in graph B too, where the ripples are spread-out.

Graph E: Ripples are spread-out. Therefore, uncertainty in position is large. However, because the graph resulted from few wavelengths being composed, wavelength can be calculated. The de Broglie hypothesis can, thus, be used to calculate the small uncertainty in **momentum**.

Graphs E and **F** depict an intermediate case, where position is known to a certain extent and momentum to a smaller degree or vice versa.

Mathematical Expression

Heisenberg's uncertainty principle (for position and momentum):

$$\Delta x \Delta p \geq \frac{1}{2}|\langle\psi|[\hat{x},\hat{p}]|\psi\rangle| \tag{2}$$

("What is—…I can't even pronounce it!" This is the only normal reaction. But things are not as tricky as they seem.)

$|\boldsymbol{\psi}\rangle$: State vector.

$\langle\boldsymbol{\psi}|$: Complex conjugate of $|\boldsymbol{\psi}\rangle$, i.e. if $|\psi\rangle = a + ib, \langle\psi| = a - ib$.

$[\hat{\boldsymbol{x}},\hat{\boldsymbol{p}}]$: Commutator of $\hat{x},\hat{p}$ defined as $[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$.

$[\hat{x},\hat{p}] \neq 0$, since $\hat{x},\hat{p}$ do not commute.

$\boldsymbol{\Delta x}$: Uncertainty in position, i.e. the standard deviation of position, the "spread" of values that position may take.

$\boldsymbol{\Delta p}$: Uncertainty in momentum, the standard deviation of momentum, the "spread" of values that momentum may take.

Since $[\hat{x},\hat{p}] \neq 0$ (and $\frac{1}{2}|\langle\psi|[\hat{x},\hat{p}]|\psi\rangle| > 0$), $\Delta x\Delta p \neq 0$. That is why uncertainty cannot be zero.

Inequality (2) simplified:

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi} \tag{3}$$

$\boldsymbol{h}$: Planck's constant.

$\boldsymbol{\pi}$: The well-known number: $\pi = 3.141592653589793$ ... —all right, that is enough.

$\boldsymbol{\frac{h}{4\pi}}$: A number ($\frac{h}{4\pi} \approx 5{,}27 \cdot 10^{-35}$ Js).

Heisenberg's Uncertainty Principle in the Macroscopic World

The baseball player can finally help!

We will prove why the uncertainty principle is ignored in the macroscopic world by calculating $\boldsymbol{\Delta x}$ for an **electron** and for a **macroscopic object.**

The baseball player offered us a **baseball**, as our macroscopic object.

A simplification:

1. The baseball and the electron have the same uncertainty in velocity, so that Δx depends exclusively on mass.
2. The baseball is a single particle having the standard mass of a baseball.

Let the uncertainty in velocity be $\Delta v = 50\frac{m}{s}$.

$\boldsymbol{\Delta x}$ for the **electron**:

$$\Delta p = m_e \cdot \Delta v = m_e \cdot 50\frac{m}{s}$$

$$\Delta x \geq \frac{h}{4\pi \cdot m_e \cdot 50\frac{m}{s}} \Rightarrow \qquad \text{(formula (3) rearranged)}$$

$$\Delta x \geq 10^{-10} m$$

The mass of the electron is very small, so Δp is very small too. Consequently, Δx is relatively large. $\Delta x \geq 10^{-10} m$ is significant on the subatomic scale, which is why Heisenberg's uncertainty principle is fundamental in quantum mechanics.

$\boldsymbol{\Delta x}$ for the **baseball**:

$$\Delta p = m_b \cdot \Delta v = m_b \cdot 50\frac{m}{s}$$

$$\Delta x \geq \frac{h}{4\pi \cdot m_b \cdot 50\frac{m}{s}} \Rightarrow$$

$$\Delta x \geq 10^{-36} m$$

Δx for a baseball is very small for the accuracy of the macroscopic measuring devices. That is why Heisenberg's uncertainty principle is not considered in classical mechanics. It is not inexistant; it is just insignificant.

TO THINK…

Does Heisenberg's uncertainty principle tell us that particles do not have a specific position and momentum simultaneously or that we cannot know it?

The Bohr Model

Introduction

In 1913, the model suggested by Danish physicist Niels Bohr was an important step in the development of atomic models and quantum mechanics. Today, while that model is still useful, it does not always depict reality.

The Contradiction with Heisenberg's Uncertainty Principle

Bohr suggested that electrons orbit the nucleus like planets at fixed distances. This implies that their trajectory is predetermined and that their position is known.

[8]

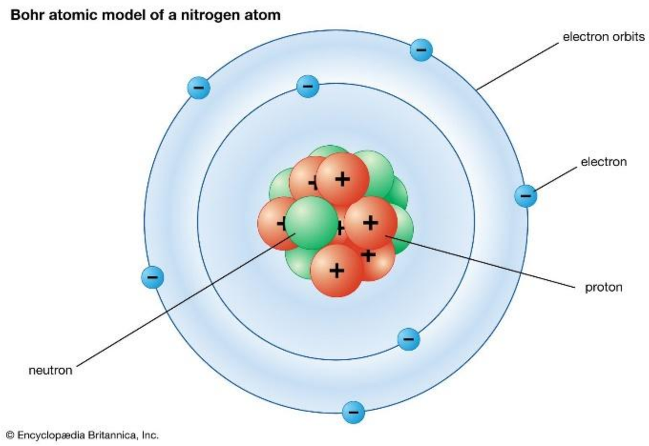


Figure 8: The Bohr Model

However, according to Heisenberg, position and momentum are uncertain.

[8] www.britannica.com/science/Bohr-model

Bohr based his model on classical mechanics, hence the contradiction. In reality, we do not know which path electrons follow. That is why we use **probability density clouds** for their position.

[9]

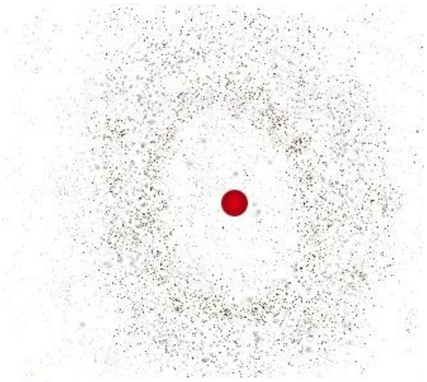


Figure 9: Electron Probability Density Cloud

The Contradiction Viewed Mathematically

Consider the Bohr model for the hydrogen atom:

1. The atom has one electron.
2. Atomic radius: $r = 5{,}3 \cdot 10^{-11} \frac{m}{s}$
3. Electron velocity: $v = 2{,}2 \cdot 10^{6} \frac{m}{s}$
4. Electron mass: $m_e = 9{,}11 \cdot 10^{-31} kg$

Let $\Delta v \approx v$ (uncertainty in velocity).

$$\Delta p = m_e \cdot \Delta v = 9{,}11 \cdot 10^{-31} kg \cdot 2{,}2 \cdot 10^{6} \frac{m}{s} \Rightarrow$$

$$\Delta p = 2 \cdot 10^{-24} kg \cdot \frac{m}{s}$$

According to Heisenberg's uncertainty principle,

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi} \Rightarrow \Delta x \geq \frac{h}{4\pi \cdot \Delta p} \Rightarrow$$

[9] www.britannica.com/science/Bohr-model

$$\Delta x \geq \frac{6.626 \cdot 10^{-34}\, Js}{4\pi \cdot 2 \cdot 10^{-24} kg \cdot \frac{m}{s}} \Rightarrow$$

$$\Delta x \geq 2{,}6 \cdot 10^{-11} m$$

Consequently, the electron can be found in an area of about $2{,}6 \cdot 10^{-10} m$ in diameter or greater. Therefore, it could be found closer or further from the nucleus than indicated by the Bohr radius. Thus, the electron not only does not orbit the nucleus at a fixed distance but exists within a probability cloud, which includes but is not limited to the Bohr radius.

The End of Our Journey

We have reached our destination! Now that we are familiar with Heisenberg's uncertainty principle, we are ready to consider a **new perspective**.

Throughout our journey, did you ever wonder if the uncertainty principle is just a concept in quantum mechanics? ("Of course not," we would all say. "All that time we were squeezing our brains to understand all that stuff!" It's true. So, we can think about it now.) Even though Heisenberg refers to a principle applicable on an unperceivable scale, we may draw conclusions about our life too. We can relate nature's probabilistic existence to our own uncertainties. Just as small uncertainty in position causes great uncertainty in momentum, choices made in life may influence or limit the available options when making another decision.

Tennis player: "Hey! We're here too! Stop philosophizing! I can't stand thinking about uncertainty anymore!"

Wait a second… Is it maybe uncertainty that stresses out athletes? Perhaps anything that can affect their performance matters, even if it is imperceptible to humans. We may never know. However, a lot of them will most probably keep their "rituals" to ensure that everything will go as planned.

[10]

Figure 10: Rafael Nadal Arranging His Bottles: "This Way, I Make Them the Same Each Time So I Can Concentrate Solely on the Game and What Lies Ahead."

[10] optimizemindperformance.com/study-shows-rituals-have-positive-impact-on-performance/

Author Biographies

Born in 2009 in Greece, Marina Giannakopoulou is a student in 11th grade (2025-2026) in Athens College, Athens, Greece. She was awarded the 2024 Athens College Junior High School "Bodossakis Athanasiadis Memorial" Award and the 2025 Merit Award. Being interested in theoretical physics, she has, among other activities, taken the course "Ten Years of the Higgs Boson" by Dr. Humberto Gilmer during the 2025 Brown Pre-College Programs, Brown University, US, and she has participated and presented her project in the 2nd Balkan Student Summer School on Quantum Physics (2025), Thessaloniki, Greece.

References

"Bohr's Atomic Model." *Libre Texts Chemistry*, Libre Texts Libraries, n.d., chem.libretexts.org/Bookshelves/Introductory_Chemistry/Introductory_Chemistry_(CK-12)/05%3A_Electrons_in_Atoms/5.06%3A_Bohr's_Atomic_Model. Accessed 5 Jan. 2025.

"Conjugate Observables in Quantum Mechanics." *The Quantum Well*, Obsidian Publish, n.d., publish.obsidian.md/myquantumwell/Quantum+Mechanics/Quantum+Measurement/Conjugate+observables+in+quantum+mechanics. Accessed 2 Jan. 2025.

Dalal, Mandeep. *A Textbook of Physical Chemistry – Volume 1*. Dalal Institute, 1 Jan. 2018, pp. 52-54, 24-25, 28, 63-64.

Dwyer, David. *Theoretical Physics*. 2023, pp. 113-118, 94-95.

Griffiths, David J., and Darrell F Schroeter. *Introduction to Quantum Mechanics*. Cambridge University Press, 2018, pp. 17-19, 35, 119, 142, 148, 151.

Hilgevoord, Jan, and Jos Uffink. "The Uncertainty Principle." *Stanford Encyclopedia of Philosophy*, The Metaphysics Research Lab, 2016, plato.stanford.edu/entries/qt-uncertainty/. Accessed 2 Jan. 2025.

Merzbacher, Eugen. *Quantum Mechanics*. John Wiley and Sons, Inc, 1998, pp. 2, 4, 8, 15-18, 217-219.

Pultarova, Tereza. "The Bohr model: The famous but flawed depiction of an atom." *Space.com*, Future US, 31 Oct. 2022, www.space.com/bohr-model-atom-structure. Accessed 5 Jan. 2025.

"Quantum mechanics." *University of Waterloo*, University of Waterloo, 2025, uwaterloo.ca/institute-for-quantum-computing/resources/quantum-101/quantum-mechanics. Accessed 2 Jan. 2025.

Schreiber, Barbara A, et al. “Uncertainty Principle.” *Britannica*, Encyclopedia Britannica, 2024, www.britannica.com/science/uncertainty-principle. Accessed 2 Jan. 2025.

Susskind, Leonard, and Art Friedman. *Quantum Mechanics*. Penguin Random House, 2015, pp. xix, xx.

Figure Credits

Arrazola, Juan Miguel. “Anyone can understand quantum mechanics — Part 1.” *Qonaom*, n.p., 2 May 2016, qonaom.wordpress.com/2016/05/02/anyone-can-understand-quantum-mechanics-part-1/. Accessed 2 Jan. 2025.

Ball, James. “An Experiment Involving Heisenberg’s Uncertainty Principle.” *OART Newsletter*, Ontario Association of Physics Teachers, 29 Jan. 2016, newsletter.oapt.ca/files/Uncertainty-Experiment.html. Accessed 2 Jan. 2025.

“Bohr Model.” *Britannica*, Encyclopedia Britannica, 19 Dec. 2024, www.britannica.com/science/Bohr-model. Accessed 5 Jan. 2025.

“3D Complex Wavefunction with Corrected Projections and Probability Density [Graph]” prompt. *Google Gemini*, Gemini 3 Pro, Google, 28 Dec. 2025, gemini.google.com.

Graham, Michael. “Rafael Nadal opens up on water-bottle obsession, saying: ‘If they aren’t right, I can’t concentrate on the game’.” *Tennis Head*, Tennishead.net, 7 Feb. 2020, tennishead.net/rafael-nadal-opens-up-on-water-bottle-obsession-saying-if-they-arent-right-i-cant-concentrate-on-the-game/. Accessed 5 Jan. 2025.

OpenStax College. “Young’s Double Slit Experiment.” *Physics*, n.p., n.d., courses.lumenlearning.com/suny-physics/chapter/27-3-youngs-double-slit-experiment/. Accessed 22 Aug. 2025.

“Research Shows Rituals Have Positive Impact on Mental Performance.” *Optimize Mind Performance*, Optimize Mind Performance, 18 May 2023, optimizemindperformance.com/study-shows-rituals-have-positive-impact-on-performance/. Accessed 5 Jan. 2025.

“Wavefunction Comparison in the Position and Momentum Space: All Cases [Graph]” prompt. *Google Gemini*, Gemini 3 Pro, Google, 29 Dec. 2025, gemini.google.com.

“Werner Heisenberg (1901- 76).” *A History of Science*, n.p., n.d., geoffneilsen.wordpress.com/navigation/twentiethcentury-draft/werner-heisenberg-1901-76/. Accessed 2 Jan. 2025.