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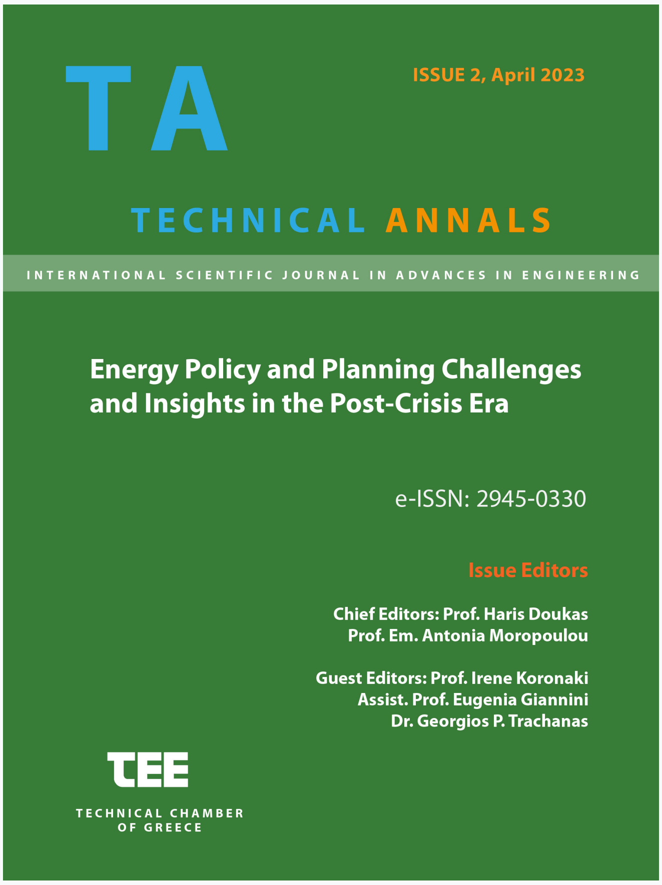

A planar geometrical non-linear bistable auxetic metamaterial mechanism

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A planar geometrically nonlinear bistable auxetic metamaterial mechanism for programmable energy-saving structures

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Abstract. The EU and its member states have agreed to reduce carbon dioxide emissions. Building operation and construction are the primary contributors to greenhouse gas emissions. Construction products have a huge environmental impact throughout their life cycle and are also the main source of waste generation. The construction sector can not embrace sustainability only by reusing and recycling materials. Adaptable multifunctional materials play a critical role towards energy-saving and green transition. In particular, adaptable structures can significantly reduce the time and cost of manufacture, transport, and construction. Mechanical metamaterials are artificial systems that can produce desired physical and mechanical properties by designing the base cell of which the lattice is composed. A structure that has negative Poisson's ratio is called auxetic and bistability is the property of having two stable equilibrium positions within the range of its motion-deformation. The present study develops a planar bistable auxetic mechanical metamaterial based on a re-entrant arrowhead auxetic topology and analyses the structure's mechanism and its properties. In the paper, we first give the geometric description and then we study the theory for the analysis of the metamaterial mechanism. Finally we present an example of the base cell.

Keywords: Adaptable structure, Mechanical Metamaterial, Mechanism, Bistable Structure, Auxetic Structure, Geometric Nonlinearity.

1 Introduction

1.1 Overview

Structural adaptability is an innovative technique that architecture focuses on to address energy concerns. One way to implement adaptability in structures is to use materials that enables shape transformation. Our structure is focused on eliminating the need for constant energy input to maintain transformation. Metamaterial mechanisms are structures with tailored physical and mechanical properties defined by their architecture rather than their chemical composition [1]. A bistable mechanism has two stable equilibrium positions within its range of motion. It achieves this behaviour by storing energy during part of its motion and then releasing it as the mechanism moves toward a second stable state [2]. The two stable states can be programmed by the base cell design. In this paper we focus on the re-entrant arrowhead auxetic topology [3] and we use it to create a two-dimensional bistable auxetic structure. Materials that have a negative Poisson's ratio when stretched, they become thicker perpendicular to the applied force. Such materials or structures are called auxetic [3]. The term auxetic derives from the Greek word αυξητικός (auxetikos) which means "that which tends to increase" and has its root in the word αύξησις, or auxesis, meaning "increase". Various structures, that present auxetic behaviour, have been studied so far [4-7]. Our structure displays auxetic properties and bistable behaviour as well. If the deformation gradient is large enough then the nonlinear terms of strain tensor cannot be overlooked and the structure exhibits geometric nonlinearity.

1.2 Programmable energy-saving structures

Shape transformation is crucial in many applications ranging from nanoscale to macro scale. There is a need for flexibility in the construction sector. Structures are expensive, energy-intensive and their skin will outlast their original use. Some buildings are more prone to demolition, while others are better suited to redevelopment. Critical to the above is the design and construction of programmable structures that can evolve according to different requirements or be erected in a more energy-efficient manner in various environments such as outer space or deep sea.

2 Base cell

Our model is a linkage-based periodic structure composed by a system of rigid bodies connected with elastic/rotary hinges (revolute joints). The rigid bodies are 1- or 2-dimensional polytopes (i.e. links, triangles) (see Fig.1,2).

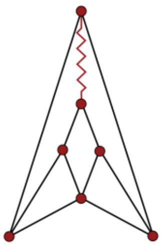

Fig.1 Base cell.

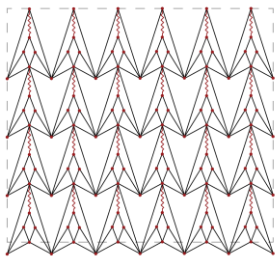

Fig.2 Lattice structure.

2.1 Degrees of freedom

The rigidity of the structure depends on the stiffness of the linear spring. If we consider the linear spring as an undeformed edge then the unit cell is a two-dimensional minimally rigid graph i.e. a Laman graph (our base cell with Nvertices has $2\mathrm{N} - 3$edges and no N'-vertex subgraph has more than $2\mathrm{N}' - 3$ edges) [8]. The rigidity can also be easily proven by a Henneberg construction (see Fig.3).

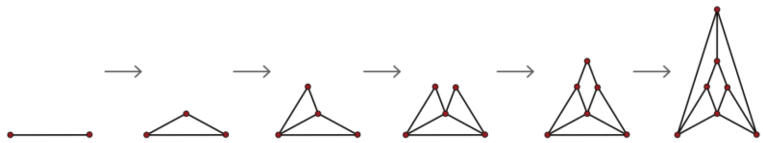

Fig.3 Henneberg construction.

The deformation mechanism is derived from the elastic rotational joints of the undeformed links and from the stiffness of the linear spring. Using the Chebychev–Grübler–Kutzbach formula ($M = 3(n-1-j) + \sum_{i=1}^{j} f_i$) we can calculate the mobility $M(DOF)$ of a system formed from $n$links and $j$joints each with $f_i, (i = 1,\dots,j)$degrees of freedom. For the present mechanical system $n = 10$, $j = 13$ and $f_i, (i = 1,\dots,13)$thus the mobility of the system is:

$$M = 3(10 - 1 - 13) + \sum_{i=1}^{13} f_i = 1 \tag{1}$$

The mechanical system has 1DoF in 2D space. So, we need only one independent parameter to define the configuration of the kinematic chain. The angle $\theta$is the independent parameter needed.

2.2 Geometry

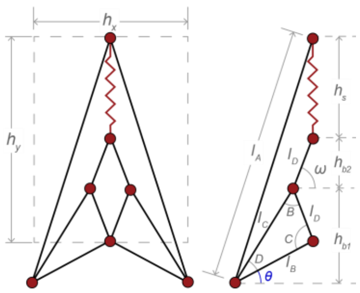

Fig.4 Base cell dimensions.

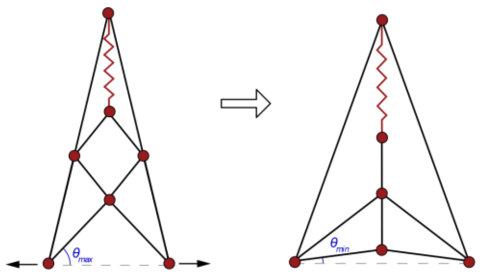

Fig.5 Mechanism's movement.

As illustrated in Fig. 4, the shape and size of this system can be described in terms of θthrough an orthogonal unit cell in $e_1 e_2$-plane with side lengths h_1, h_2given by:

$$h_1 = 2 l_B cos\theta,\ h_2 = \sqrt{{l_A}^2 - (l_B cos\theta)^2} - l_B sin\theta \tag{2}$$

Based on the desired outcome of the bistable state, the user can set the following parameters: $l_A, l_B, l_C, \widehat{D}, k_r, k_l$. During the deformation and for physically realistic structures where the triangles do not overlap (see Fig.5), the range $(\theta_{min}, \theta_{max})$of the angle $\theta$is:

$$\theta_{min} = \frac{\pi}{2} - \widehat{D} - \widehat{B}, \theta_{max} = arctan\left(\frac{1}{tan(\widehat{D})} - \frac{l_B}{l_A sin(\widehat{D})}\right)$$

$$\widehat{B} = \begin{cases} arcsin\left(\frac{l_B}{l_D} sin\widehat{D}\right) & if \frac{l_B}{l_C} cos\widehat{D} < 1 \\ \frac{\pi}{2} & if \frac{l_B}{l_C} cos\widehat{D} = 1 \\ \pi - arcsin\left(\frac{l_B}{l_D} sin\widehat{D}\right) & if \frac{l_B}{l_C} cos\widehat{D} > 1 \end{cases} \tag{3}$$

Also, the following manufacturing parameter restrictions must apply:

$$l_B \leq l_A, \widehat{D} \leq arccos\left(\frac{l_B}{l_A}\right), l_C \leq \frac{h_y + 2l_B sin\theta}{2sin(\theta + \widehat{D})} \tag{4}$$

3 Geometrically nonlinear strain theory

3.1 Strain tensor

The displacement vector $(u)$of the mechanism during its deformation without rigid-body translation is the difference between the deformed (x) and the undeformed (X) configuration $(u_i = x_i - X_i)$. Thus, the displacement gradient tensor is $\nabla u = u_{i,j} = \frac{\partial(x_i - X_i)}{\partial x_j} = \frac{\partial x_i}{\partial x_j} - \frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} = \delta_{ij} - X_{i,j}$. If the displacement gradient is large enough $(\nabla u > 10^{-3})$to invalidate the assumptions of the infinitesimal strain theory $\left(\frac{\partial u_i}{\partial x_j} \neq \frac{\partial u_i}{\partial X_j}\right)$then the body exhibits geometric nonlinearity. According to the above, the finite strain tensor is defined as:

$$\varepsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - X_{k,i}X_{k,j}\right) = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{5}$$

The lattice of the present model is a planar structure that cannot shear during deformation $\left(\frac{\partial X_1}{\partial x_2} = \frac{\partial X_2}{\partial x_1} = 0\right)$. Thus, the strain tensor has the form:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1 - \left(\frac{\partial X_1}{\partial x_1}\right)^2 - \left(\frac{\partial X_2}{\partial x_1}\right)^2\right) \\ \frac{1}{2}\left(1 - \left(\frac{\partial X_1}{\partial x_2}\right)^2 - \left(\frac{\partial X_2}{\partial x_2}\right)^2\right) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1 - \left(\frac{\partial X_1}{\partial x_1}\right)^2\right) \\ \frac{1}{2}\left(1 - \left(\frac{\partial X_2}{\partial x_2}\right)^2\right) \\ 0 \end{bmatrix} \tag{6}$$

The domain of $\theta$is a subset of $\left[0, \frac{\pi}{2}\right]$ $\left([\theta_{min}, \theta_{max}] \subseteq \left[0, \frac{\pi}{2}\right]\right)$. Utilizing the strain tensor of the continuum body (∂X) to our planar $(h_1 x h_2)$ mechanism during $\theta_\alpha = 0$ to θ_{max}deformation we get:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{2}\left(1 - \left(\frac{h_1(\theta_\alpha)}{h_1(\theta)}\right)^2\right) = \frac{1}{2}\left(1 - \left(\frac{2l_B cos0}{2l_B cos\theta}\right)^2\right) = \frac{1}{2}\left(1 - \frac{1}{cos^2\theta}\right) \\ &= \frac{-tan^2\theta}{2} < 0, \qquad \forall\theta \in \left(0, \frac{\pi}{2}\right] \end{aligned} \tag{7}$$

$$\varepsilon_2 = \frac{1}{2}\left(1 - \left(\frac{h_2(\theta_\alpha)}{h_2(\theta)}\right)^2\right) = \frac{1}{2}\left(1 - \frac{{l_A}^2 - {l_B}^2}{\left(\sqrt{{l_A}^2 - (l_B cos\theta)^2} - l_B sin\theta\right)^2}\right) \tag{8}$$

$$\varepsilon_2 < 0 \Rightarrow \frac{1}{2}\left(1 - \frac{{l_A}^2 - {l_B}^2}{\left(\sqrt{{l_A}^2 - (l_B cos\theta)^2} - l_B sin\theta\right)^2}\right) < 0 \Rightarrow l_B < l_A \tag{9}$$

This inequality is a manufacturing parameter restriction of the structure (Eq.4). When we deform the mechanism from $\theta_a = 0$to $\theta_{max}$the strain tensor becomes $\varepsilon_1 < 0$and $\varepsilon_2 < 0$. Respectively it turns out that by deforming the mechanism from $\theta_b = \frac{\pi}{2}$towards θ_{min}we get $\varepsilon_1 > 0$and $\varepsilon_2 > 0$. Therefore:

$$\frac{\varepsilon_1}{\varepsilon_2} > 0, \frac{\varepsilon_2}{\varepsilon_1} > 0 \ \forall\theta \in [\theta_{min}, \theta_{max}] \tag{10}$$

3.2 Poisson's ratio

The Poisson's ratio $(v)$for a stable, isotropic, linear elastic material must be between $-1.0$and $+0.5$due to the requirement that the modulus of elasticity $(E)$the shear modulus $(G)$and the bulk modulus (B), have positive values, but this is not binding for anisotropic elastic materials [9,10]. In the small strain regime the Poisson's ratio is constant, in large strain this ratio is a scalar function that varies with strain [11, 12]. For our structure according to Eq. 10 the Poisson's ratio is defined as:

$$v_{12} = \frac{-\varepsilon_2}{\varepsilon_1} < 0, v_{21} = \frac{-\varepsilon_1}{\varepsilon_2} < 0, \ \forall\theta \in [\theta_{min}, \theta_{max}] \tag{11}$$

4 Mechanism analysis

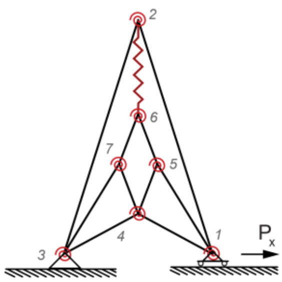

Fig.6 2D pseudo rigid body model.

4.1 Potential energy

According to the first law of thermodynamics, in an isolated system where no losses are created in the form of heat $(Q)$and assuming that there is no conversion into kinetic energy $(K)$the total potential energy $(\Pi)$is defined as:

$$\Pi = U - W \tag{12}$$

Where $(U)$is the internal strain energy and $W$is the work done by the external forces.

4.2 Strain energy

To design the bistable structure we construct a planar system of six rigid bodies connected with seven elastic/rotary hinges (revolute joints), as well as one elastic link (i.e. linear spring). According to the 2D pseudo rigid body model (see Fig.6) the strain energy of the base cell consists of the sum of the energy stored in each spring (torsional $U_r$and linear U_l).

$$U = U_l + U_r \tag{13}$$

Strain energy of linear spring. $U_l$is the strain energy from the linear spring, $k_l \left[\frac{N}{m}\right]$is the stiffness constant of the linear spring and $\Delta\psi[m]$is the deformation of the spring.

$$U_l = \frac{1}{2} k_l (\Delta\psi)^2 \tag{14}$$

$$\Delta\psi = \psi(\theta) - \psi(\theta_0),\ \psi = \sqrt{{l_A}^2 - (l_B cos\theta)^2} + l_B sin\theta - 2 l_C sin(\theta + \widehat{D}) \tag{15}$$

Strain energy of torsional springs. The strain energy from the torsional springs is the sum of the energy of each individual torsional spring. $k_l \left[\frac{N*m}{rad}\right]$is the stiffness constant of the torsional spring and $\Delta\varphi$is the deflection angle of each spring based on the initial undeformed place $\varphi(\theta_0)$.

$$\begin{gathered} U_r = \tfrac{1}{2}\textstyle\sum_{i=1}^{7} k_r (\Delta\varphi_i)^2, \Delta\varphi_i = \varphi_i - \varphi_{i0}, i{:}\{1,2,.|.7\}, \\ \varphi_1 = \varphi_3 = \theta, \\ \varphi_2 = arcsin\left(\frac{l_B}{l_A} cos\theta\right) \\ \varphi_4 = \pi - 2\theta, \\ \varphi_5 = \varphi_7 = 2(\hat{C} - \theta) \end{gathered} \tag{16}$$

4.3 Mechanism's equilibrium

Bistable structure. The storage and release of energy defines the structure's stable equilibrium positions. Equilibrium is established at a point in configuration space when no external forces $P$are required to maintain the structure's position (the work done by the external forces $W$is equal to zero). In these positions the total potential energy of the system has an extrema. $\frac{\partial\Pi}{\partial\theta}\,|_{\theta=\theta_0} = 0$

The equilibrium position is characterized as stable if for every possible small displacement from the equilibrium position the system tends to return to the same position, i.e.

the total potential energy is at a local minimum and thus increases during the displacement. $\frac{\partial \Pi}{\partial \theta}|_{\theta=\theta_0} > 0$

The equilibrium position is characterized as unstable if for every possible small displacement from the equilibrium position the system tends to move even further away from the initial position, i.e. the total potential energy is at a local maximum and thus decreases during the displacement. $\frac{\partial \Pi}{\partial \theta}|_{\theta=\theta_0} < 0$

There are also mechanisms where if the system shifts to a new position, it will remain in that position, that is, each position is an equilibrium position. This balance is characterized as neutral. The potential energy of such a system does not change. To investigate the precise stability of this system, higher order derivatives must be examined.

The above is the energy method and it is based on the Lagrange-Dirichlet theorem, which states that "when the potential energy has a minimum for an equilibrium position, the equilibrium position is stable" [13]. For a structure to be bistable in a given configuration, it must meet three criteria: (α) The function of the potential energy must have three critical points (extrema). (β) The second derivative of potential energy must be positive in two of these solutions, indicating two stable states, while it must be negative in all other solutions, indicating unstable positions. (γ) The two stable positions as well as at least one of the unstable positions must be viable positions (inside the range of its defined motion) [14].

Strain energy of torsional springs. If we remove the linear spring from the structure ($\boldsymbol{U_l = 0}$)and $\boldsymbol{k_r}$is chosen to be the only non-zero spring constant, then the internal strain energy consists only of the energy stored in the torsional springs. So, the equilibrium position where the potential energy of the system has an extrema is:

$$\frac{\partial \Pi}{\partial \theta} = 0 \Rightarrow \frac{\partial U_r}{\partial \theta} = 0 \Rightarrow \sum_{i=1}^{7} k_r(\varphi_i - \varphi_{i0})\frac{\partial \varphi_i}{\partial \theta} = 0 \Rightarrow \varphi_i = \varphi_{i0} \tag{17}$$

Therefore, for this type of configuration (if we remove the linear spring) the initial undeformed state is the only equilibrium position the mechanism has. The structure does not have a bistable behaviour.

Strain energy of linear spring. If $\boldsymbol{k_l}$is chosen to be the only non-zero spring constant, then the total strain energy results from the strain energy of the linear spring ($\boldsymbol{U = U_l}$). So, the potential energy of the system presents critical points where $\boldsymbol{\frac{\partial \Pi}{\partial \theta}}$ becomes zero or is not defined.

$$\frac{\partial \Pi}{\partial \theta} = 0 \Rightarrow \frac{\partial U_l}{\partial \theta} = 0 \Rightarrow k_l(\psi - \psi_0)\frac{\partial \psi}{\partial \theta} = 0 \Rightarrow \left| \begin{array}{l} \psi - \psi_0 = 0 \\ \frac{\partial \psi}{\partial \theta} = 0 \end{array} \right. \tag{18}$$

$$\psi - \psi_0 = 0 \Rightarrow \theta = \theta_0\text{: initial undeformed state.} \tag{19}$$

$$\frac{\partial \psi}{\partial \theta} = 0 \Rightarrow \frac{{l_B}^2 sin(2\theta)}{2\sqrt{{l_A}^2 - (l_B cos\theta)^2}} + l_B cos\theta - 2l_C cos(\theta + \widehat{D}) = 0 \Rightarrow \left| \begin{array}{l} \theta = \theta_1 \\ \theta = \theta_2 \end{array} \right. \tag{20}$$

The three critical points of the function are at $\theta_{0,}\theta_{1,}\theta_2$. Then with the second derivative of the potential energy $\frac{\partial^2 \Pi}{\partial \theta^2}$we find the maxima and minima and thus the stable and unstable equilibrium as stated before.

Total strain energy $\boldsymbol{k_r, k_l > 0}$. In a third case where the stiffness constants of both the torsional and the linear spring are different from zero, the internal strain energy of the mechanism is obtained according to Eq.13. Therefore, for the structure to present bi-stable behaviour the ratio $\boldsymbol{k_r/k_l}$is:

$$\frac{\partial}{\partial\theta}U = 0 \Rightarrow \frac{k_r}{k_l} = -\Delta\psi\frac{\partial\psi}{\partial\theta}\left(\sum_{i=1}^{7}\Delta\varphi_i\frac{\partial\varphi_i}{\partial\theta}\right)^{-1} \tag{21}$$

4.4 Deformation load

In our structure the strain energy and the complementary strain energy are equal. Thus, from Castigliano's first theorem will get:

$$P = \frac{\partial U}{\partial(x-X)} \Rightarrow P_i = \frac{\partial U}{\partial(h_i(\theta) - h_i(\theta_0))} = \frac{\frac{\partial U}{\partial\theta}}{\frac{\partial h_i(\theta)}{\partial\theta}} \tag{22}$$

5 Material properties

5.1 Density

Our structure is defined by geometric topological principles without being limited by the length scale of the mechanism. So, we can analyse our structure as a porous medium with the mechanical metamaterial part considered as the skeletal portion of the continuum body and the space in between as the pore network. We will set our base cell as the representative elementary volume of the material which includes the volume of both "phases":

V_α: volume of the mechanism
V_β: volume of the "empty" space
$V_\alpha + V_\beta = V$: bulk volume of the continuum body

For our structure the volume $(V)$is given by: $V = h_1 * h_2 * 1$. Thus, from Eq.2 the expression of the volume becomes:

$$V = 2l_B cos\theta\sqrt{{l_A}^2 - (l_B cos\theta)^2} - l_B^2 sin(2\theta) \tag{23}$$

A particularly important property of materials is the percentage of the volume occupied by their matter. This is the relative density and is the percentage of the mechanism's volume to the bulk volume $(\rho = V_\alpha/V)$. In our structure the volume of the mechanism (V_α)remains constant throughout its deformation. However, the same is not true for bulk volume. So, the relative density fraction in our two stable equilibrium states is:

$$\frac{\rho(\theta_2)}{\rho(\theta_0)} = \frac{V(\theta_0)}{V(\theta_2)} \tag{24}$$

Density and porosity affects properties of materials related to transport phenomena (water absorption, air permeability, thermal and electrical conductivity), mechanical properties and more.

5.2 Stiffness

During the movement of the mechanism the forces acting on it are in equilibrium.

$$\partial U = \partial W \Rightarrow \partial U = P\partial u \Rightarrow \frac{\partial U}{V} = \sigma\partial\varepsilon \tag{25}$$

The mechanical properties of the metamaterial that are necessary to create the transition from one stable position to the second are defined as:

$$\frac{\partial U}{V} = \sigma\partial\varepsilon \Rightarrow \sigma_{ij} = \frac{\partial U}{V\partial\varepsilon_{ij}} \Rightarrow \partial\sigma_{ij} = \frac{\partial^2 U}{V\partial\varepsilon_{ij}} \Rightarrow C_{ijkl}\partial\varepsilon_{kl} = \frac{\partial^2 U}{V\partial\varepsilon_{ij}} \Rightarrow C_{ijkl} = \frac{\partial^2 U}{V\partial\varepsilon_{ij}\partial\varepsilon_{kl}} \Rightarrow C_{ijkl} = \frac{\frac{\partial^2 U}{\partial\theta^2}}{V\frac{\partial\varepsilon_{ij}\partial\varepsilon_{kl}}{\partial\theta\partial\theta}} \tag{26}$$

6 Application

6.1 Geometry

The parameters that define the geometry are:

$$l_A = 15 * 10^{-3}m, l_B = 3 * 10^{-3}m, l_C = 5 * 10^{-3}m, \widehat{D} = \frac{\pi}{6,} k_r = 0 \tag{27}$$

Under the restrictions of the manufacturing parameters (Eq.4) we can state that this is a valid configuration. Based on the above parameters (Eq.27) and Eq.3 we derive that the domain of deformation is:

$$\theta \in [0.49,0.92](rad) \tag{28}$$

The side lengths (Eq.2) of the base cell at the two extreme positions are:

$$\begin{aligned} h_1 &= 2l_B cos\theta \Rightarrow \left| \begin{matrix} h_1(\theta_{min}) = 5.29mm \\ h_1(\theta_{max}) = 3.60mm \end{matrix} \right. \\ h_2 &= \sqrt{{l_A}^2 - (l_B cos\theta)^2} - l_B sin\theta \Rightarrow \left| \begin{matrix} h_2(\theta_{min}) = 13.35mm \\ h_2(\theta_{max}) = 12.50mm \end{matrix} \right. \end{aligned} \tag{29}$$

$$\begin{aligned} \left|\frac{\Delta h_1}{h_1}\right| &= \left|\frac{h_1(\theta_{min}) - h_1(\theta_{max})}{h_1(\theta_{max})}\right| = 0.32 > 10^{-3} \\ \left|\frac{\Delta h_2}{h_2}\right| &= \left|\frac{h_2(\theta_{min}) - h_2(\theta_{max})}{h_2(\theta_{max})}\right| = 0.064 > 10^{-3} \end{aligned} \tag{30}$$

As can be seen, the structure during tension (from the initial undeformed position $\theta_0 \equiv \theta_{max}$we go to θ_{min}) undergoes large deformations.

6.2 Equilibrium positions

Since only the linear spring contributes to the strain energy, then according to Eq.18 the equilibrium positions of the mechanism are:

$$\frac{\partial \Pi}{\partial \theta} = 0 \Rightarrow \frac{\partial U_l}{\partial \theta} = 0 \Rightarrow k_l(\psi - \psi_0)\frac{\partial \psi}{\partial \theta} = 0 \Rightarrow \begin{vmatrix} \theta_0 = 0.92 \\ \theta_1 = 0.79 \\ \theta_2 = 0.69 \end{vmatrix} \tag{31}$$

$$\left.\frac{\partial^2 \Pi}{\partial \theta^2}\right|_{\theta=\theta_0} > 0, \quad \left.\frac{\partial^2 \Pi}{\partial \theta^2}\right|_{\theta=\theta_1} < 0, \quad \left.\frac{\partial^2 \Pi}{\partial \theta^2}\right|_{\theta=\theta_2} > 0 \tag{32}$$

In this case our structure will have two stable positions (θ_0, θ_2) and one unstable (θ_1) within its range of motion. It stores energy during part of its motion $\theta \in [0.92,0.79]$ and then releasing it as the mechanism moves toward a second stable state $\theta \in [0.79,0,69]$.

6.3 Poisson's ratio

During the deformation at the bistable domain the strain tensor from Eq.6 becomes:

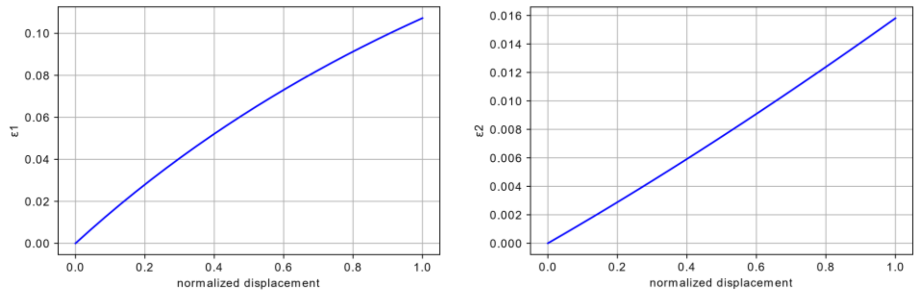

Fig.7 Strain tenso

Therefore the function of Poisson's ratio is:

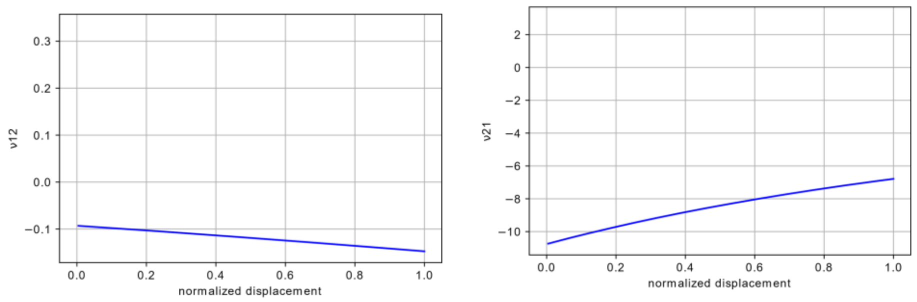

Fig.8 Poisson's ratio

6.4 Deformation load

$$P_i = \frac{\frac{\partial U}{\partial \theta}}{\frac{\partial h_i(\theta)}{\partial \theta}} = \frac{k_l \Delta\psi \frac{\partial \psi}{\partial \theta}}{\frac{\partial h_i(\theta)}{\partial \theta}} \Rightarrow \left| \begin{array}{l} \frac{P_1}{k_l} = \frac{\Delta\psi \frac{\partial \psi}{\partial \theta}}{\frac{\partial (h_1(\theta))}{\partial \theta}} \\ \frac{P_2}{k_l} = \frac{\Delta\psi \frac{\partial \psi}{\partial \theta}}{\frac{\partial (h_2(\theta))}{\partial \theta}} \end{array} \right. \quad (33)$$

The ratio of force by the stiffness of the linear spring as a function of normalized displacement is shown in the graph below (Fig.9). The force does not monotonically increase with displacement. The saw-tooth serrations are caused by snap-through buckling, from which the metamaterial bistability originates.

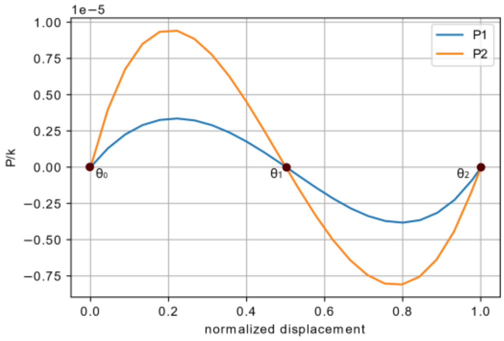

Fig.9 Deformation load.

6.5 Relative density

From Eq. 23 we get the bulk volume in the three equilibrium positions:

$$V(\theta_0) = 59.74mm^3 V(\theta_1) = 53.71mm^3 V(\theta_2) = 45.45mm^3 \quad (34)$$

So, the relative density fraction in our two stable states is:

$$\frac{\rho(\theta_2)}{\rho(\theta_0)} = \frac{V(\theta_0)}{V(\theta_2)} = 1.31 \quad (35)$$

6.6 Stiffness

The stiffness is negative between the maximum and minimum points of force.

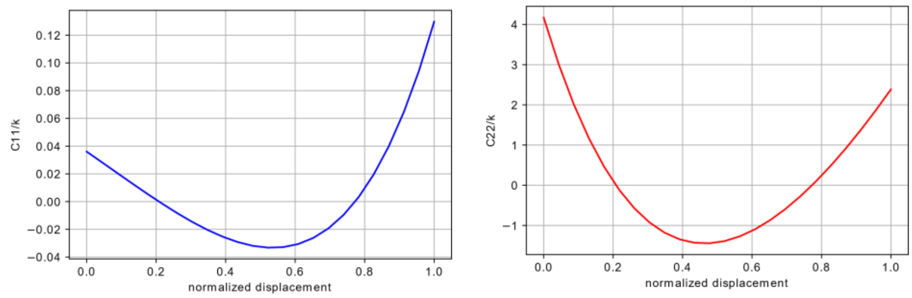

Fig.10 Stiffness tensor

7 Conclusion

In this paper, we presented a novel planar mechanical metamaterial that exhibits bistable and auxetic behaviour. The combined structural bistability and negative Poisson's ratio has not been previously observed in the re-entrant arrowhead topology. We believe the proposed mechanism enable the design of new programmable structures across scales and the reduction of energy consumption in various sectors.

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